MECHANICS

Part I · STATICS

By J. L. Meriam

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Part II · Dynamics

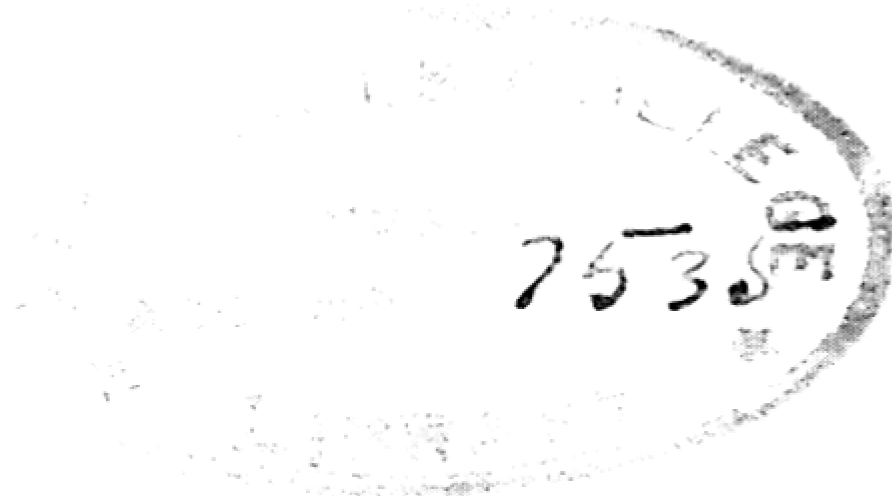
MECHANICS

Part I · STATICS

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PREFACE

To the Student

Engineering analysis hinges directly on the basic principles of mechanics. The study of mechanics welds the tools of physics, mathematics, and graphics into an effective weapon of attack on engineering problems. The emphasis in this book is on engineering or applied mechanics, and it has been designed as a text for the basic mechanics courses in the normal engineering curriculum.

The representation of real situations by mathematical and graphical symbols constitutes an ideal descriptive model which approximates but never quite equals the actual situation. A useful understanding of mechanics requires a dual process of repeated transition of thought between the physical situation and its symbolic representation. The development of ability to make this transition of thought freely is one of the major aims of this book.

Mechanics is based on a surprisingly few fundamental principles. The more important equations are set in boldface type, and much secondary detail has been reduced or eliminated. Care has been taken not to sacrifice the rigor of the development, and attempt has been made to present the principles in clear and concise terms. The student will find that firm progress can be made only by an understanding of the physical and mathematical principles jointly and not by mere memorization of formulas with mechanical substitution of values therein.

Success in analysis depends to a surprisingly large degree on a well-disciplined method of attack from hypothesis to conclusion where a straight path of rigorous application of principles has been followed. The student is urged to develop ability to represent his work in a clear, logical, and neat manner. Very often the mere adherence to good form and procedure will in itself prove to be the needed guide toward a successful solution. The basic training in mechanics is a most excellent place for early development of this disciplined approach which is so necessary in most of the engineering work which follows.

As in all subjects the student learns more when his interest is stimulated. The author hopes that the reader will find interest and stimula-

tion in many of the real and practical situations included in the problems. Arrangement of problems is generally in the order of increasing difficulty, and the most difficult ones are starred.

J. L. MERIAM

PREFACE

To the Instructor

The natural learning process begins with simple situations. Thus in mechanics the average student is best initiated by exposure to simplified, symbolic problems where irrelevant factors have been omitted and where attention is focused on the conditions which are pertinent. Problems presented in this way are already partially analyzed. A full and useful appreciation of mechanics does not come, however, until the analysis of real situations and actual working conditions is made. Here the student must be taught to define the problem by first isolating the pertinent factors and discarding the irrelevant ones. Principles are then applied and conclusions drawn. It is only when the principles of mechanics are applied to *practical* problems involving real situations that the full significance of mechanics can be seen. There has been a trend in the treatment of mechanics to avoid real problems of a practical and interesting nature in favor of the ideal symbolic problems which are stripped of reality, practical value, and interest. Such presentation places in jeopardy one of the most fundamental objectives of instruction in mechanics, namely, to develop ability in problem formulation where the connection between actuality and symbolic representation is required. It is true that many practical problems involve too many complicating factors for early exposure to the student. However, there is a wealth of problems which describe real, practical, and interesting situations that are not overly complex and which can enrich the experience and develop the ability of the student of mechanics far more than is possible with the overly idealized problems.

It is the purpose of this book to present a large selection of problems which illustrate wide application to the various fields of engineering and which will lead the student from the idealized and symbolic representation to the more practical, real, and interesting engineering situation. The author feels strongly that reality brought into the illustrations is of great help to the student in making the transfer of thought from the physical to the mathematical description. Consequently the book has been profusely illustrated, and effort has been made to produce reality and clarity. The problems in each set represent a considerable range of difficulty and are presented generally in order of increasing difficulty. Those problems which are considered the most difficult are starred. All problems have been worked and checked and are believed to be free of error. Computations have been made with the slide rule so that some disagreement in the third figures may be expected. Answers to approximately two thirds of the problems are given.

Full use of graphic procedures is made whenever they are of advantage. The graphic methods have not been isolated from the remainder of the treatment since it is felt that graphical representation is an invaluable aid to the interpretation of physical problems and should be an integral part of the development.

Chapter I presents a summary of the foundations of mechanics. An overall introduction to both statics and dynamics is quite appropriate at the outset and can be understood by the average student who has had an introduction to mechanics in his physics course. A second reading of Chapter I would be helpful before starting the work on dynamics.

Chapters II and III present the development of the resultants of force systems and the equilibrium of force systems with a minimum of attention to the many separate categories and special cases which many texts in mechanics differentiate so elaborately. Emphasis on the properties common to all cases is the best way to focus attention on the relatively few fundamental ideas involved. Furthermore this approach is made without excessive generality for the average student.

Chapter VI covers the equilibrium of statically determinate beams. No analysis of the distribution of force over the beam cross section is made since this aspect of the problem involves the properties of the material and more properly belongs in a course in strength of materials. Traditionally the statics problem of determining shear-force and bending-moment diagrams has been included in the strength-of-materials course, but the author feels that there is every reason to discuss a statics problem in the course in statics.

Chapter VIII covers the principles of virtual work and shows the type of problem for which this method is superior to the equilibrium method based on force and moment summation. In this chapter further advantage is taken of the opportunity to illustrate the use of the calculus.

Moments of inertia of areas and mass are treated in Appendix A and may be included in either the statics or dynamics part of the course as desired.

The collection of problems is almost entirely original. The author has, however, obtained ideas for problems from many published sources. The critical review and valuable comments of A. L. Hale and K. E. Barnhart are especially acknowledged along with numerous helpful suggestions made by others. The conscientious checking of all problems by a number of students should also be mentioned. For this help the author is deeply grateful.

J. L. Meriam

Berkeley, California
January 1952

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CHAPTER I

Principles of Mechanics

1. Mechanics. Mechanics is that physical science which deals with the state of rest or motion of bodies under the action of forces. There is no one subject which plays a greater rôle in engineering analysis than does mechanics. The early history of this subject is synonymous with the very beginnings of engineering. Modern research and development in the fields of vibrations, stability, strength of structures and machines, engine performance, fluid flow, electrical machines and apparatus, and molecular, atomic, and subatomic behavior are highly dependent upon the basic principles of mechanics. A thorough understanding of this subject is an absolute prerequisite for work in these and many other fields.

Mechanics is undoubtedly the oldest of the physical sciences. The earliest recorded writings in this field are those of Archimedes (287–212 B.C.) which concern the principle of the lever and the principle of buoyancy. Substantial progress awaited the formulation of the laws of vector combination of forces by Stevinus (1548–1620), who also formulated most of the principles of statics. The first investigation of a dynamic problem is credited to Galileo (1564–1642) in connection with his experiments with falling stones. The accurate formulation of the laws of motion including the law of gravitation was made by Newton (1642–1727), who also conceived the idea of the infinitesimal in mathematical analysis. Substantial contributions to the development of the theory of mechanics were made subsequently by Varignon, D'Alembert, Lagrange, Laplace, and others.

Before 1905 the laws of Newtonian mechanics had been verified by innumerable physical experiments and were considered the final description of the motion of bodies. The concept of *time*, considered an absolute quantity in the Newtonian theory, received a basically different interpretation in the theory of relativity announced by Einstein in 1905. The new concept called for a complete reformulation of the accepted laws of mechanics. The theory of relativity was subject to early ridicule but has had experimental check and is now universally accepted by physicists the world over. Although the difference between the mechanics of Newton

and that of Einstein is basic, there is no practical difference in the results given by the two theories except when velocities of the order of the speed of light (186,000 mi./sec.) are encountered.* Certain problems in atomic physics and celestial mechanics, for example, involve calculations based on the principles of relativity.

The subject of mechanics is logically divided into two parts, *statics*, which concerns the equilibrium of bodies under the action of forces, and *dynamics*, which concerns the motion of bodies. Dynamics in turn includes *kinematics*, which is the study of the motion of bodies without reference to the forces which cause the motion, and *kinetics*, which relates the forces and the resulting motions. *Theoretical mechanics* is primarily the concern of the physicist, while *applied* or *engineering mechanics* is of concern mainly to the engineer. There is no distinct separation between the two, however, since the question is one of emphasis.

2. Mathematical and Graphical Description. The science of mechanics is based on certain physical laws and utilizes freely the languages of mathematics and graphics. Mathematics is used as a description of the physical situation. It establishes the relationships between the various quantities involved and enables the prediction of effects to be made from these relations. The student must recognize that a dual thought process is necessary. He must think in terms of the physical situation, and he must also think in terms of the corresponding mathematical description. Analysis of every problem will require the repeated transition of thought between the physical and the mathematical. Without question one of the greatest difficulties that students have with mechanics is the inability to make this transition freely by interconnecting these two thought processes. The student should make a strong effort to connect each physical thought with its corresponding mathematical expression. He should recognize that the mathematical formulation of a physical problem represents an ideal limiting description or model which is approximated but never quite reached by the actual physical situation.

The language of graphics is an important tool of mechanics and serves in three capacities. First, it enables the representation of a physical system to be made on paper. This geometrical expression is vital to physical interpretation and aids greatly in visualizing the three-dimensional aspects of many problems. In the second place, graphics often affords a means of solving physical relations without the use of an algebraic

* By the principles of relativity a clock carried by the pilot of an airplane flying at a constant speed of 500 mi./hr. and making a 500 mi. flight away from an airport and returning nonstop at the same speed would be slow, compared with the airport clock, by 0.00000000202 sec.

solution. Graphical solutions not only provide practical means for obtaining results, but they also aid greatly in making the transition of thought between the physical situation and the mathematical expression because both are represented simultaneously. A third use of graphics is in the display of results. Proper graphical representation is an invaluable aid to interpretation.

3. Basic Concepts. There are certain definitions and concepts which are basic to the study of mechanics, and they should be understood at the outset.

Space is a region extending in all directions. Position in space is determined relative to some reference system by linear and angular measurements. The basic frame of reference for the laws of Newtonian mechanics is the *primary inertial system* or *astronomical frame of reference* which is an imaginary set of rectangular axes attached to the mean position of the so-called "fixed" stars. Measurements show that relative to this reference system the laws of Newtonian mechanics are valid as long as any velocities involved are negligible compared with the speed of light. Measurements made with respect to this reference are said to be *absolute*, and this reference system is considered to be "fixed" in space. A reference frame attached to the surface of the earth has a somewhat complicated motion in the primary system, and the same laws of mechanics are found not to hold with respect to measurements relative to the earth's reference frame. For most practical purposes, however, the errors are extremely small and may be neglected. Thus we are justified in assuming the fundamental laws of mechanics to be valid for measurements made relative to the earth, and for almost all problems the word *absolute* may be used in a practical sense to refer to such measurements.

Time is a measure of the succession of events and is considered an absolute quantity in Newtonian mechanics. The unit of time is the second, which is a convenient fraction of the period of the earth's rotation.

Force is the action of one body on another. A force tends to move a body in the direction of its action upon it. Considerably more will be said about force in Chapter II.

Matter is substance which occupies space.

Inertia is the property of matter causing a resistance to change in motion.

Mass is the quantitative measure of inertia.

A *body* is matter bounded by a closed surface.

A *particle* is a body of negligible dimensions. In some cases a body of finite size may be treated as a particle, or at other times the particle may be a differential element.

A *rigid body* is one wherein exists no relative deformation between its parts. This is an ideal hypothesis since all real bodies will change shape to a certain extent when subjected to forces. When such changes are small, the body may be termed rigid without appreciable error. With the exception of deformable springs this book is a treatment of the mechanics of rigid bodies only. A body is considered *deformable* when the relations between the applied forces and the resulting deformations are investigated. This problem is taken up in the studies of strength of materials and the theories of elasticity and plasticity.

4. Scalars and Vectors. The quantities dealt with in mechanics are of two kinds. *Scalar* quantities are those with which a magnitude only is associated. Examples of scalars are time, volume, density, speed, energy, and mass. Quantities with which direction as well as magnitude is associated are called *vectors*. Examples of vectors are displacement, velocity, acceleration, force, moment, and momentum. Scalars may be combined according to the ordinary laws of algebra, whereas combination of vectors requires a particular form of algebra to account for both magnitude and direction.

A vector quantity V is represented by a straight line, Fig. 1, having the direction of the vector and having an arrowhead to indicate the sense. The length of the directed line segment represents to some convenient scale the magnitude of V, and the direction of the vector is specified by the angle θ measured from some convenient reference line. The negative of V is a vector $-V$ directed in the opposite sense to V as shown. There are three kinds of vectors, *free, sliding,* and *fixed.* A free vector is one which may be represented by a vector arrow anywhere in space as long as the magnitude and direction remain fixed. If a body moves in a straight line without rotation, then the movement of any point in the body may be taken as a vector, and this vector will describe equally well the motion of every point in the body. Thus the displacement of such a body may be represented by a free vector. Velocity and acceleration are examples also of free vectors.

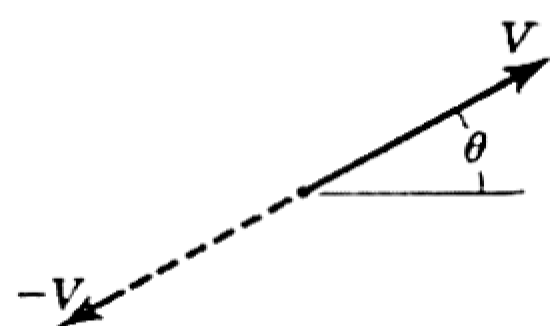


Fig. 1

A sliding vector is one for which a unique line in space must be maintained along which the quantity acts. When dealing with the action of a force on a rigid body, the force may be applied at any point along its line of action without changing its effect on the body as a whole * and thus may be considered a sliding vector.

A fixed vector is one for which a unique point of application is speci-

* This is the so-called principle of transmissibility which will be discussed in Art. 11, Chapter II.

fied, and therefore the vector occupies a fixed position in space. The action of a force on a nonrigid body must be specified by a fixed vector at the point of application of the force. In this problem the forces and movements internal to the body will be a function of the point of application of the force as well as its line of action.

Vectors may be added and subtracted according to the triangle and parallelogram laws. The two free vectors V_1 and V_2 in Fig. 2*a* may be

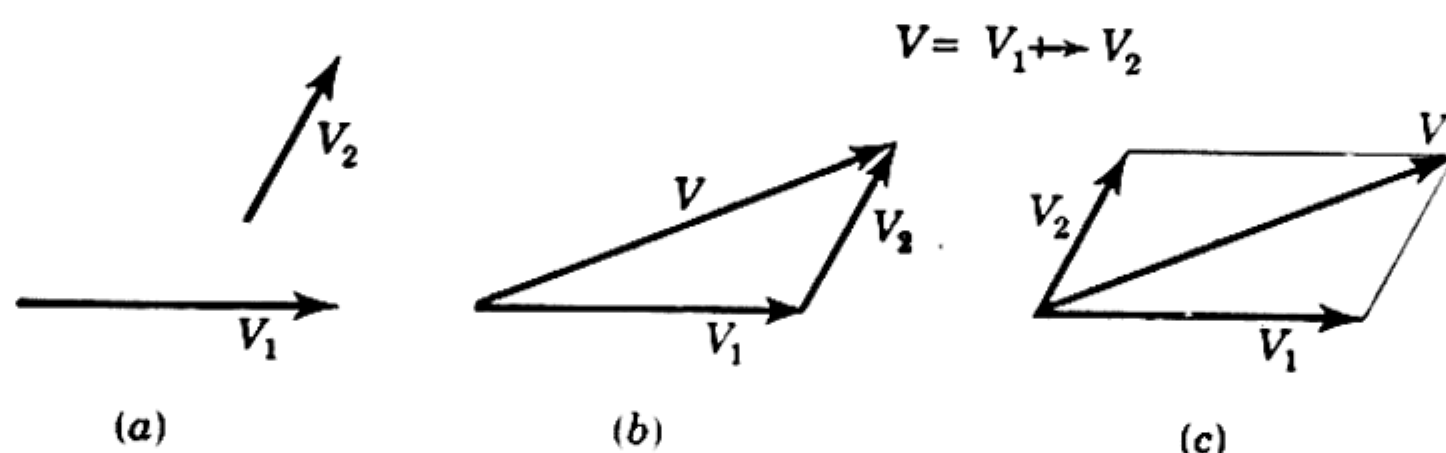


FIG. 2

added tip-to-tail to obtain their sum as shown in the *b*-part of the figure for the triangle law. The order of their combination does not affect their sum. The identical result in magnitude and direction is obtained by completing the parallelogram as shown in the *c*-part of the figure. In each case this vector addition is expressed symbolically by the equation

$$V = V_1 \nrightarrow V_2,$$

where the symbol $\nrightarrow$ is used to denote *vector addition* in contrast to the + sign used for scalar addition.

The difference V' between the vectors V_1 and V_2 may be obtained by either the triangle or the parallelogram procedure as shown in Fig. 3. It is necessary only to add the negative of V_2 to V_1 in order to obtain the vector difference. This difference is indicated symbolically by the equation

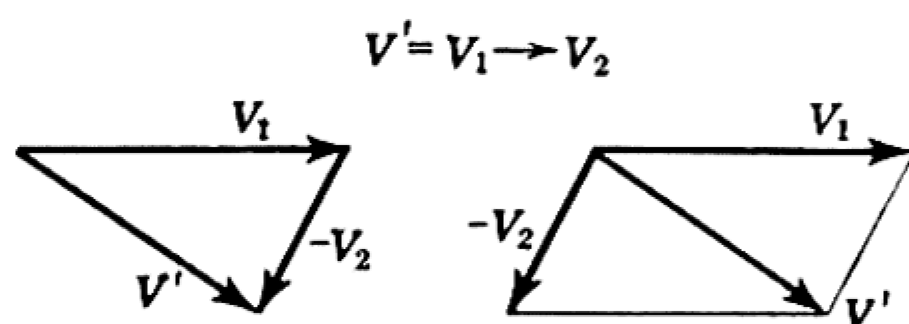


FIG. 3

$$V' = V_1 \rightarrow V_2,$$

where the symbol $\rightarrow$ is used to denote *vector subtraction* as distinguished from the − sign used for scalar subtraction.

Any two or more vectors whose sum equals a certain vector V are said to be the *components* of that vector. Thus the vectors V_1 and V_2 in Fig. 4*a* are the components of V in the directions 1 and 2, respectively. It is usually more convenient to deal with vector components which are mutually perpendicular, and these are called *rectangular components*. The

vectors V_x and V_y in Fig. 4b are the x- and y-components, respectively, of V. Likewise, in Fig. 4c, $V_{x'}$ and $V_{y'}$ are the x'- and y'-components of V. When expressed in rectangular components, the direction of the vector with respect to, say, the x-axis is clearly specified by

$$\theta = \tan^{-1} \frac{V_y}{V_x}.$$

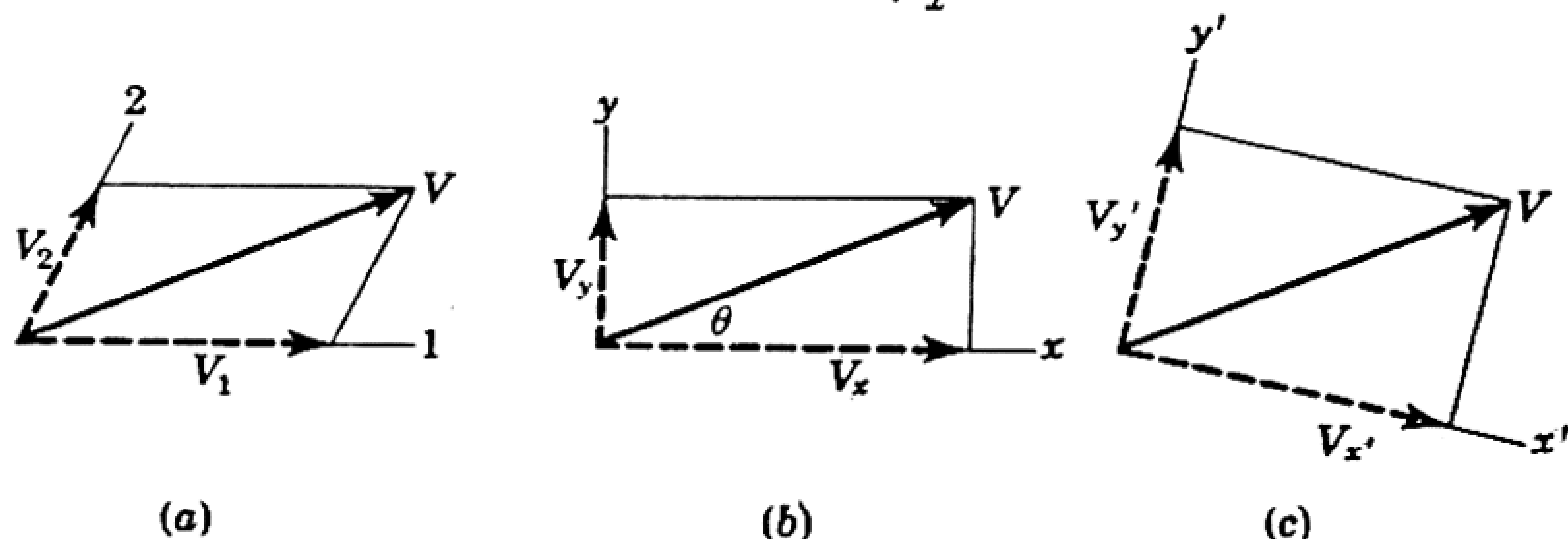


Fig. 4

5. Newton's Laws. Sir Isaac Newton was the first to state correctly the basic laws governing the motion of a particle and to demonstrate their validity.* Slightly reworded, these laws are as follows:

Law I. A particle remains at rest or continues to move with a uniform velocity if there is no unbalanced force acting on it.

Law II. The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.†

Law III. The forces of action and reaction between contacting bodies are equal in magnitude, opposite in direction, and collinear.

The correctness of these laws has been verified by innumerable accurate physical measurements. The first two laws hold for measurements made in an absolute frame of reference but are subject to slight correction when the motion is measured relative to a reference system having acceleration, such as the earth's surface.

Newton's second law forms the basis for most of the analysis in mechanics. As applied to a particle of mass m it may be stated as

$$F = ma, \tag{1}$$

where F is the resultant force acting on the particle and a is the resulting

* Newton's original formulations may be found in the translation of his *Principia* (1687) revised by F. Cajori, University of California Press, 1934.

† To some it is preferable to interpret Newton's second law as meaning that the resultant force acting on a particle is proportional to the time rate of change of momentum of the particle and that this rate of change is in the direction of the force. Both formulations are equally correct.

acceleration. This equation is a *vector* equation since the direction of F must be equal to the direction of a in addition to the equality in magnitudes of F and ma. Newton's first law is a consequence of the second since there is no acceleration when the force is zero, and the particle either is at rest or moves with a constant velocity. The first law adds nothing new to the description of motion but is included since it was a part of Newton's classical statements.

The third law is basic to our understanding of force. It states that forces always occur in pairs of equal and opposite forces. Thus the downward force exerted on the desk by the pencil is accompanied by an upward force of equal magnitude exerted on the pencil by the desk. This principle holds for all forces, variable or constant, regardless of their source and holds at every instant of time during which the forces are applied. Lack of careful attention to this basic law is the cause of frequent error by the beginner. In analyzing bodies under the action of forces it is absolutely necessary to be clearly aware of which of the pair of forces is being considered. It is first of all necessary to *isolate* the body under consideration and then to consider only the one force of the pair which acts *on* the body in question.

In addition to formulating the laws of motion for a particle Newton was also responsible for stating the law which governs the mutual attraction between bodies. This *law of gravitation* is expressed by the equation

$$F = \gamma \frac{m_1 m_2}{r^2}, \tag{2}$$

where F = the mutual force of attraction between two particles,
γ = a universal constant known as the constant of gravitation,
m_1, m_2 = the masses of the two particles,
r = the distance between the centers of the particles.

The mutual forces F obey the law of action and reaction since they are equal and opposite and are directed along the line joining the centers of the particles. Experiment yields the value of $\gamma = 6.67 \times 10^{-8}$ cm.^{3}/(gm. sec.^{2}) for the gravitational constant. Gravitational forces exist between every pair of bodies. On the surface of the earth the only gravitational force of appreciable magnitude is the force due to the earth's attraction. Thus each of two iron spheres 4 in. in diameter is attracted to the earth with a force of 8.90 lb. which is called its *weight*. On the other hand the force of mutual attraction between them if they are just touching is 0.0000000234 lb. This force is clearly negligible compared with the earth's attraction of 8.90 lb., and consequently the

gravitational attraction of the earth is the only gravitational force of any magnitude which need be considered for experiments conducted on the earth's surface.

The weight of a body is the force of attraction of the body to the earth and depends on the position of the body relative to the earth. An object weighing 10 lb. at the earth's surface will weigh 9.99500 lb. at an altitude of 1 mi., 9.803 lb. at an altitude of 40 mi., and 2.50 lb. at an altitude of 4000 mi. or a height approximately equal to the radius of the earth. It is at once apparent that the variation in the weight of high-altitude rockets must be accounted for.

Every object which is allowed to fall in a vacuum at a given location on the earth's surface will have the same acceleration g as can be seen by combining Eqs. (1) and (2) and cancelling the term representing the mass of the falling object. This combination gives

$$g = \frac{\gamma m_0}{r^2},$$

where m_0 is the mass of the earth and r is the radius of the earth.* The mass m_0 and mean radius r of the earth have been found by experiment to be 5.98×10^{27} gm. and 6.38×10^8 cm., respectively. These values together with the value for γ already cited when substituted into the expression for g give

$$g = 980 \text{ cm./sec.}^2 \qquad \text{or} \qquad g = 32.2 \text{ ft./sec.}^2$$

A more accurate determination must account for the fact that the earth is actually an oblate spheroid with flattening at the poles. The value of g has been found equal to 32.09 ft./sec.^{2} at the equator, 32.17 ft./sec.^{2} at a latitude of 45 deg., and 32.26 ft./sec.^{2} at the poles. The proximity of large land masses will also influence the local value of g to a small but detectable amount. It is sufficiently accurate in almost all engineering calculations to use the value of 32.2 ft./sec.^{2} for g.

The mass m of a body may be calculated from the results of the simple gravitational experiment. If the gravitational force or *weight* is W, then, since the body falls with an acceleration g, Eq. (1) gives

$$W = mg \qquad \text{or} \qquad m = \frac{W}{g}. \tag{3}$$

* It can be proved that, for this purpose, the earth may be considered a particle with its entire mass concentrated at its center.

6. Units. There are a number of systems of units used in relating force, mass, and acceleration. Four of these systems are defined in the following table.

Systems of Units

Type of System (fundamental quantities)	Gravitational (length, force, time)		Absolute (length, mass, time)	
Name of System	British or FPS	MKS	British or FPS	CGS
length L	foot (ft.)	meter (m.)	foot (ft.)	centimeter (cm.)
force F	pound (lb.)	kilogram (kg.)	poundal (pdl.)	dyne
time T	second (sec.)	second (sec.)	second (sec.)	second (sec.)
mass M	lb. ft.$^{-1}$ sec.^{2}	kg. m.$^{-1}$ sec.^{2}	pound (lb.)	gram (gr.)
System in use by	Engineers in English-speaking countries	Engineers in non-English-speaking countries	Physicists (occasionally)	Physicists everywhere

Engineers use a gravitational system in which length, force, and time are considered fundamental quantities and the units of mass are derived. Physicists use an absolute system in which length, mass, and time are considered fundamental and the units of force are derived. Either system, of course, may be used with the same results. The engineer prefers to use force as a fundamental quantity because most of his experiments involve direct measurement of force. The British or FPS gravitational system is the one used in this book. The engineer has not adopted a unit for mass which is universally used although *slug* and less often *g-pound* are seen occasionally in the literature. One slug (or g-pound) is the mass of a body which weighs 32.2 lb. at the earth's surface.

It is frequently necessary to convert a quantity from one set of units to another. During the process of conversion it is essential that the dimensions of the quantity remain unchanged. In order to convert a velocity of 30 mi./hr., for example, to the equivalent number of centimeters per second it is first necessary to know that

5280 ft. are contained in 1 mi.,
30.48 cm. " " " 1 ft.,
3600 sec. " " " 1 hr.

The conversion is then

$$\left(30\,\frac{\text{mi.}}{\text{hr.}}\right)\left(5280\,\frac{\text{ft.}}{\text{mi.}}\right)\left(30.48\,\frac{\text{cm.}}{\text{ft.}}\right)\left(\frac{1}{3600}\,\frac{\text{hr.}}{\text{sec.}}\right) = 1340\,\frac{\text{cm.}}{\text{sec.}}.$$

The units mi., hr., and ft. cancel, leaving cm./sec.

It is quite customary to write expressions stating equivalents in units such as

$$5280 \text{ ft.} = 1 \text{ mi.},$$

or

$$30 \text{ mi./hr.} = 1340 \text{ cm./sec.}$$

The meaning of these equivalents is clear, but it is important to note that these expressions are *not* algebraic equations. Here the equals sign does not mean mathematical equality since $5280 \neq 1$ and ft. $\neq$ mi. The equals sign when used in this way actually means "are contained in."

7. Dimensions. A given dimension such as length can be expressed in a number of different units such as feet, centimeters, or miles. Thus the word *dimension* is distinguished from the word *unit*. Physical relations must always be dimensionally homogeneous, that is, the dimensions of each term in an equation must be the same. It is customary to use the symbols L, F, T, and M, to stand for length, force, time, and mass, respectively. In the engineer's or gravitational system mass is derived. From Eq. (1) mass has the dimensions of force divided by acceleration or

$$M = \frac{F}{L/T^2} = FL^{-1}T^2.$$

One important use of the theory of dimensions is found in checking the dimensional correctness of some derived physical relation. In deriving the expression for the velocity v of a body of mass m which is moved from rest a horizontal distance x by a force F the following equation results:

$$Fx = \tfrac{1}{2}mv^2,$$

where the $\frac{1}{2}$ is a dimensionless coefficient resulting from integration. This equation is dimensionally correct since substitution of L, F, and T gives

$$[FL] = [FL^{-1}T^2][LT^{-1}]^2 = [FL].$$

Dimensional homogeneity is a necessary condition for correctness, but it is not sufficient since the correctness of dimensionless coefficients cannot be checked in this way.

A second important use for dimensional theory is in the prediction of full-scale performance from the results of experiments on models. There

are many problems, such as the flow resistance of ships and airplanes and the behavior of loaded structures of complex shape, where a mathematical solution is not feasible by reason of the great complexity involved. The form of the relation which describes a physical problem certainly does not depend on the size of the units employed, and therefore a physical relation should describe equally well the behavior of a model or its prototype. A full discussion of this use of dimensional analysis is beyond the scope of this book,* and only one simple example of the procedure followed is given here.

Let it be desired to determine the expression for the period τ of vibration for a simple pendulum consisting of a small mass m suspended by a cord of length l. Guided by observation, it will be assumed that the period is a function of the length l, the acceleration of gravity g, and the mass m. Next, it is assumed that this functional relationship is given by the products of these quantities raised to unknown powers α, β, γ, or

$$\tau = kl^{\alpha}g^{\beta}m^{\gamma},$$

where k is a dimensionless constant to account for the units used. Expressing this relation in dimensional symbols gives

$$\begin{aligned}[T] &= [L]^{\alpha}[LT^{-2}]^{\beta}[FL^{-1}T^{2}]^{\gamma},\\ &= [L^{\alpha+\beta-\gamma}][T^{-2\beta+2\gamma}][F^{\gamma}].\end{aligned}$$

In order that the equation be dimensionally homogeneous it is necessary for the exponents of each of the three fundamental dimensions to be the same on both sides of the equation. Equating the exponents of T, L, and F in that order gives

$$1 = -2\beta + 2\gamma,$$

$$0 = \alpha + \beta - \gamma,$$

$$0 = \gamma.$$

The solutions are clearly $\gamma = 0$, $\beta = -\frac{1}{2}$, $\alpha = \frac{1}{2}$, and the assumed relation becomes

$$\tau = kl^{1/2}g^{-1/2} = k\sqrt{l/g}.$$

Dimensional considerations disclose that the period does not depend on the mass m. One carefully executed experiment for small amplitudes of vibration will give measurements for τ and l. Substitution of these measured values along with the known value of g will give $k = 6.283$. Therefore the equation

$$\tau = 6.283\sqrt{l/g}$$

* See *Dimensional Analysis* by P. W. Bridgman, Yale University Press, 1932.

may be used to describe the period for *any* similar pendulum of a different size as long as a consistent set of units is used. In the case of the simple pendulum direct solution will disclose the fact that $k = 2\pi$ for small amplitudes.

8. Accuracy. The number of significant figures shown in an answer should be no greater than that which corresponds to the least number of significant figures in the given data. Thus the cross-sectional area of a shaft whose diameter, 0.25 in., say, was measured to the nearest hundredth of an inch should be written as 0.049 in.^{2} and not 0.0491 in.^{2} as would be indicated when the numbers were multiplied out.

When calculations involve small differences in large quantities, greater accuracy must be achieved. Thus it is necessary to know the numbers 4.2503 and 4.2391 to an accuracy of five significant figures in order that their difference 0.0112 be expressed to three-figure accuracy. It is often difficult in somewhat lengthy computations to know at the outset the number of significant figures needed in the original data to insure a certain accuracy in the answer.

Sliderule accuracy, usually three significant figures, is considered satisfactory for the majority of engineering calculations. The decimal point should be located by a rough longhand approximation which also serves as a check against large sliderule error.

9. Mathematical Limits and Approximations. The essential purpose of applied mechanics is the mathematical description of engineering situations, and as such it is extremely necessary to understand and be able to apply certain limiting and approximating mathematical relations.

The *order* of differential quantities is the subject of frequent misunderstanding by students who are making application of the calculus for the first time. Higher-order differentials may always be neglected compared with lower-order differentials. As an example the element of volume dV of a right circular cone of altitude h and base radius r may be taken to be a circular slice a distance x from the vertex and of thickness dx. It can be verified that the complete expression for the volume of the element may be written as

$$dV = \frac{\pi r^2}{h^2}\left[x^2\,dx + x\,(dx)^2 + \frac{1}{3}\,(dx)^3\right].$$

It should be recognized that, when passing to the limit in going from ΔV to dV, the terms in $(dx)^2$ and $(dx)^3$ drop out, leaving merely

$$dV = \frac{\pi r^2}{h^2}\,x^2\,dx,$$

which is an exact expression.

In using trigonometric functions of differential quantities it is well to call attention to the following relations which are true in the mathematical limit:

$$\sin d\theta = \tan d\theta = d\theta,$$

$$\cos d\theta = 1.$$

The angle $d\theta$ is, of course, expressed in radian measure. When dealing with small but finite angles it is often convenient to replace the sine by the tangent or either function by the angle itself. Likewise the cosine of a small angle may often be approximated satisfactorily by unity. As an example, for an angle of 1 deg.,

$$\sin 1° = 0.0174524 \qquad \text{and} \qquad 1° \text{ is } 0.0174533 \text{ radian.}$$

The error in replacing the sine by the angle for 1 deg. is only 0.005 per cent. For 5 deg. the error is 0.13 per cent, and for 10 deg. the error is still only 0.51 per cent.

A few of the mathematical relations which are useful in mechanics are listed in Table B3, Appendix B.

10. Method of Problem Solution. An understanding of the method of attack on engineering problems is an essential aspect of their solution. This method involves a logical sequence of steps from hypothesis to conclusion and should include the following:

(*a*) given data,
(*b*) statement of results desired,
(*c*) necessary diagrams,
(*d*) statement of principles and basic equations which apply,
(*e*) application of principles and equations,
(*f*) answers or conclusions.

Presentation of information in this order represents a logical sequence, and all problem work should follow this general pattern. It is also important that the arrangement of work be neat and orderly. Careless solutions which cannot be easily read by others are of little or no value. It will be found that the discipline involved in adherence to good form will in itself be an invaluable aid to the development of the powers of analysis. Many problems which at first may seem difficult and complicated become clear and simple once they are begun with a logical and disciplined method of attack.

The science of mechanics is based on a surprisingly few fundamental concepts and involves mainly the application of these basic relations to a variety of situations. In this application the *method* of analysis is all-important. In solving a problem it is essential that the laws which apply

be carefully fixed in mind and that these principles be applied literally and exactly. In applying the principles which define the requirements for forces acting on a body it is essential that the body in question be *isolated* from all other bodies so that complete and accurate account of all forces which act on this body may be taken. This *isolation* should exist mentally as well as be represented on paper. The drawing of such an isolated body with the representation of *all* external forces acting on it is called a *free-body diagram*. It has long been established that the *free-body diagram* method is the key to the understanding of mechanics. This is so because the *isolation* of a body is the tool by which *cause* and *effect* are clearly separated and by which attention on the literal application of a principle is accurately focused. The technique of drawing free-body diagrams will be covered in Chapter III where they are first used.

In applying physical laws to the solution of a problem numerical values of the quantities may be used directly in proceeding toward the solution. On the other hand algebraic symbols may be used to represent the quantities involved, and the answer left as a formula. In the first scheme the magnitude of all quantities expressed in their particular units is evident at each stage of the calculation. This is often an advantage when the practical significance of the magnitude of the terms is appraised. The second method, or symbolic solution, has several advantages over the numerical solution. In the first place the abbreviation achieved by the use of symbols aids in focusing attention on the interconnection between the physical situation and its related mathematical description. Secondly, a symbolic solution permits a dimensional check to be made at every step, whereas dimensional homogeneity may not be checked when numerical values are used. Furthermore a symbolic solution may be used repeatedly for obtaining answers to the same problem when different sets and sizes of units may be involved. Facility with both methods of solution is essential, and ample practice with each should be sought in the problem work.

CHAPTER II

Force Systems

11. Force. Before dealing with a group or *system* of forces it is necessary to examine the properties of a single force in some detail. A force has been defined as the action of one body on another. It is evident that force is a vector quantity since its effect depends on the direction of the action as well as on the magnitude. Furthermore it is necessary to know where it acts. In Fig. 5 the effect of the force P on the bracket will depend on the magnitude of P, the angle θ, and the location of the point of

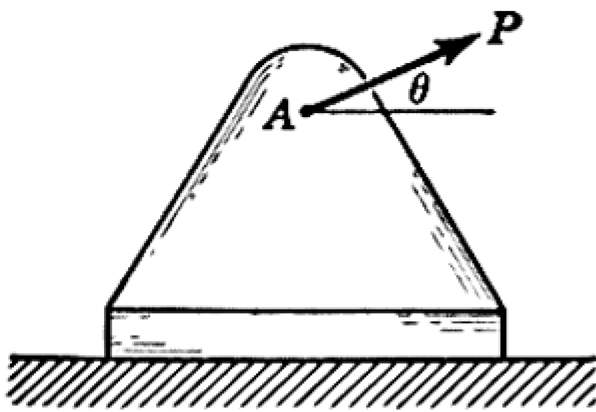


FIG. 5

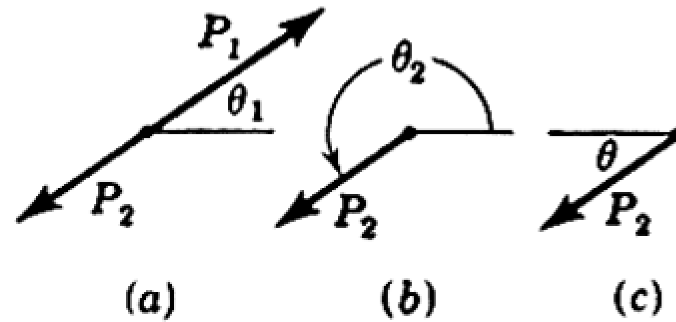


FIG. 6

application A. Changing any one of these three specifications will alter the effect on the bracket. The sense of a force along its line of action is a fourth specification which may be added to distinguish between two forces having identical lines of action but opposite sense, such as with the forces P_1 and P_2 of Fig. 6*a*. The need for the fourth specification disappears if the obtuse angle θ_2 is used as in Fig. 6*b*. Since directions of lines are most frequently measured and represented by acute angles, the specification of an obtuse angle is often inconvenient. The question is most easily resolved if the direction of the force is indicated by an acute angle such as θ in Fig. 6*c*. As long as the force and angle are clearly represented on a sketch or are adequately described, there is need for only the three specifications of force: *magnitude, direction*, and *point of application*.

Force is applied either by direct mechanical contact or by remote action. Gravitational and electrical forces are the two examples of force applied by remote action. All other actual forces are applied through direct physical contact.

The action of a force on a body can be separated into two effects, *external* and *internal.* For the bracket of Fig. 5 the effects of P external to the bracket are the reactions or forces exerted on the bracket by the foundation in consequence of the action of P. Forces external to a body are then of two kinds, applied or *active* forces and resulting or *reactive* forces. The effects of P internal to the bracket are the resulting internal movements and forces distributed throughout the material of the bracket. The relation between internal forces and internal movements involves the material properties of the body and is studied in the subjects of strength of materials, elasticity, and plasticity.

In dealing with the mechanics of rigid bodies where concern is given only to the net *external* effects of forces, experience shows that it is not necessary to restrict the action of an applied force to a given point. Thus the force P acting on the rigid bracket in Fig. 7 may be applied at A or at B or at any other point on its action line, and the net external effect of P on the bracket will not change. This situation is described by the *principle of transmissibility,* which states that a force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts. When the resultant external effects only of a force are to be investigated, the force may be treated as a *sliding* vector, and it is necessary and sufficient to specify the *magnitude, direction,* and *line of action* of the force. Since this book deals essentially with the mechanics of rigid bodies, almost all forces will be treated as sliding vectors.

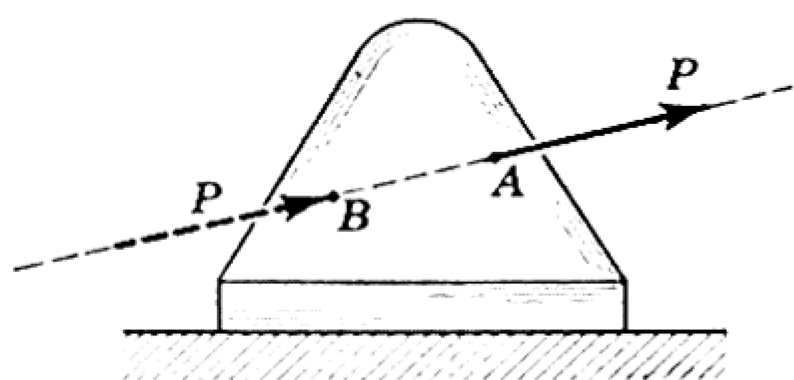


Fig. 7

Forces may be either concentrated or distributed. Actually every contact force is applied over a finite area and is therefore distributed. When the dimensions of the area are negligible compared with the other dimensions of the body, the force may be considered as concentrated at a point. Force may be distributed over an area, as in the case of mechanical contact, or it may be distributed over a volume when gravity or magnetic force is acting. The "weight" of a body is the force of gravity distributed over its volume and may be taken as a concentrated force acting through the center of gravity. The position of the center of gravity is usually obvious from considerations of symmetry. If the position is not clear, then a separate calculation, explained in Chapter V, will be necessary to locate the center of gravity.

A force may be measured either by comparison with other known forces, using a mechanical balance, or by the calibrated movement of an

elastic spring. All such comparisons or calibrations have as their basis a primary standard. The standard pound for the United States is legally defined as 0.4535924277 times the international kilogram and is that force required to support this portion of the standard kilogram in a vacuum and under the standard conditions of sea level and a latitude of 45 deg.*

The characteristic of a force expressed by Newton's third law must be carefully observed. The action of a force is always accompanied by an equal and opposite reaction. It is essential to fix clearly in mind which force of the pair is involved. The answer is always clear when the body in question is *isolated* and the force exerted *on* that body (not *by* the body) is represented. It is very easy to make a careless mistake and consider the wrong force of the pair unless careful distinction between every action and reaction is made.

12. Addition. Two concurrent forces F_1 and F_2 may be added by the parallelogram law to obtain their sum or *resultant* R as shown in Fig.

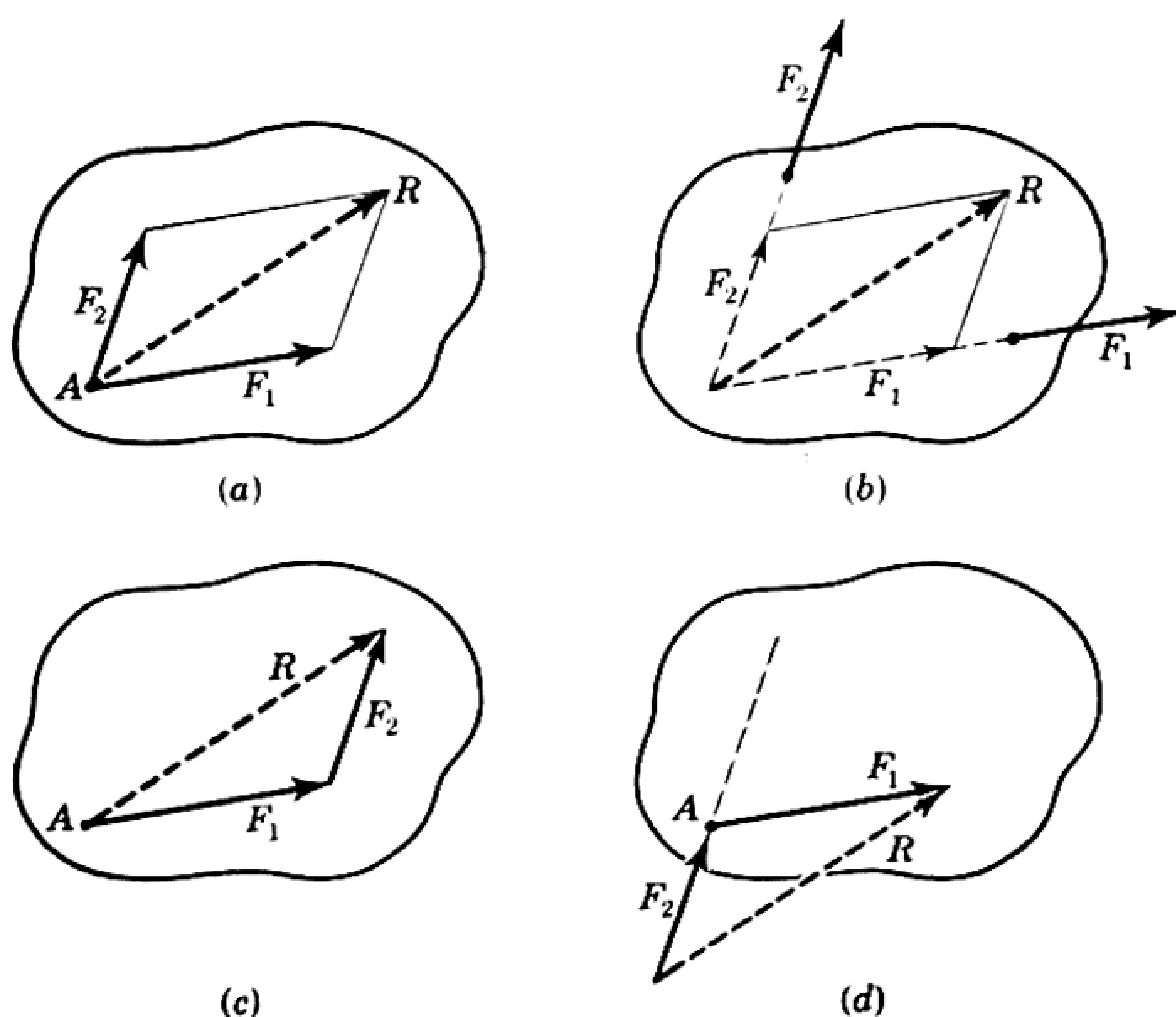


FIG. 8

8*a*. If the two forces lie in the same plane but are applied at two different points as in Fig. 8*b*, they may be moved along their lines of action and their vector sum R completed at the point of concurrency. The resultant R may replace F_1 and F_2 without altering the external effects on

* More precisely the "standard conditions" refer to any location at which the acceleration of gravity is 32.1740 ft./sec.^{2}

the body. The triangle law may also be used to obtain R, but it will require moving the line of action of one of the forces as shown in Fig. 8c. In Fig. 8d the same two forces are added, and although the correct magnitude and direction of R are preserved, the correct line of action is lost. This type of combination should be avoided. Mathematically the sum of the two forces may be written by the vector equation

$$F_1 \nrightarrow F_2 = R.$$

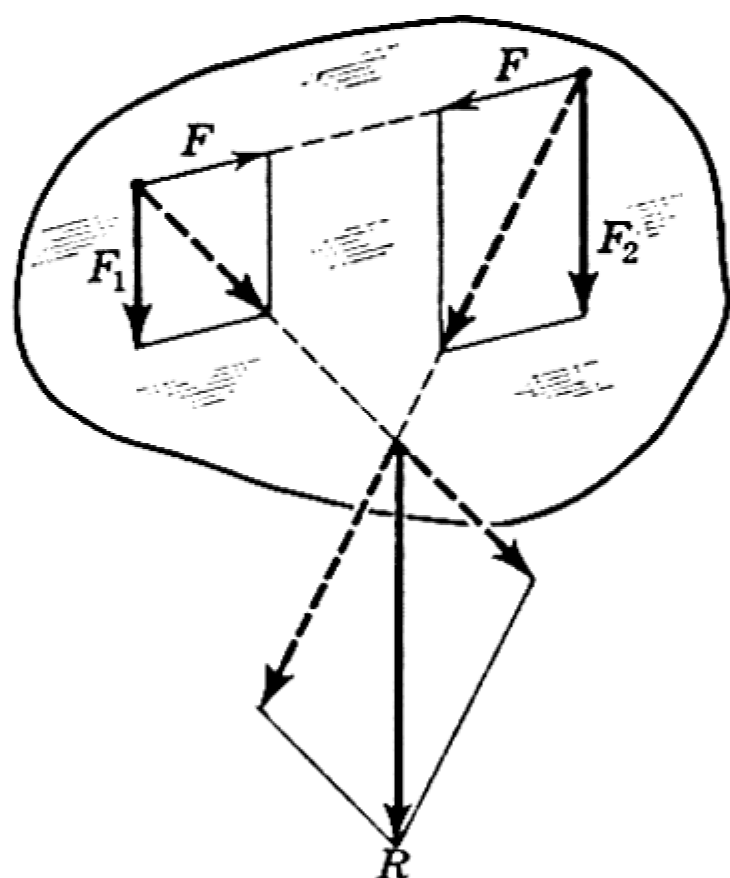
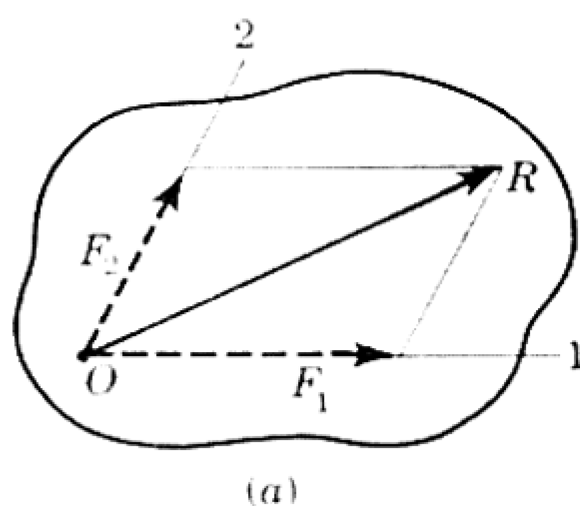


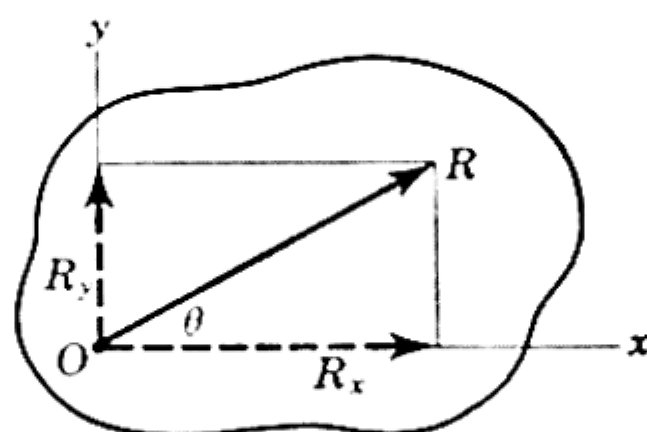
FIG. 9

A special case of addition is presented when the two forces F_1 and F_2 are parallel, Fig. 9. Graphically they may be combined by first adding two equal, opposite, and collinear forces F of convenient magnitude which together produce no external effect on the body. Adding F_1 and F and combining with the sum of F_2 and F yield the resultant R correct in magnitude, direction, and line of action. The procedure here is also useful in obtaining a graphical combination of two forces which are almost parallel and hence may have a remote point of concurrency.

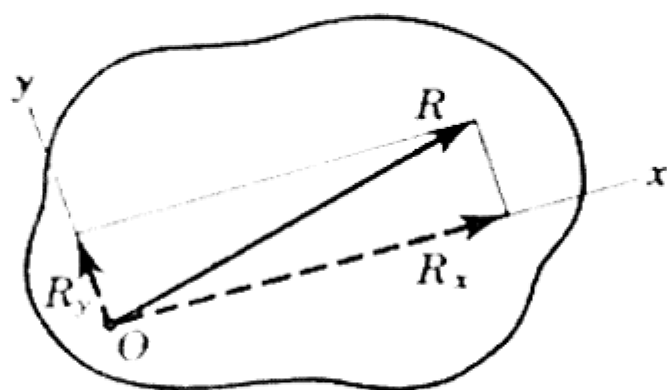
13. Resolution. The force R of Fig. 10a is said to have the *components* F_1 and F_2 or to be *resolved* in the directions O–1 and O–2, re-



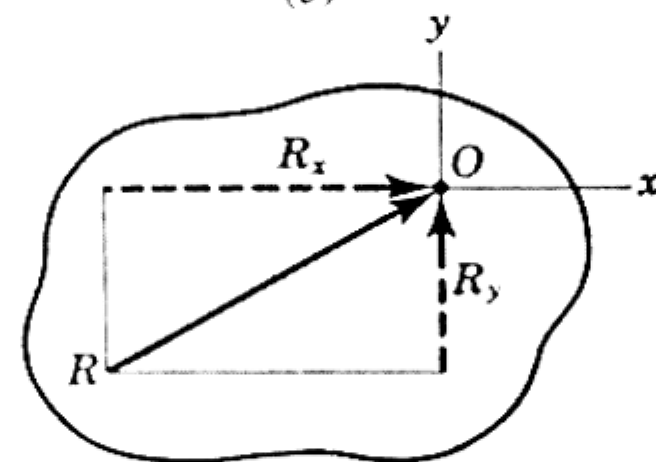
(a)



(b)



(c)



(d)

FIG. 10

spectively. When the directions of resolution are mutually perpendicular, the force R is said to have *rectangular* components R_x and R_y as in Fig. 10*b*. It is at once evident from the figure that

$$R_x = R\cos\theta, \qquad R_y = R\sin\theta,$$
$$R = \sqrt{R_x^2 + R_y^2}, \qquad \theta = \tan^{-1}\frac{R_y}{R_x}. \tag{4}$$

The orientation of axes is arbitrary, and a selection as in Fig. 10*c* could be made. The action of a force and its components at the point of application may also be represented as in Fig. 10*d*.

In order to eliminate ambiguity between the representation of a force and its components it is desirable to show the components in dotted lines and the resultant in a full line, as in Fig. 10, or else to show the components in full lines and their resultant in a dotted line, as in Fig. 8. With this understanding it will always be apparent that a force and its components are represented and not three separate forces.

Many problems are three-dimensional, and it becomes necessary to consider the three mutually perpendicular components of a force. In Fig. 11 force F is resolved into its rectangular components F_x, F_y, F_z by forming the edges of the rectangular parallelepiped of which F is a diagonal. When the angles made by F with the x-, y-, and z-axes are designated by θ_x, θ_y, and θ_z, respectively, it is evident from the right triangles involved that

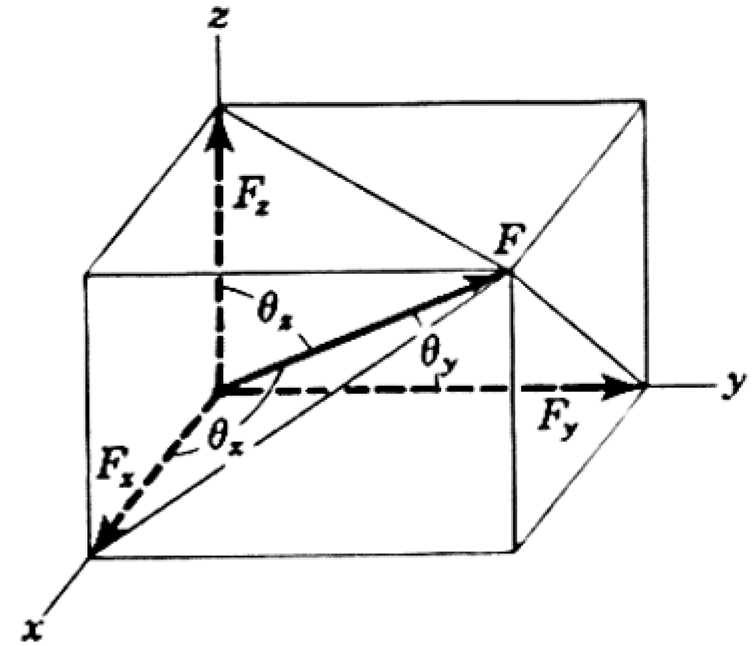


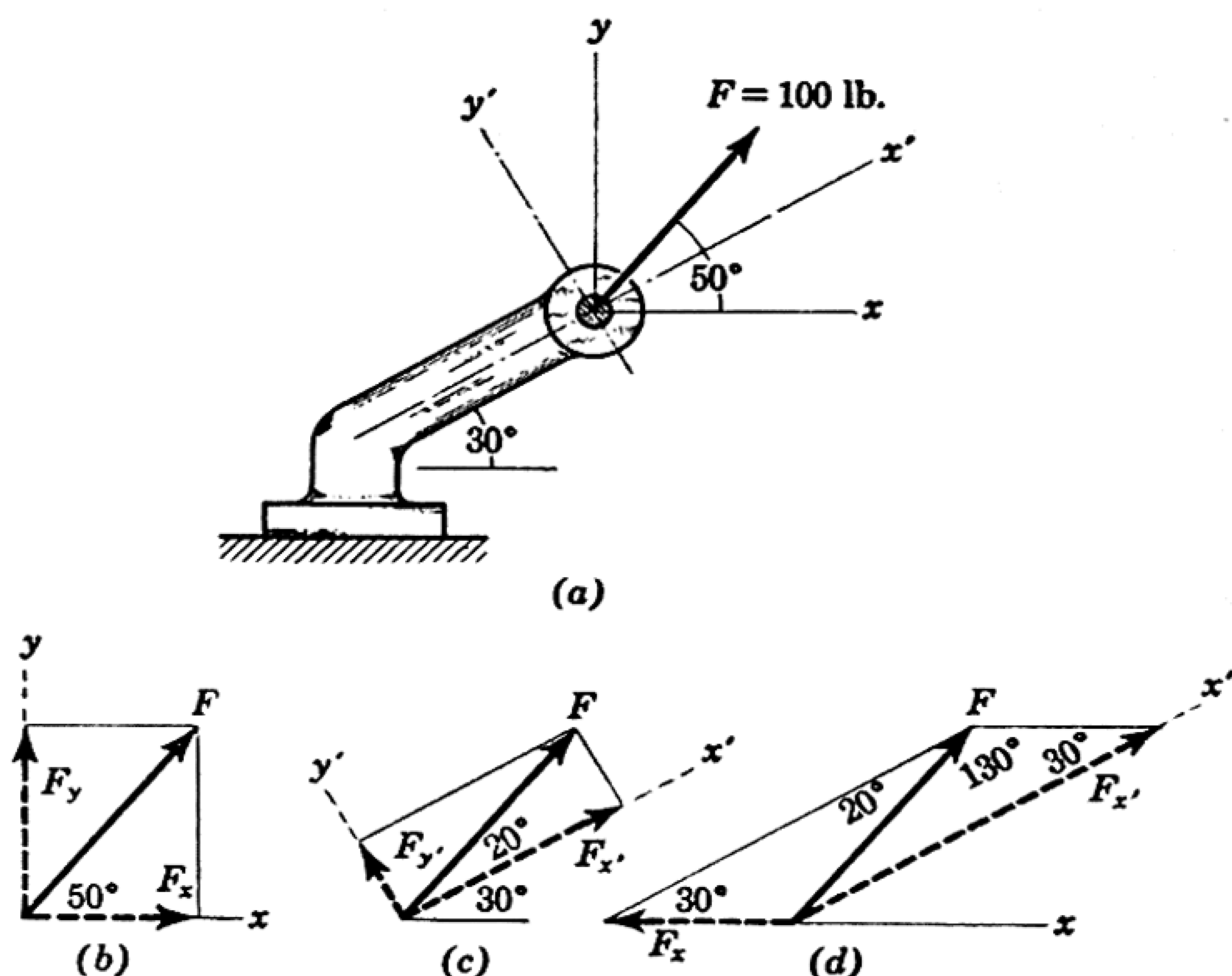
FIG. 11

$$\left.\begin{aligned} F_x &= F\cos\theta_x \\ F_y &= F\cos\theta_y \\ F_z &= F\cos\theta_z \end{aligned}\right\} \quad \text{and} \quad F = \sqrt{F_x^2 + F_y^2 + F_z^2}. \tag{5}$$

The direction cosines of F are $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$. The choice of the orientation of the axes is quite arbitrary, and considerations of convenience usually determine this selection.

SAMPLE PROBLEMS

1. The 100 lb. force is applied to the bracket as shown in the *a*-part of the figure accompanying this problem. Determine the rectangular components of F in (1) the x- and y-directions and (2) the x'- and y'-directions. Also (3) find the components of F in the x- and x'-directions.



PROB. 1

Solution: Part (1): The x- and y-components of F are shown in the *b*-part of the figure and are

$$F_x = F\cos\theta_x = 100\cos 50° = 64.3 \text{ lb.},$$

$$F_y = F\sin\theta_x = 100\sin 50° = 76.6 \text{ lb.} \qquad \textit{Ans.}$$

Part (2): The x'- and y'-components of F are shown in the *c*-part of the figure and are

$$F_{x'} = F\cos\theta_{x'} = 100\cos 20° = 94.0 \text{ lb.},$$

$$F_{y'} = F\sin\theta_{x'} = 100\sin 20° = 34.2 \text{ lb.} \qquad \textit{Ans.}$$

Part (3): The components of F in the x- and x'-directions are obtained by completing the parallelogram as indicated in the *d*-part of the figure. F_x and $F_{x'}$ in the directions indicated are obtained from the law of sines. Thus

$$\frac{F_x}{\sin 20°} = \frac{F}{\sin 30°}; \qquad F_x = \frac{0.342}{0.500} \times 100 = 68.4 \text{ lb.};$$

$$\frac{F_{x'}}{\sin 130°} = \frac{F}{\sin 30°}; \qquad F_{x'} = \frac{0.766}{0.500} \times 100 = 153 \text{ lb.} \qquad \textit{Ans.}$$

2. If the force F in Fig. 11 is 100 lb. and passes through a point whose x-, y-, and z-coordinates are 4, 5, and 3, respectively, determine the rectangular components of F. Also find the component of F in the x–y plane.

Solution: The direction cosines are

$$\cos\theta_x = \frac{4}{\sqrt{4^2+5^2+3^2}} = 0.566,$$

$$\cos\theta_y = \frac{5}{\sqrt{4^2+5^2+3^2}} = 0.707,$$

$$\cos\theta_z = \frac{3}{\sqrt{4^2+5^2+3^2}} = 0.424.$$

The components of F are then

$$F_x = F\cos\theta_x = 100 \times 0.566 = 56.6 \text{ lb.},$$

$$F_y = F\cos\theta_y = 100 \times 0.707 = 70.7 \text{ lb.},$$

$$F_z = F\cos\theta_z = 100 \times 0.424 = 42.4 \text{ lb.}$$ *Ans.*

The cosine of the angle θ_{xy} made by F with the x–y plane is

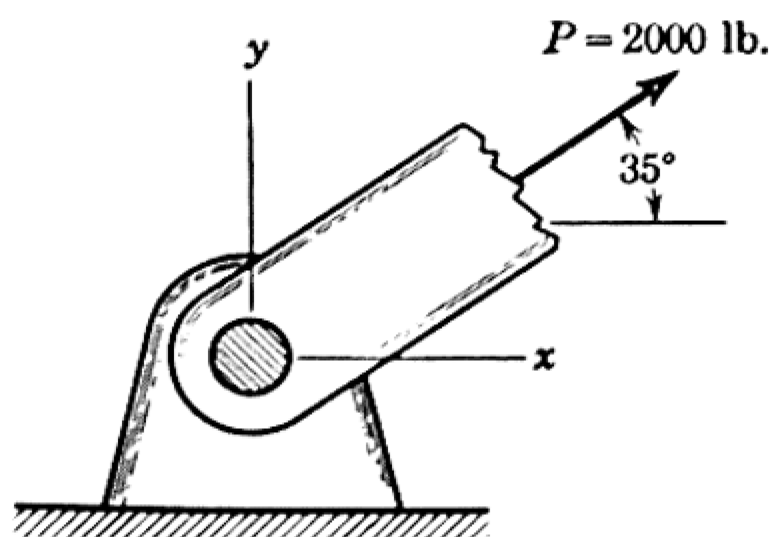
$$\cos\theta_{xy} = \frac{\sqrt{4^2+5^2}}{\sqrt{4^2+5^2+3^2}} = 0.906,$$

and the component of F in the x–y plane is

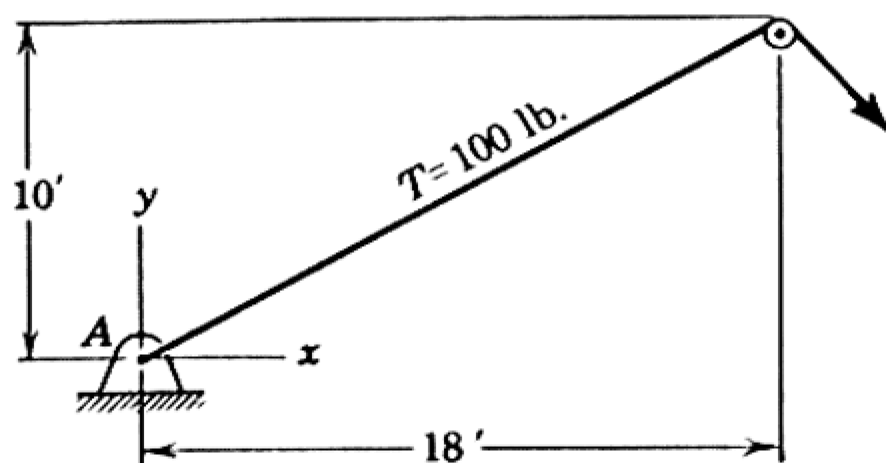
$$F_{xy} = F\cos\theta_{xy} = 100 \times 0.906 = 90.6 \text{ lb.}$$ *Ans.*

PROBLEMS

3. The member is under a tensile load of $P = 2000$ lb. as shown. How much force is exerted by this member on the pin in the x- and y-directions?



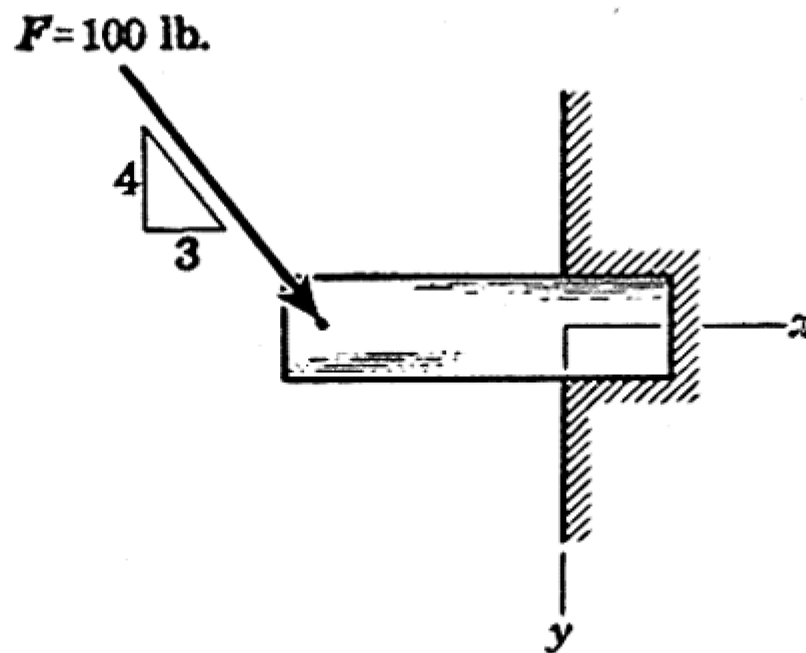
PROB. 3



PROB. 4

4. Determine the x- and y-components of the force exerted on the pin at A by the 100 lb. cable tension T. *Ans.* $T_x = 87.4$ lb., $T_y = 48.6$ lb.

5. Find the x- and y-components of the force F.

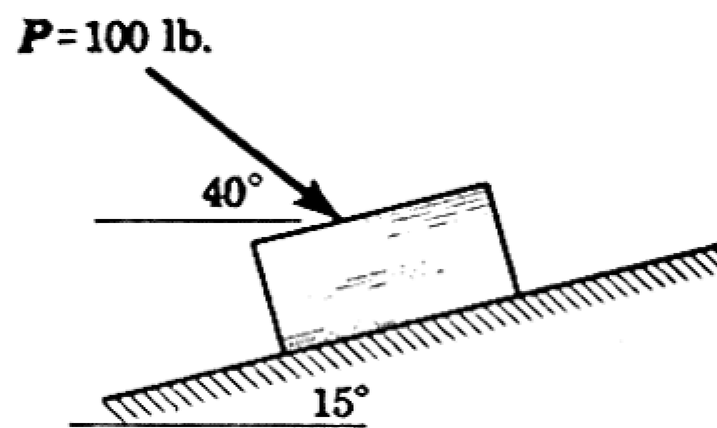


PROB. 5

6. A force F which acts in the x-y plane has a magnitude of 100 lb. and a direction $\theta_x = 115$ deg. Find the x- and y-components of F.

7. Represent the action of P on the block by a component P_t acting along the incline and a component P_n normal to the incline.

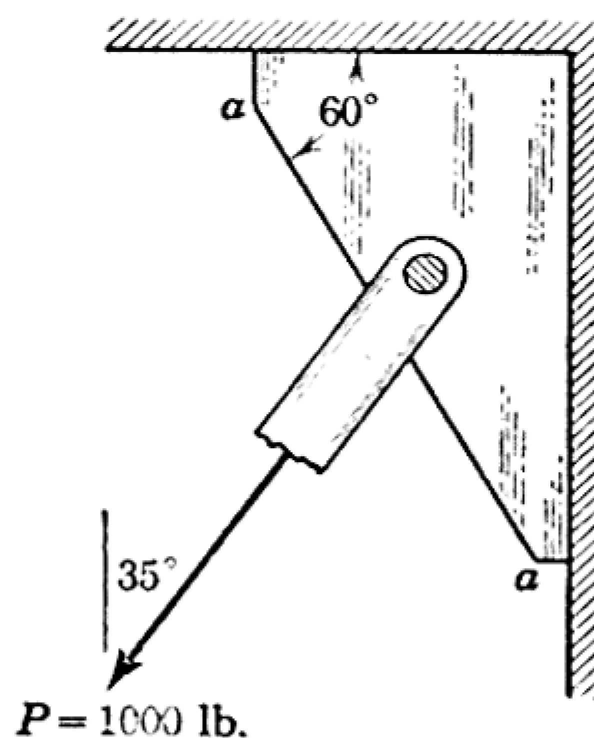
Ans. $P_t = 57.4$ lb., $P_n = 81.9$ lb.



PROB. 7

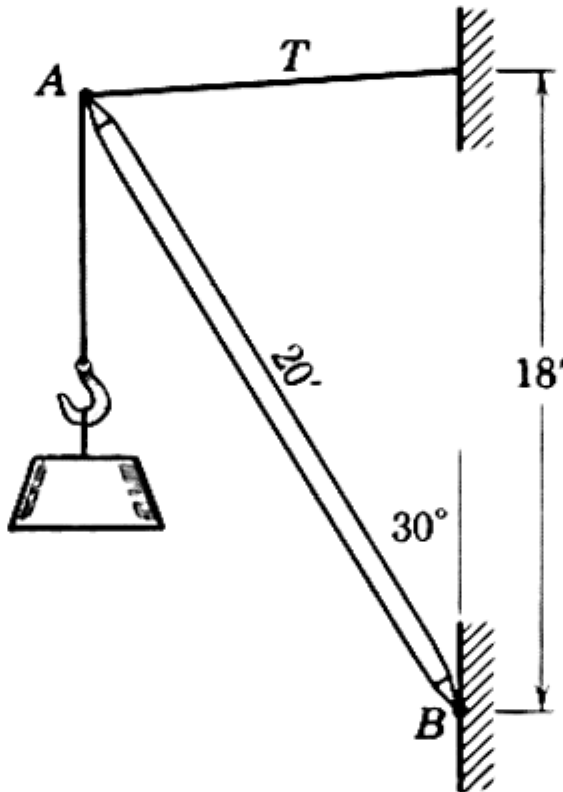
8. How much of the force P is exerted in the direction parallel to the surface a–a?

Ans. $P_a = 423$ lb.

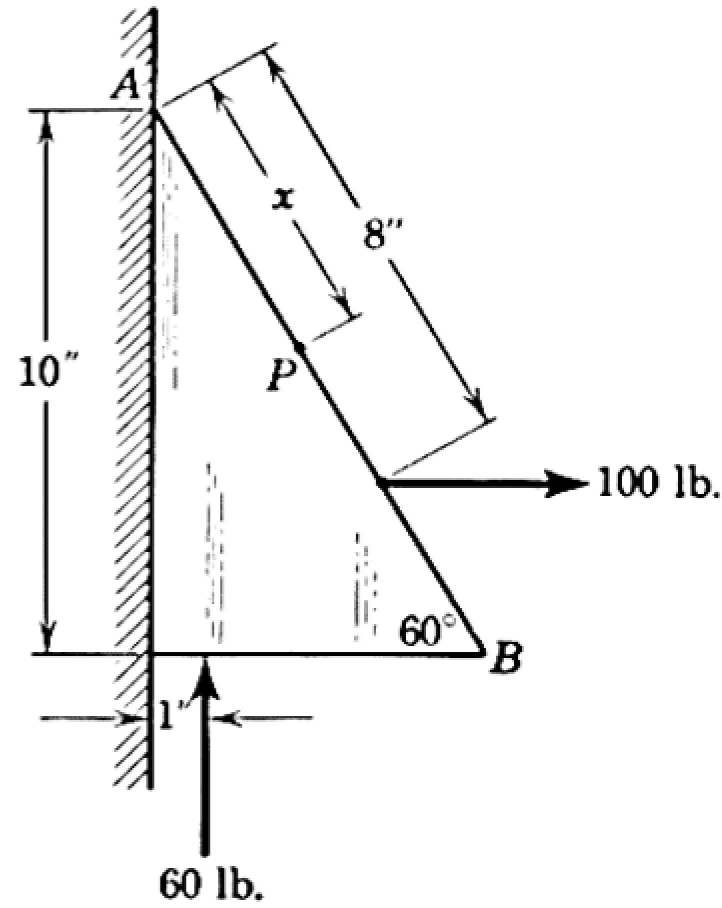


PROB. 8

9. The tension T in the supporting cable of the 20 ft. boom is 2000 lb. Resolve T into two forces applied at A: one, F_n, normal to the boom, and the other, F_t, along the boom.



Prob. 9

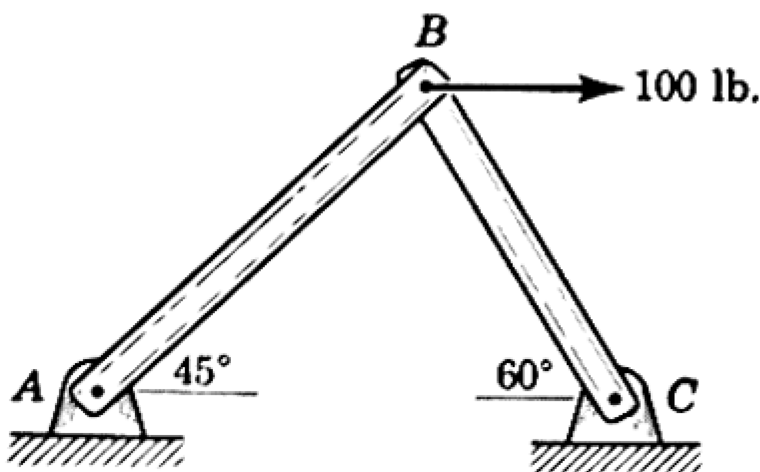


Prob. 10

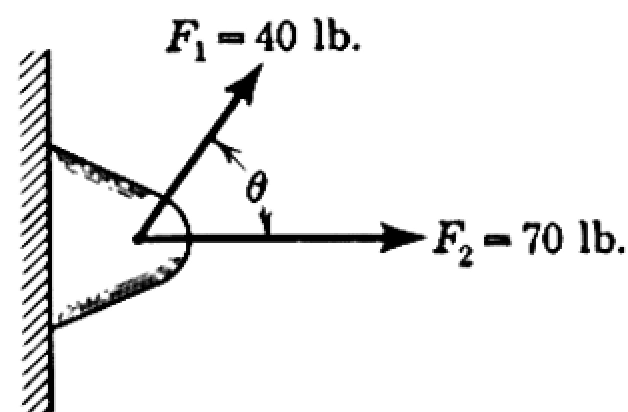
10. The two forces shown are to be replaced by an equivalent force R applied at a point P on the surface AB. Locate P by finding its distance x from A and specify the magnitude of R and the angle θ it makes with the horizontal.

Ans. $x = 6.44$ in., $R = 117$ lb., $\theta = 30° 58'$

11. In finding the forces exerted on the pins A and C the 100 lb. force is resolved into two forces, one along the line AB and the other along BC. Find these components.



Prob. 11



Prob. 12

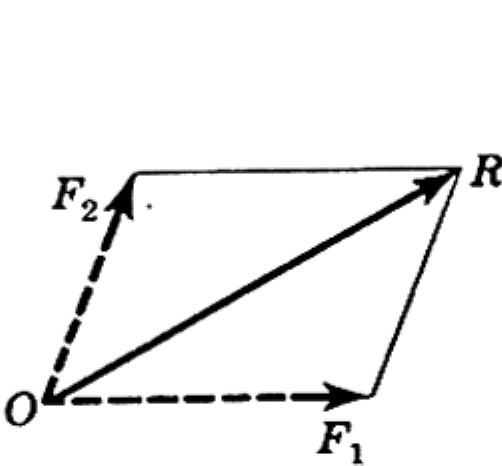
12. At what angle θ must F_1 be applied in order that the resultant R of F_1 and F_2 be equal to 100 lb.? For this condition what will be the angle β between R and the horizontal?

Ans. $\theta = 51° 19'$, $\beta = 18° 12'$

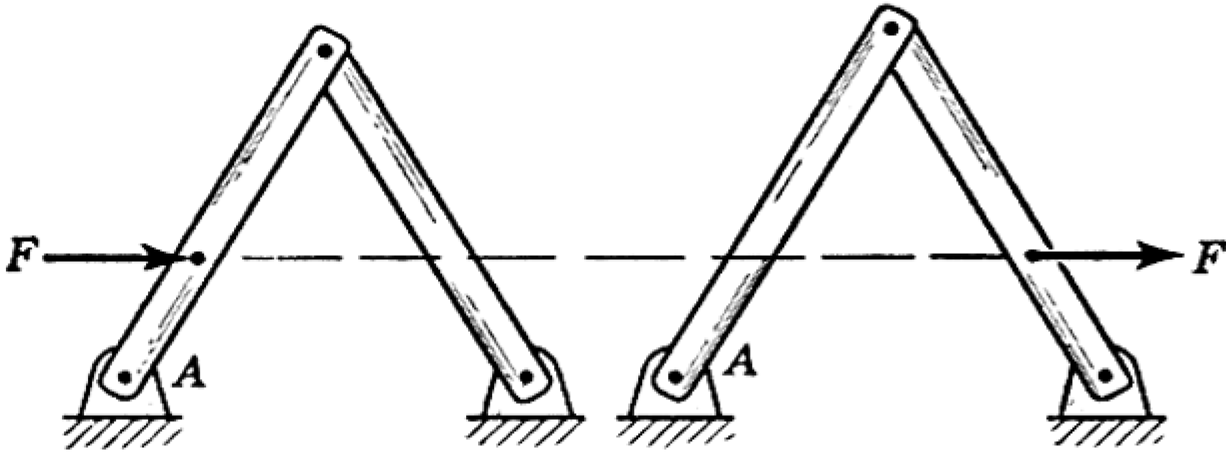
13. A 100 lb. force which makes an angle of 30 deg. with the horizontal x-axis is to be replaced by two forces, a horizontal force F and a second force of 60 lb. magnitude. Find F.

Ans. $F = 119.8$ lb. or $F = 53.4$ lb.

14. The force R has components F_1 and F_2 acting through O as indicated. Show graphically that R may also be represented by two components through O having the same magnitudes as F_1 and F_2 but differing in direction.



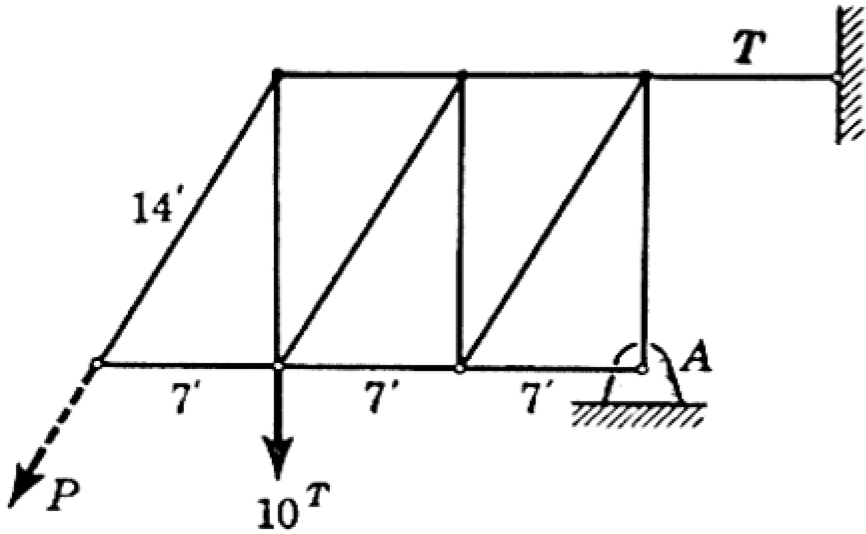
PROB. 14



PROB. 15

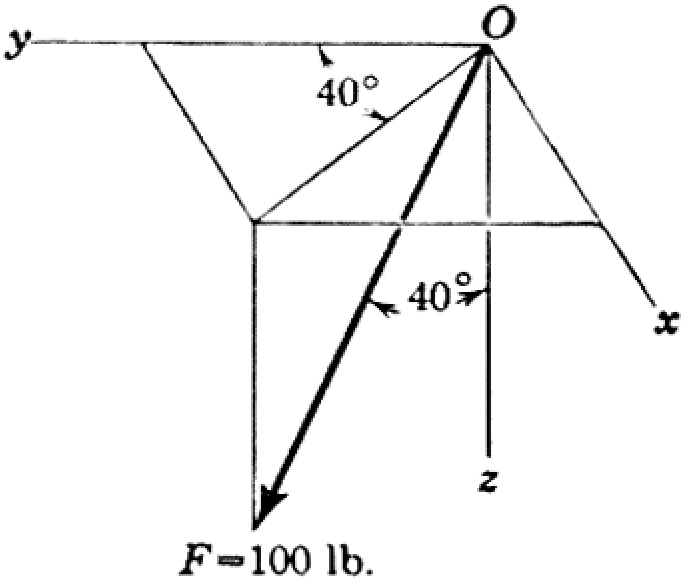
15. Can it be concluded from the principle of transmissibility that the pin reactions at A are the same for the two identical frames shown? Explain.

16. A 10 ton load is applied to the truss as shown, and a certain force on the pin connection at A results. If the 10 ton load is removed and a force P is applied as indicated by the dotted line, determine the magnitude of P and the corresponding increase ΔT in cable tension so that the reaction on the pin A will remain unchanged. *Ans.* $P = 11.55$ tons, $\Delta T = 5.77$ tons



PROB. 16

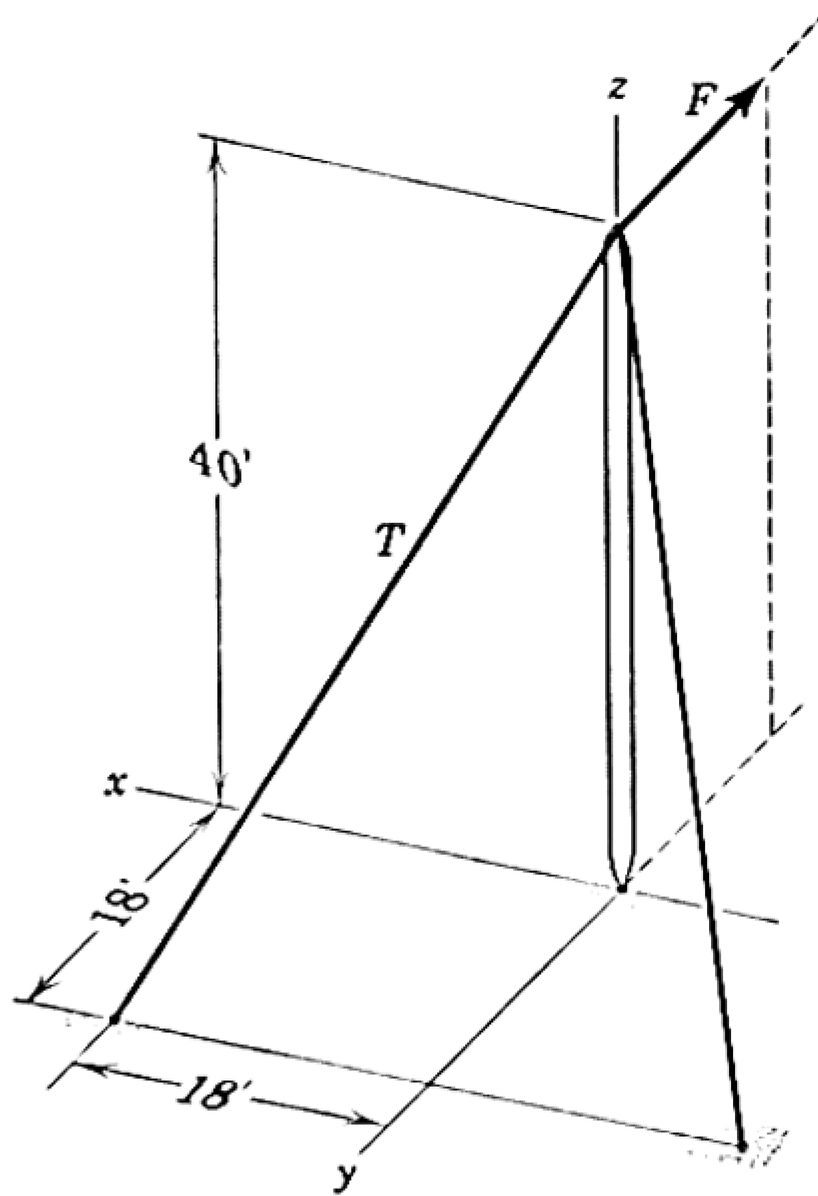
17. A force F of 100 lb. applied to a body at point O has the magnitude and direction shown. Find the x-component of F. *Ans.* $F_x = 41.3$ lb.



PROB. 17

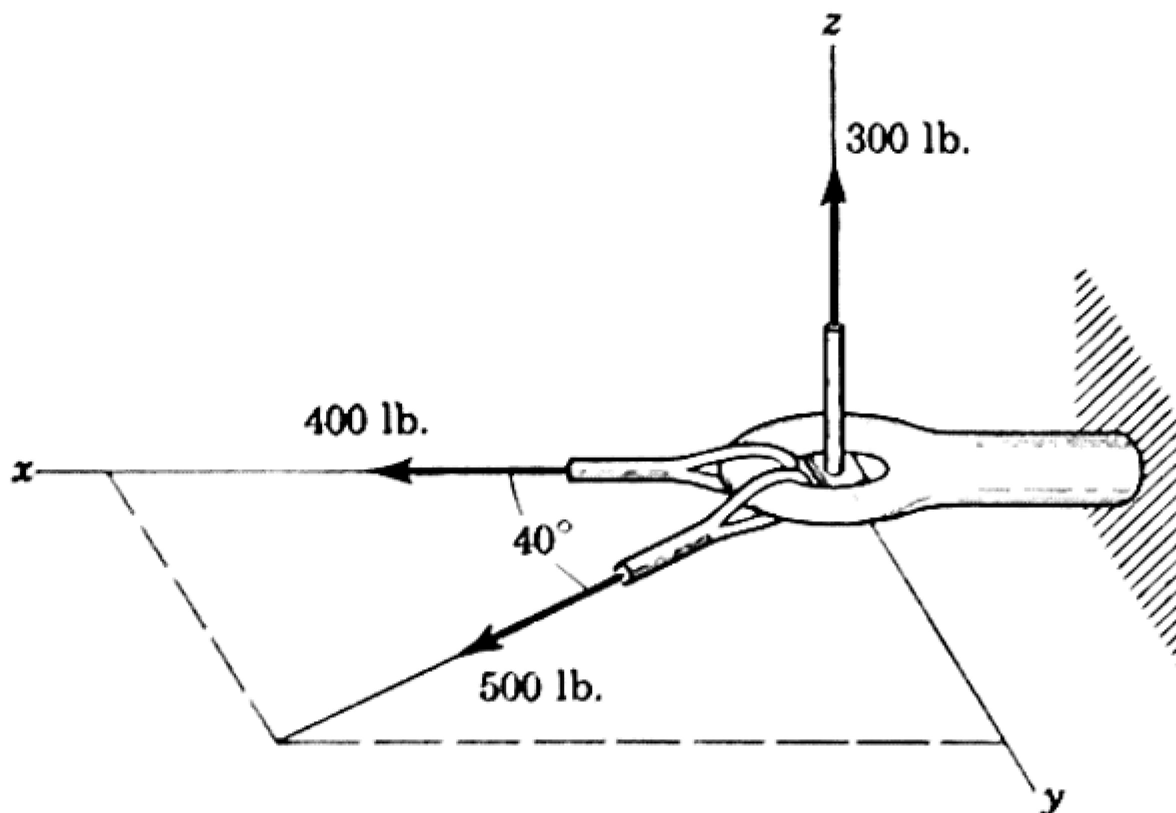
18. Determine the direction cosines of the 100 lb. force of Prob. 17.

19. The tension T in one of the two supporting cables for the mast is 500 lb. Resolve the force exerted by this cable on the mast into its x-, y-, and z-components. *Ans.* $T_x = T_y = 190$ lb., $T_z = -422$ lb.



Prob. 19

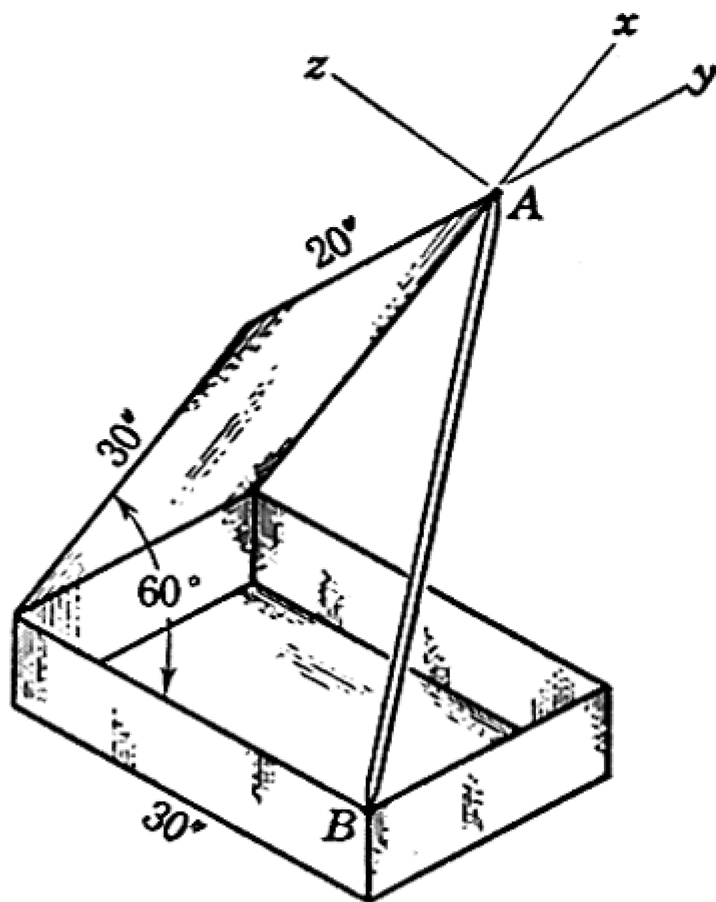
20. An eyebolt is subjected to three cable tensions as shown. If a single cable is to replace the three cables without altering the effect on the eyebolt, find the correct tension T and its direction cosines.
Ans. $T = 898$ lb., $\cos\theta_x = 0.872$, $\cos\theta_y = 0.358$, $\cos\theta_z = 0.334$



Prob. 20

* **21.** The lid of the box shown is supported by the strut AB which exerts a 10 lb. force on the lid at A in the direction of the strut. Resolve this force into components F_x, F_y, F_z along the edges of the lid and normal to the lid.

Ans. $F_x = 4.16$ lb., $F_y = 5.54$ lb., $F_z = 7.20$ lb.



PROB. 21

14. Moment. The tendency of a force to rotate the body upon which it acts about a certain axis is known as the *moment* of the force about that axis. In Fig. 12*a* consider the moment M about the axis O–O of a force

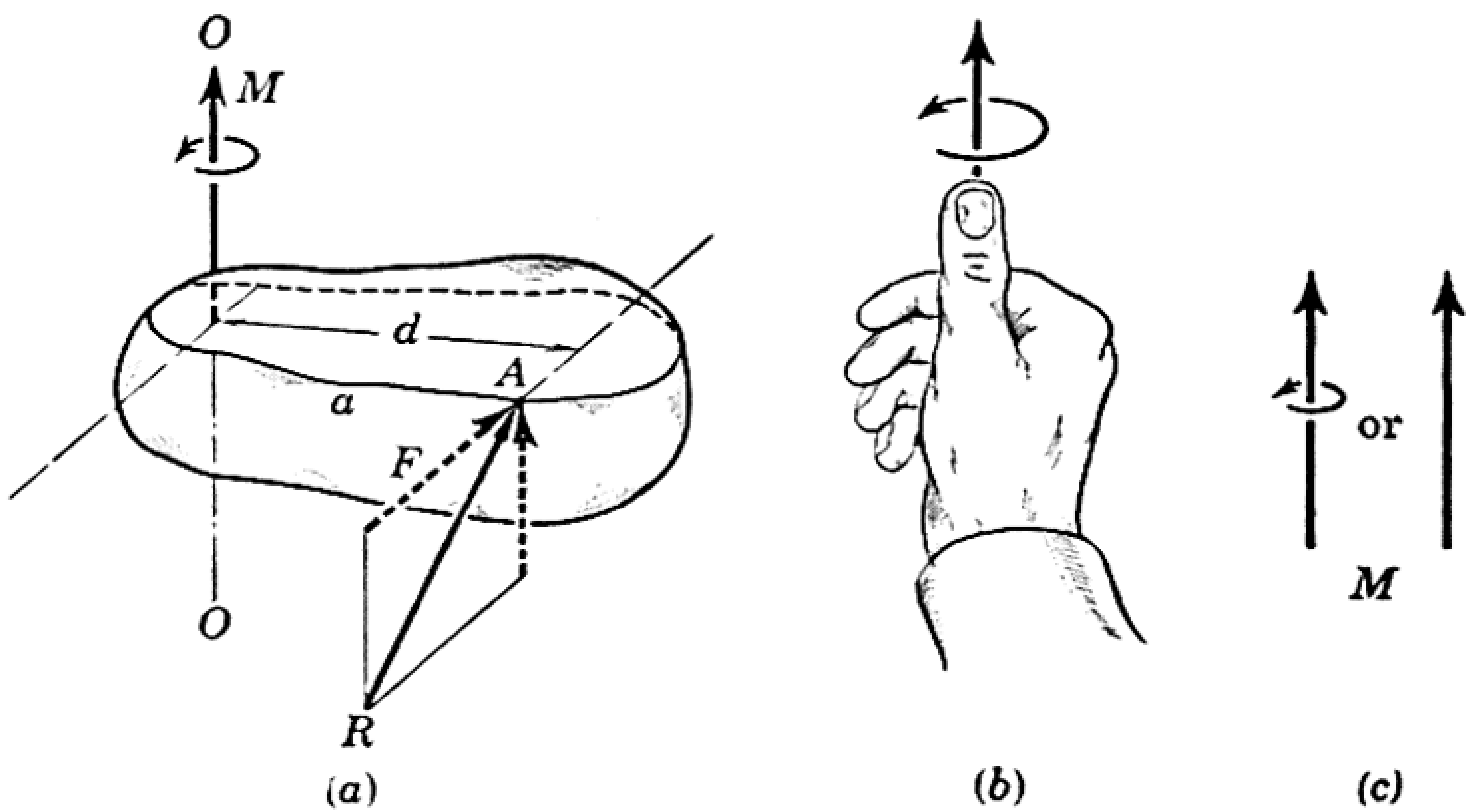


FIG. 12

R applied at any point A on the body as shown. This moment is due entirely to the component of R in a plane normal to the axis and equals this component F multiplied by the perpendicular distance or moment arm d from the line of action of F to the axis O–O. Thus

$$M = Fd. \tag{6}$$

The component of R normal to F is parallel to O–O and hence exerts no tendency to rotate the body about this axis.

Moment is a vector quantity. The vector direction is along the axis about which the moment is taken, and the sense of the vector is specified by the arbitrary but universally used *right-hand rule.* To represent the direction of the moment of R about O–O in Fig. 12*a* the fingers of the right hand are curled in the direction of the tendency to rotate as shown in the *b*-part of the figure. The thumb then points in the direction of the vector M. This vector may be represented in either of the two ways shown in Fig. 12*c*. The small curl is sometimes used to distinguish a moment vector from a force vector. A moment vector obeys all the rules of vector combination and may be considered to be a sliding vector with a line of action coinciding with the moment axis.

When dealing with forces all of which act in a given plane it is customary to speak of the moment about a point. Actually the moment with respect to an axis normal to the plane and passing through the point is implied. Thus the moment of force F about point O in Fig. 13 is

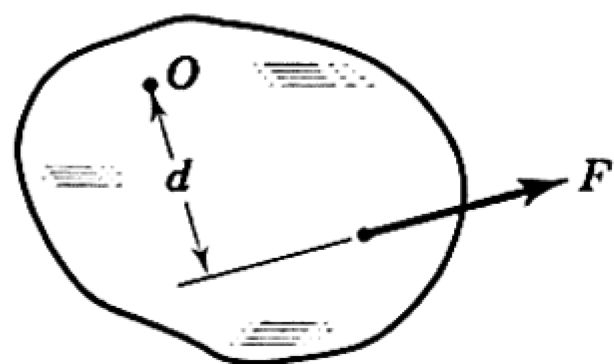


Fig. 13

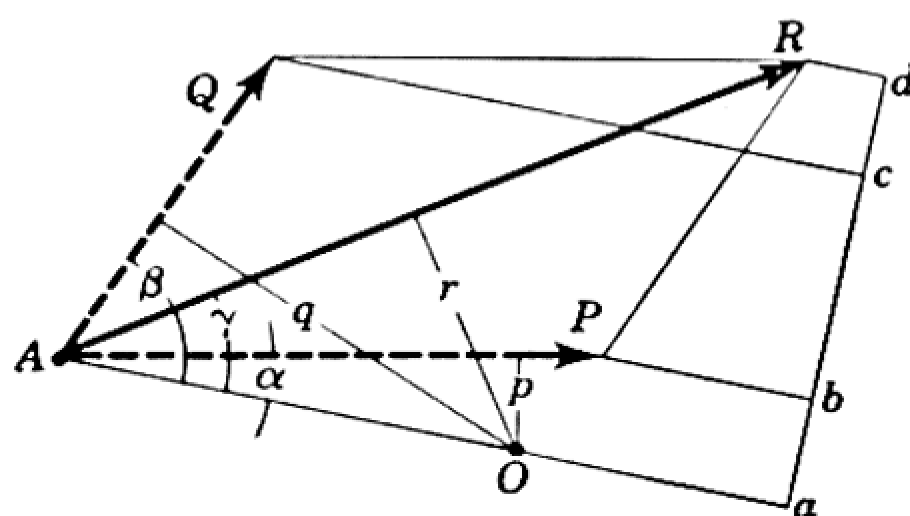


Fig. 14

$M_O = Fd$ and is counterclockwise. Vector representation of moments for coplanar forces is not convenient since the vectors are either out from the paper (counterclockwise) or into the paper (clockwise). Since the addition of parallel free vectors may be accomplished with *scalar* algebra, the moment directions may be accounted for by using a plus sign (+) for counterclockwise moments and a minus sign (−) for clockwise moments, or vice versa. It is necessary only to be consistent within a given problem in using either sign convention.

One of the most important principles of mechanics is *Varignon's theorem,* which states that the moment of a force about any point is equal to the sum of the moments of its components about the same point. To prove this statement consider a force R and two components P and Q acting at point A, Fig. 14. Point O is selected arbitrarily as the moment center. Construct the line AO and project the three vectors onto the normal to this line. Also construct the moment arms p, q, r of the three

forces to point O and designate the angles of the vectors to the line AO by α, β, γ as shown in the figure. Since the parallelogram whose sides are P and Q requires that $\overline{ac} = \overline{bd}$, it is evident that

$$\overline{ad} = \overline{ab} + \overline{bd} = \overline{ab} + \overline{ac},$$

or

$$R \sin \gamma = P \sin \alpha + Q \sin \beta.$$

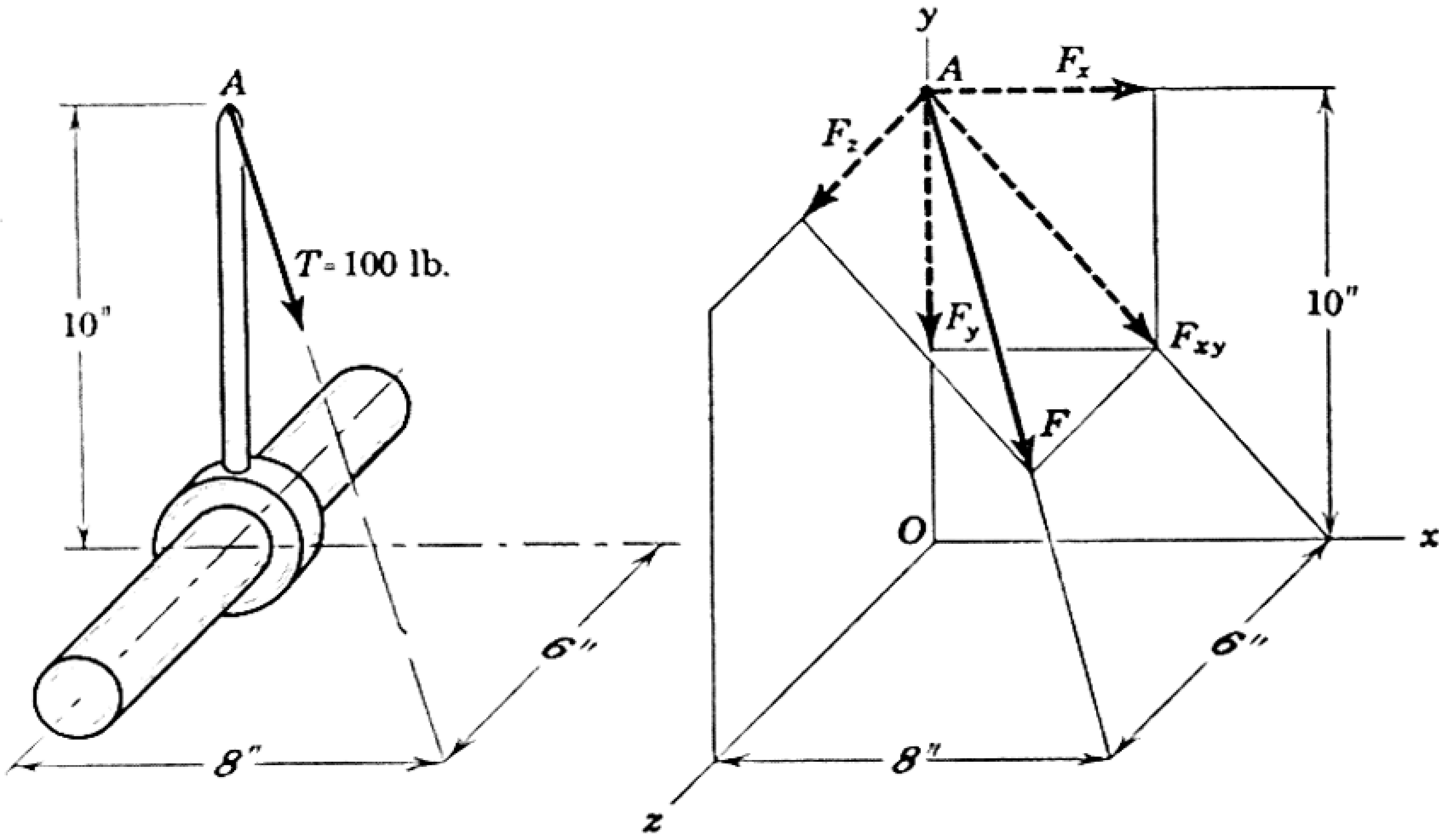
Multiplying by the distance $\overline{AO}$ and substituting the values of p, q, r give

$$Rr = Pp + Qq,$$

which proves that the moment of a force about any point equals the sum of the moments of its two components about the same point. Varignon's theorem need not be restricted to the case of only two components but applies equally well to three or more since it is always possible by direct combination to reduce the number of components to two for which the theorem was proved. The theorem may also be applied to the moments of vectors other than force vectors.

SAMPLE PROBLEM

22. A force $T = 100$ lb. is applied to the arm shown, which is attached to the rigid shaft. Determine the moment M of T about the shaft axis.



PROB. 22

Solution: In the right-hand part of the figure the force F is shown resolved into components F_{xy} in the x–y plane which is normal to the shaft axis and the

component F_z. The moment M of F about the shaft or z-axis is

$$M = F_{xy}d,$$

where d is the perpendicular distance from F_{xy} to O. The cosine of the angle between F and F_{xy} is $\sqrt{8^2 + 10^2}/\sqrt{8^2 + 6^2 + 10^2} = 0.906$, and therefore

$$F_{xy} = 100 \times 0.906 = 90.6 \text{ lb.}$$

The moment arm d equals OA multiplied by the sine of the angle between F_{xy} and OA, or

$$d = 10 \times \frac{8}{\sqrt{8^2 + 10^2}} = 6.25 \text{ in.}$$

Hence the moment of F about the z-axis is

$$M = 90.6 \times 6.25 = 566 \text{ lb. in.} \qquad \textit{Ans.}$$

Calculation of the moment is somewhat simplified by resolving F_{xy} into F_x and F_y. It is clear that F_y exerts no moment about the z-axis so that F_x only need be considered. The direction cosine of T with respect to the x-axis is $8/\sqrt{8^2 + 6^2 + 10^2} = 0.566$ so that $F_x = 100 \times 0.566 = 56.6$ lb. Thus

$$M = 56.6 \times 10 = 566 \text{ lb. in.}$$

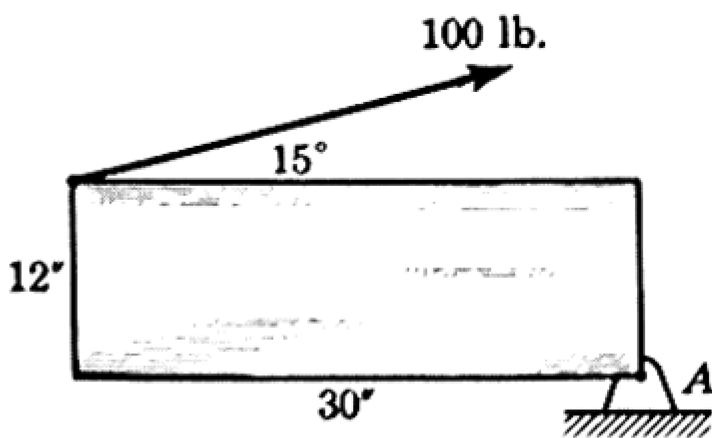
Expressed as a vector, M would be in the negative z-direction.

PROBLEMS

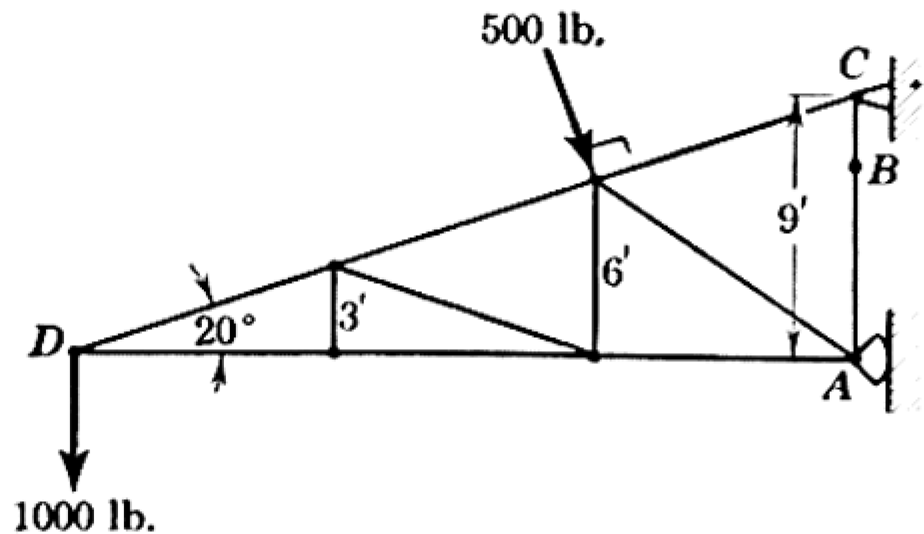
23. Prove that the moment of a force about a point equals twice the area of the triangle defined by that point and the two extremities of the force vector.

24. A force F is applied at a point whose coordinates are x, y, z. Show that the moment of F about the x-axis is $M_x = F_zy - F_yz$ and write the expressions for the moments M_y and M_z about the y- and z-axes, respectively.

25. Determine the moment M of the 100 lb. force shown about the pivot A.
Ans. $M = 1935$ lb. in.

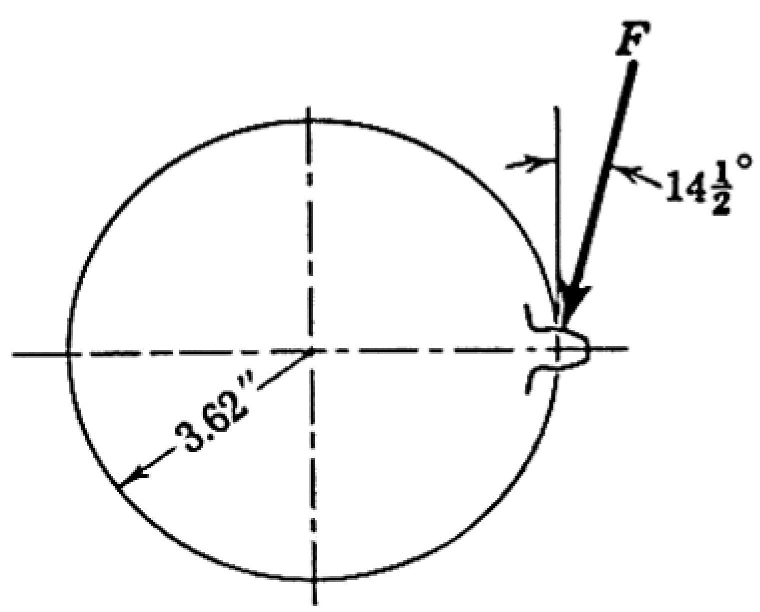


PROB. 25



PROB. 26

26. Determine the moment M of the 1000 lb. force about point B for the overhanging truss.
Ans. $M = 24{,}700$ lb. ft.



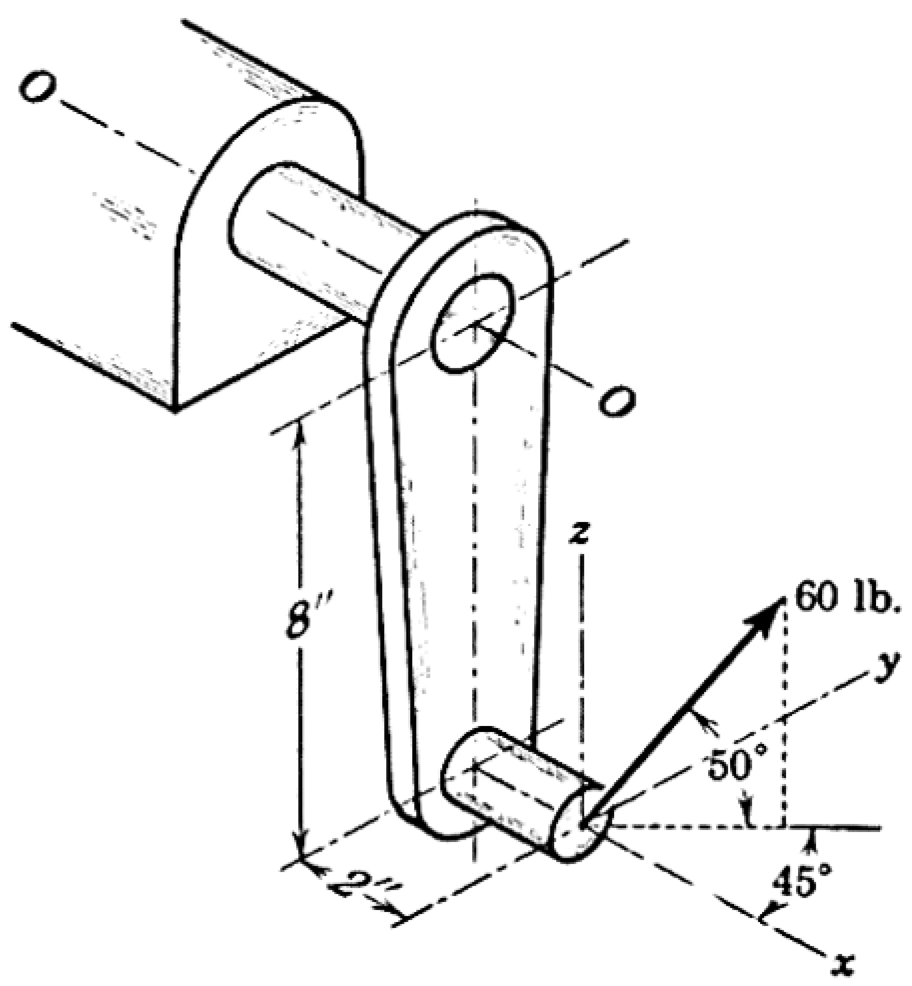
Prob. 27

27. The spur gear shown has a pressure angle of 14½ deg. and transmits a torque of 300 lb. in. Assume that the contact between the gear and its mating pinion (not shown) is all on one tooth at any instant and find the value of the contact force F between the gear teeth.

28. Determine the moment M about point A of the 500 lb. force on the truss in Prob. 26.

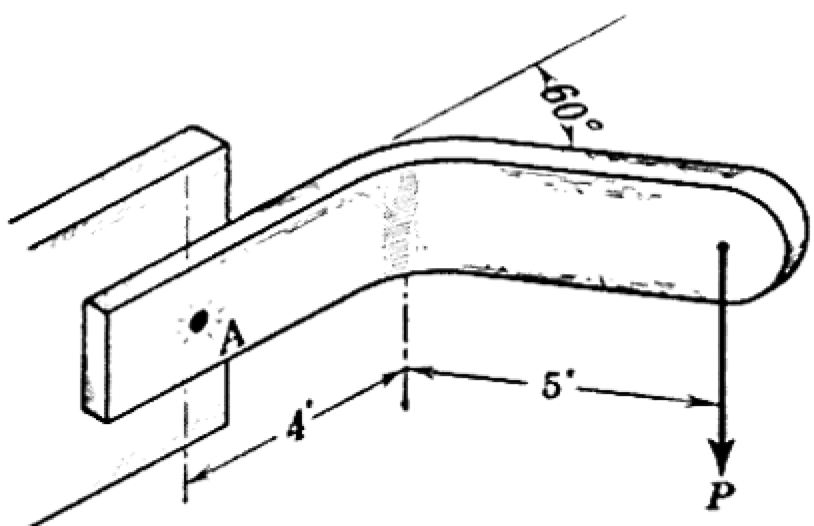
Ans. $M = 2840$ lb. ft.

29. Determine the moment M about the shaft axis O–O due to the 60 lb. force applied to the crank handle as shown.



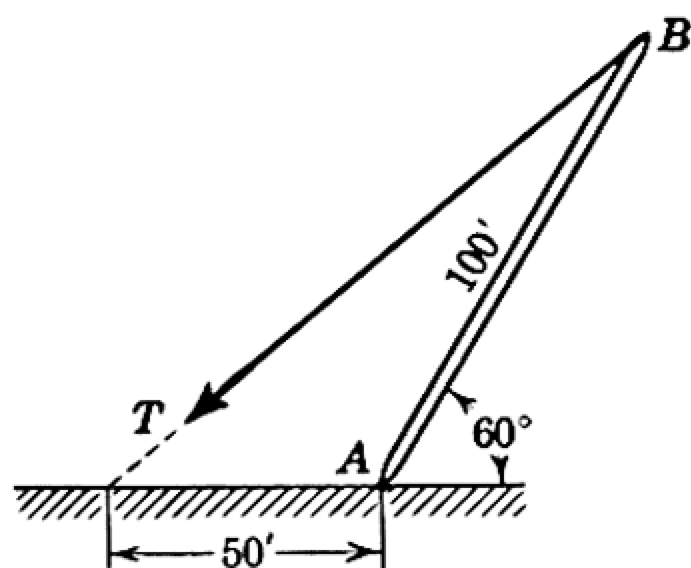
Prob. 29

30. The spot weld which holds the bracket to the plate at point A as shown in the figure can withstand a maximum twist in the plane of the plate of 100 lb. ft. Determine the maximum load P.

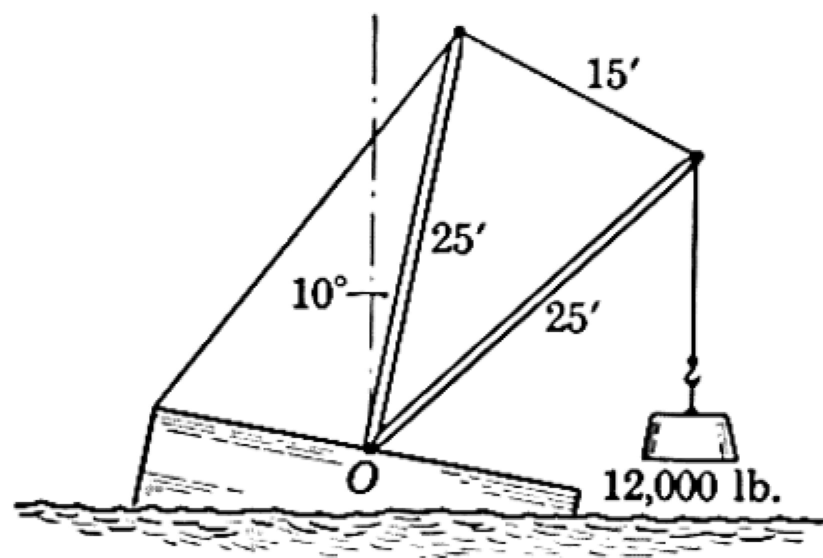


Prob. 30

31. In raising the mast AB from the position shown the tension T in the cable must supply a moment about A of 25,000 lb. ft. Find T. *Ans.* $T = 764$ lb.



Prob. 31

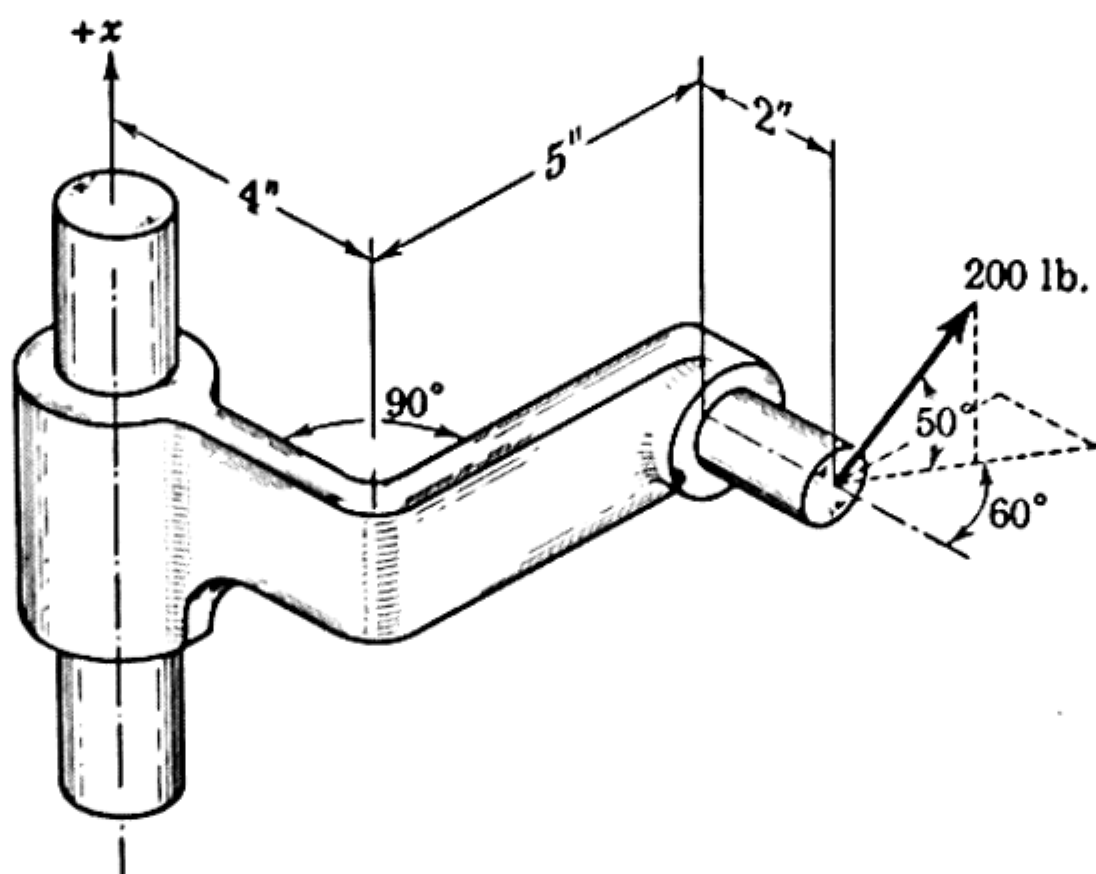


Prob. 32

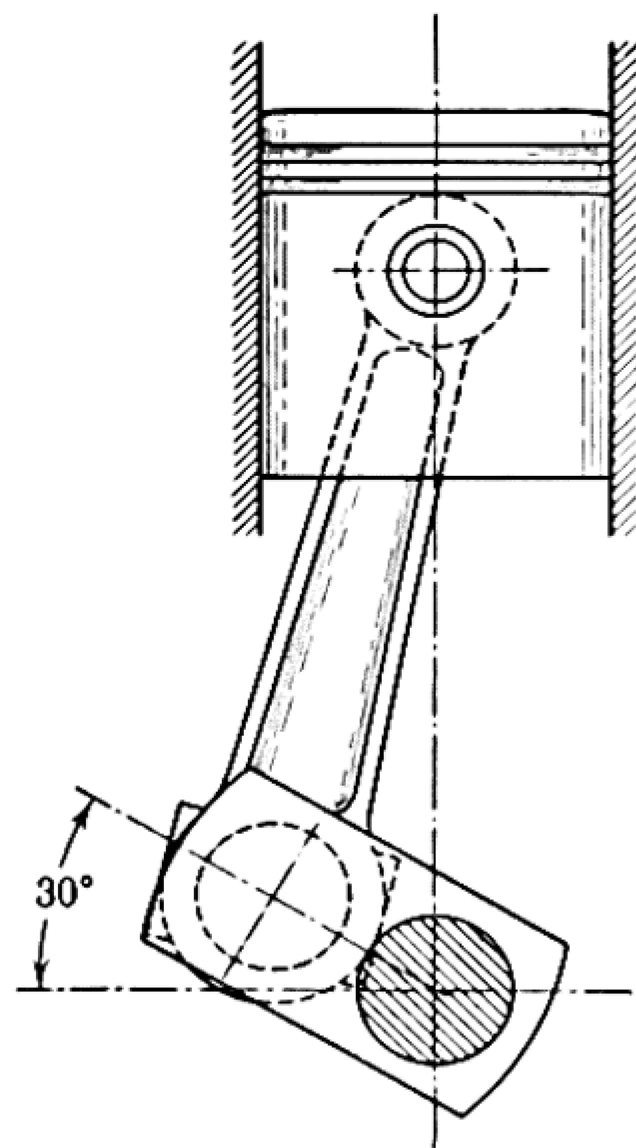
32. The derrick on the barge acquires a 10 deg. list when supporting a 12,000 lb. load. What is the moment M of this force about O?

Ans. $M = 212{,}000$ lb. ft.

33. Determine the moment M_x about the shaft axis of the 200 lb. force applied to the bracket shown.



Prob. 33



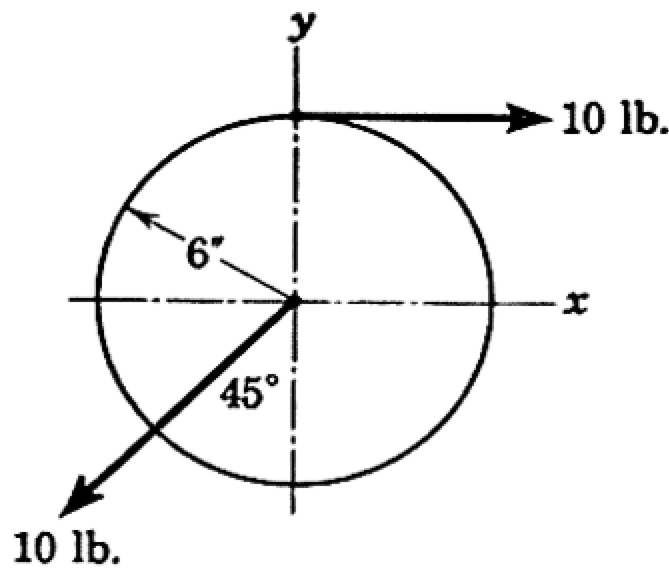
Prob. 34

34. The single-cylinder gasoline engine represented in the figure has a stroke of 8 in., and the connecting rod is 14 in. between bearing centers. In the position shown the rod is under a compression (in the direction of the rod) of 4500 lb. Determine the resultant moment M of this force about the crankshaft axis.

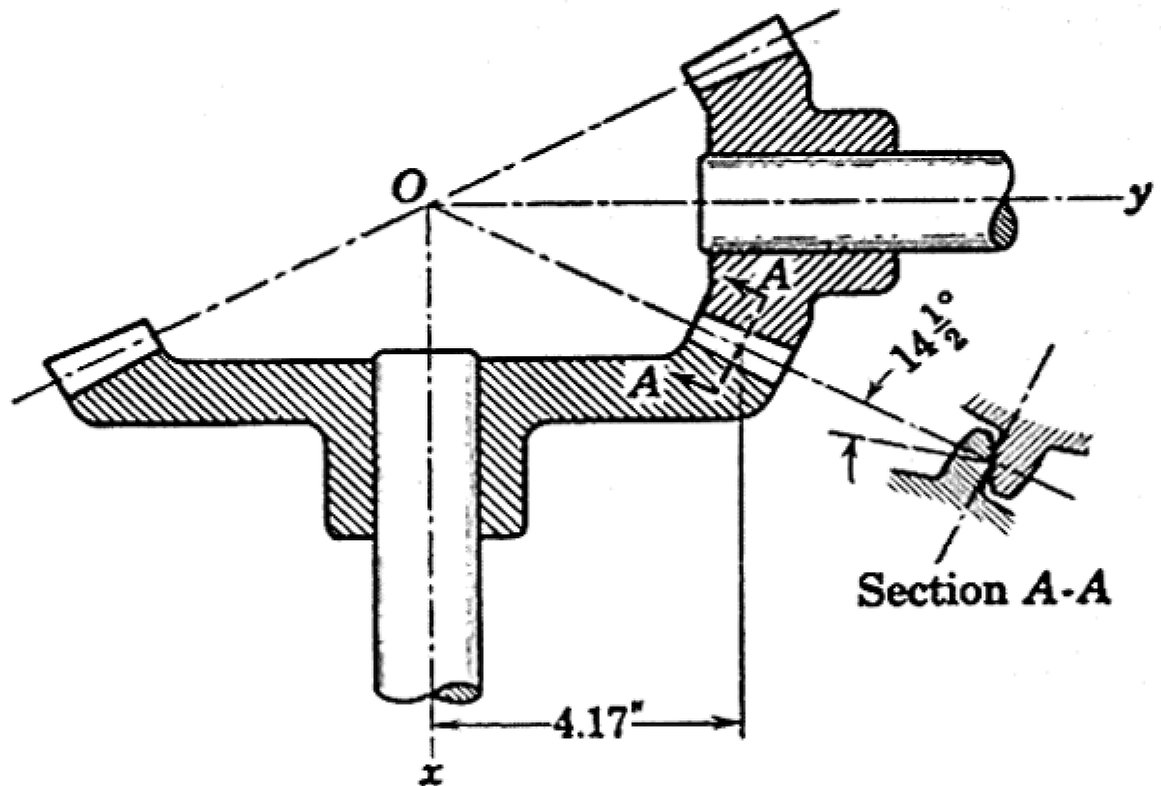
Ans. $M = 17{,}330$ lb. in.

35. A wheel is subjected to two forces as shown in the figure. Determine graphically the coordinates of the point on the wheel about which the combined moment M of the two forces is a maximum. Find M for this point.

Ans. $x = -5.5$ in., $y = -2.3$ in., $M = 106$ lb. in.



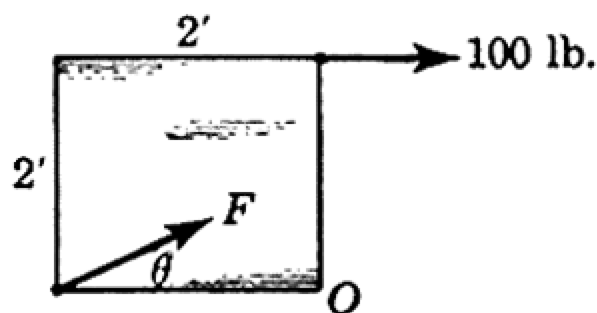
PROB. 35



PROB. 36

* **36.** Contact between the two bevel gears shown, which have a speed ratio of 2:1, may be assumed to occur at a point 4.17 in. from the axis of the larger gear. If the contact force is normal to the tooth surfaces shown in the auxiliary view and is 100 lb., determine the magnitude of its moment M about each gear axis.

Ans. $M_x = 404$ lb. in., $M_y = 202$ lb. in.



PROB. 37

* **37.** If the combined moment about O of the two forces shown is 320 lb. ft. clockwise and the forces are the components of a single 150 lb. force, find F and θ. *Ans.* $F = 70.7$ lb., $\theta = 58° 6'$; or $F = 245$ lb., $\theta = 165° 49'$

* **38.** Solve Prob. 35 algebraically.

15. Couple. Two parallel forces which are equal in magnitude, opposite in direction, and not collinear constitute a *couple*. Consider the action of two such forces as shown in Fig. 15*a*. These two forces cannot be combined into a single force since their sum in every direction is zero. Their effect is entirely one of producing a tendency of rotation. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O is

$$M = F(a + d) - Fa,$$

or

$$M = Fd,$$

in a counterclockwise direction. This expression is the value of the couple and does not contain any reference to the dimension a which lo-

cates the forces with respect to the moment center O. It follows that the value of a couple is the same for *all* moment centers.

A couple may be represented by the *free* vector M, as shown in Fig. 15*b*, where the direction of M is normal to the plane of the couple and the sense of the vector is established by the right-hand convention. A couple

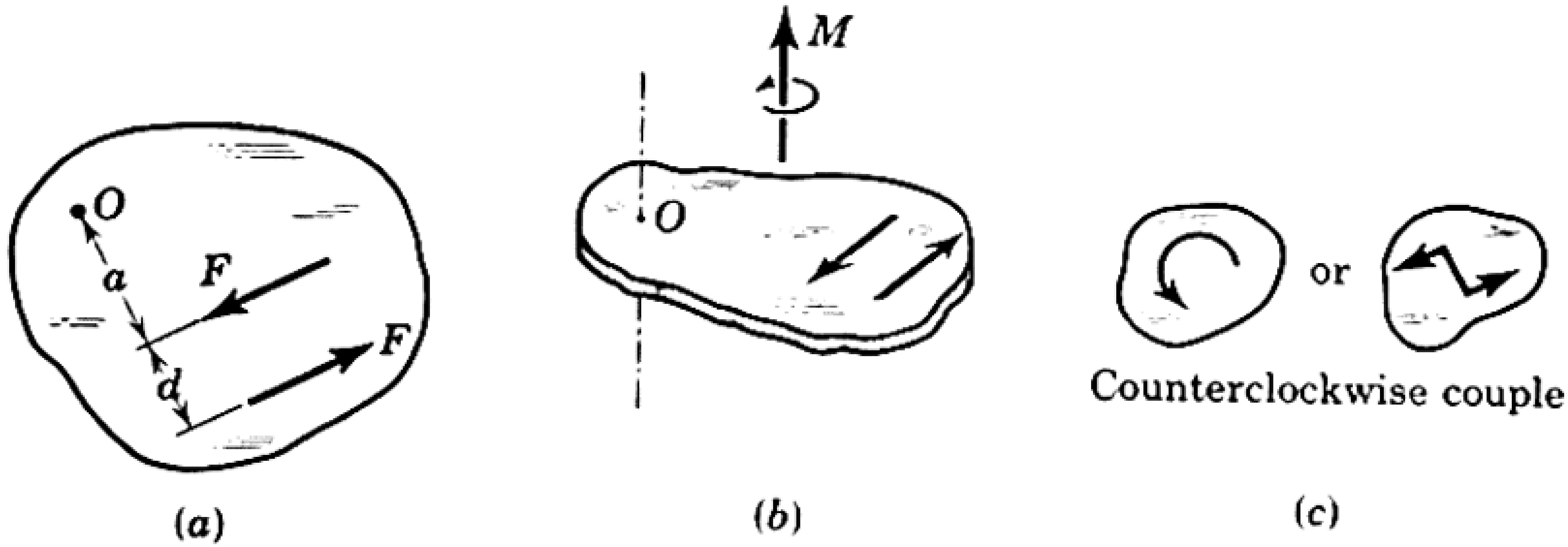


Fig. 15

is unchanged as long as the magnitude and direction of its vector remain constant. A given couple will not be altered by changing the values of F and d as long as their product remains the same. Likewise a couple is not affected by allowing the forces to act in any one of parallel planes. In Fig. 16 are shown four different configurations of the same couple $M = Fd$. When dealing with couples due to forces all of which act in the

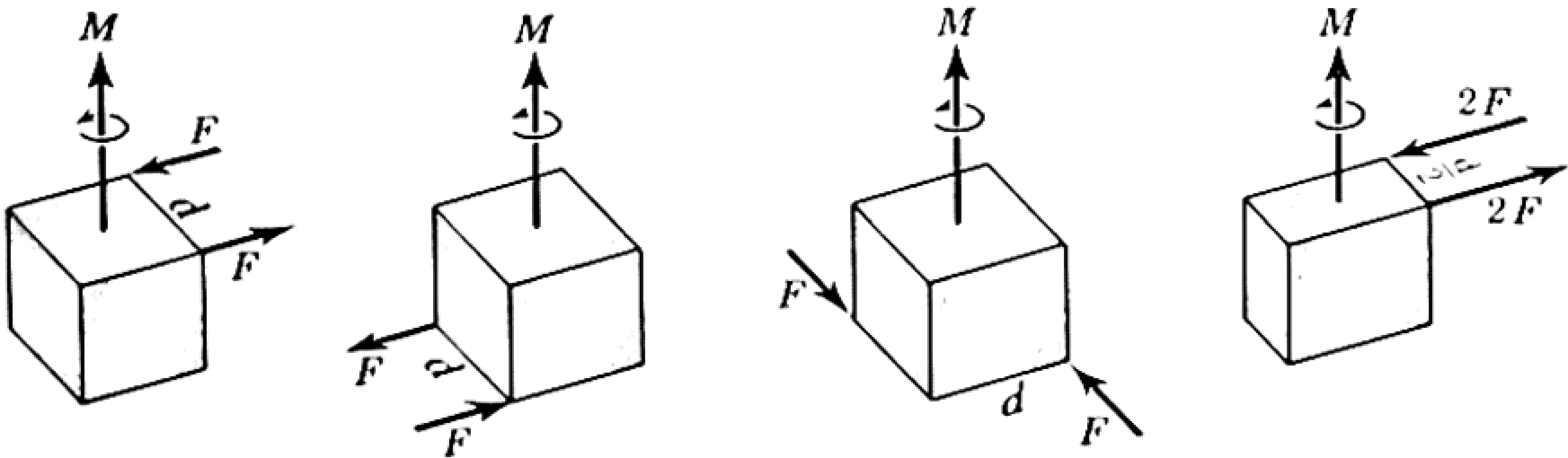


Fig. 16

same or parallel planes, the couple vectors will be perpendicular to the plane or planes. In this case it is more convenient to represent such a couple by either of the conventions shown in Fig. 15*c*, where the counterclockwise couple may be taken as positive and a clockwise couple negative or vice versa.

Couples which act in nonparallel planes may be added by the ordinary rules of vector combination. Thus the two couples M_1 and M_2 in

Fig. 17*a* due to forces which act in the two planes indicated may be replaced by their vector sum M shown in Fig. 17*b*, which represents a couple due to forces in a plane normal to M.

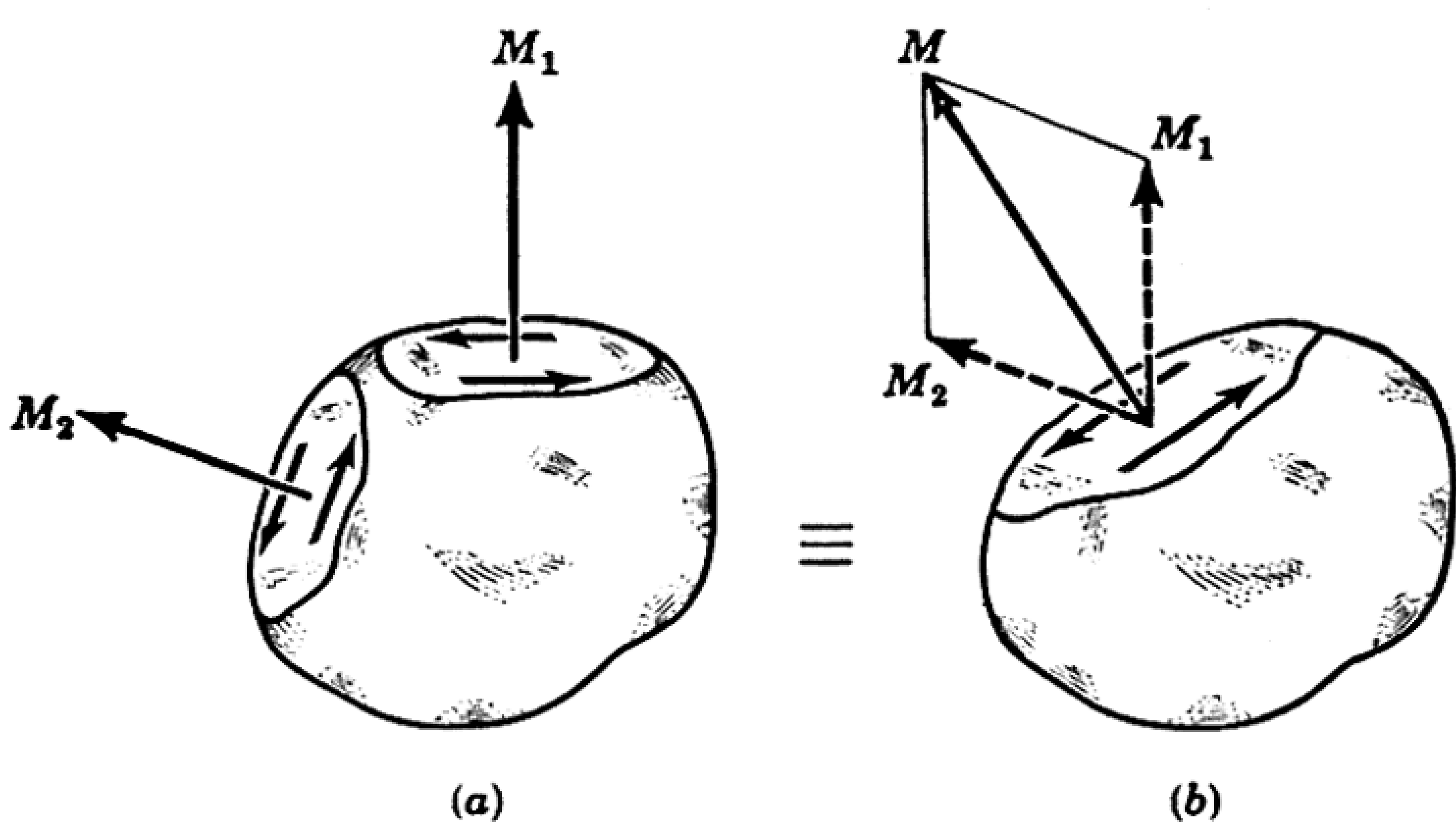


FIG. 17

16. Resolution of a Force into a Force and a Couple. Consider a body with a force F acting at point A as shown in Fig. 18*a*. At any other point B two equal and opposite forces F may be applied as in Fig. 18*b* with no external effect on the body. If these forces are parallel to the original force F, then a couple $M = Fd$ is formed by the original F and the force F in the opposite direction at B. In representing the couple by the curled arrow in Fig. 18*c*, it becomes evident that the original force

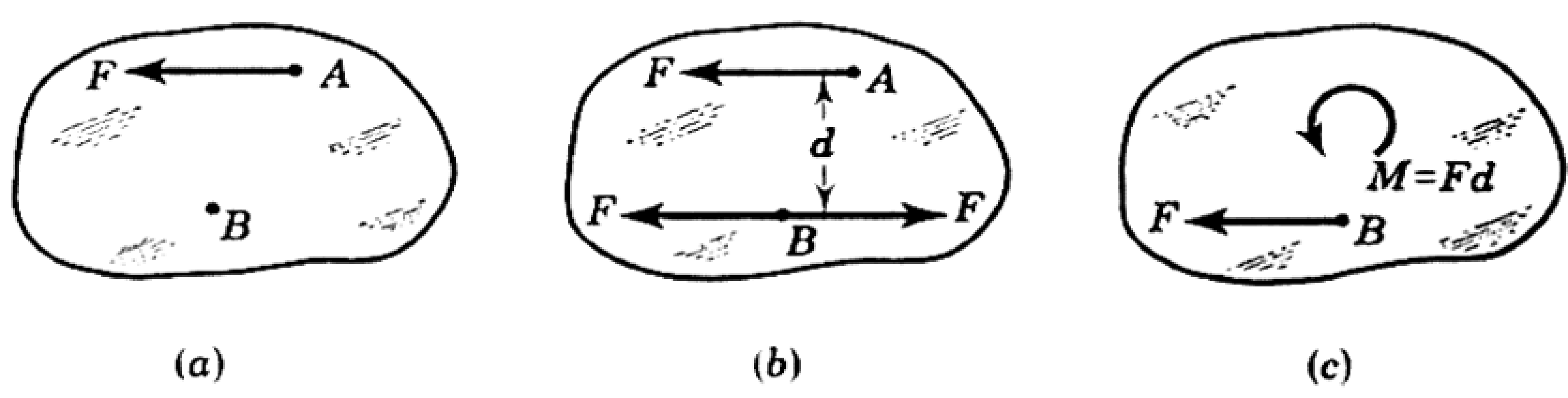


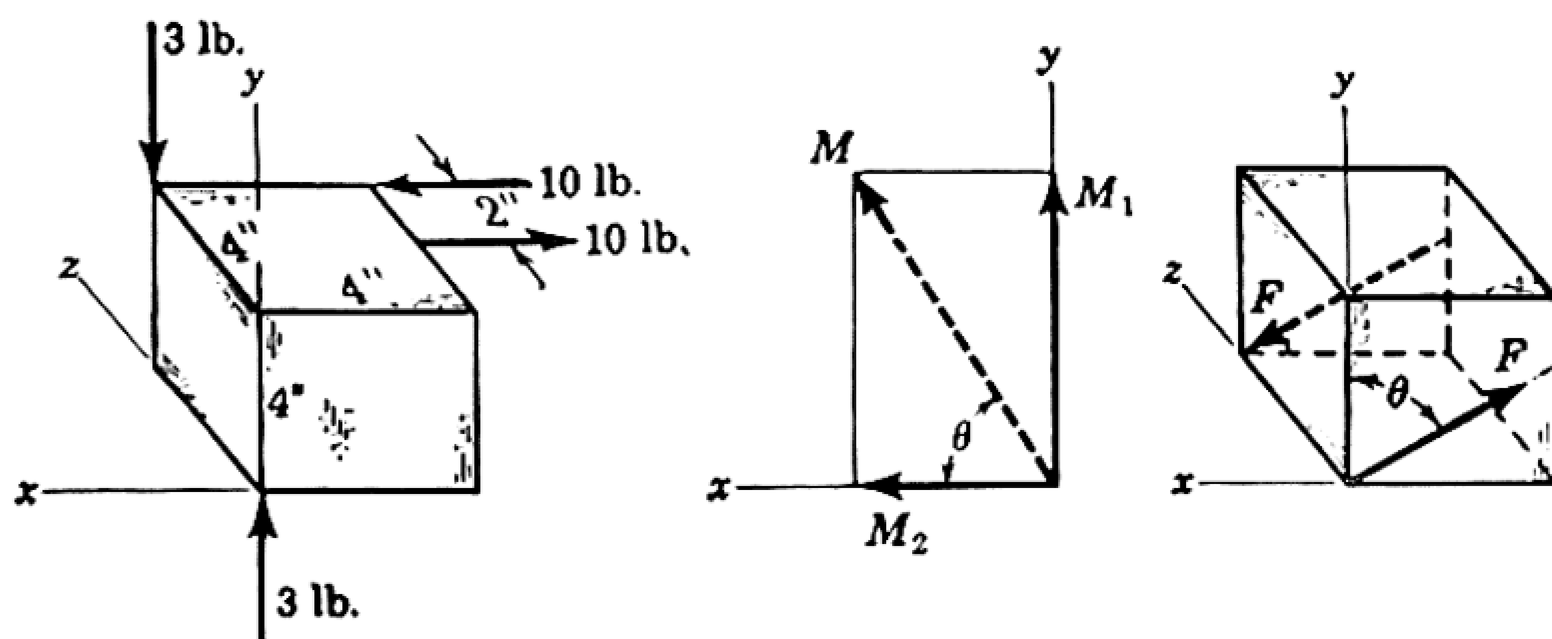
FIG. 18

F has been replaced by or resolved into a force of the same magnitude and direction as F, but having a different line of action, and a couple. The magnitude of the couple equals the value of the force multiplied by the distance through which its line of action has been shifted. The direction of the couple is the same as the direction of the moment of the original force about a point on its new line of action. A force may always be replaced by an equal force having any parallel line of action and the corresponding couple. It follows that a given force and couple which act in

the same plane may be combined to yield a single equal force having a unique line of action. The representation of a force by a force and couple has many useful applications.

SAMPLE PROBLEMS

39. Determine the magnitude and direction of the vector M which represents the resultant of the two given couples. Find the two forces F applied in the two faces of the cube parallel to the x–y plane which may replace the four forces shown.



PROB. 39

Solution: The couple due to the 10 lb. forces is $M_1 = 10 \times 2 = 20$ lb. in., and the vector is in the y-direction by the right-hand convention. The couple due to the 3 lb. forces is $M_2 = 3 \times 4 = 12$ lb. in., and the vector is in the x-direction. The vector sum is

$$M = \sqrt{M_1^2 + M_2^2} = \sqrt{(20)^2 + (12)^2} = 23.3 \text{ lb. in.,} \qquad \textit{Ans.}$$

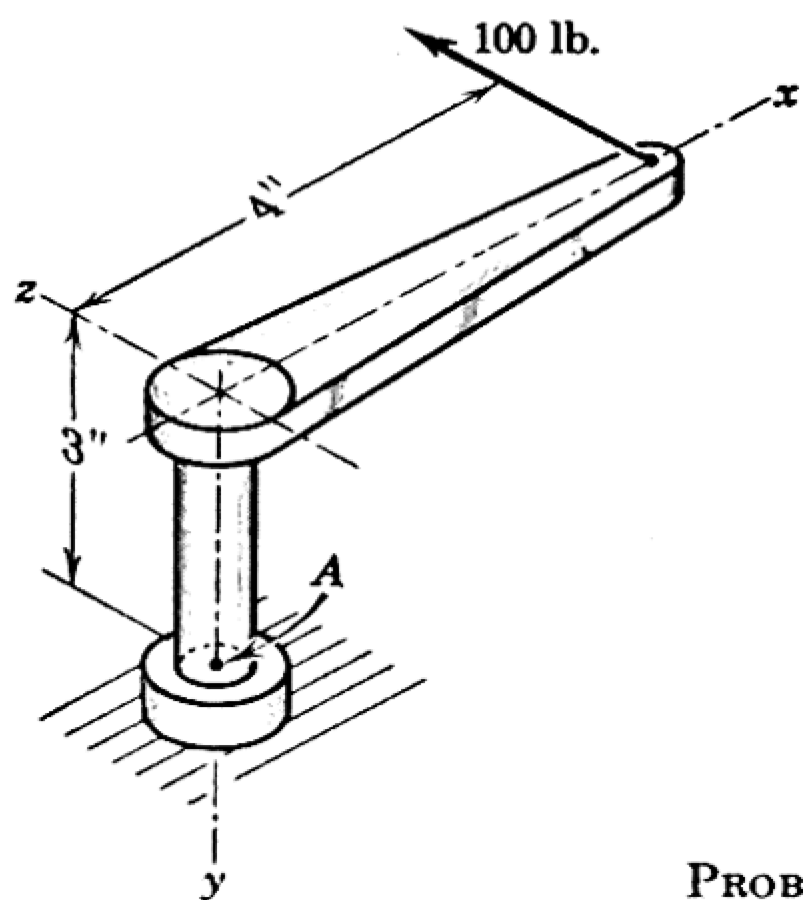
and the direction of M with the x-axis is given by

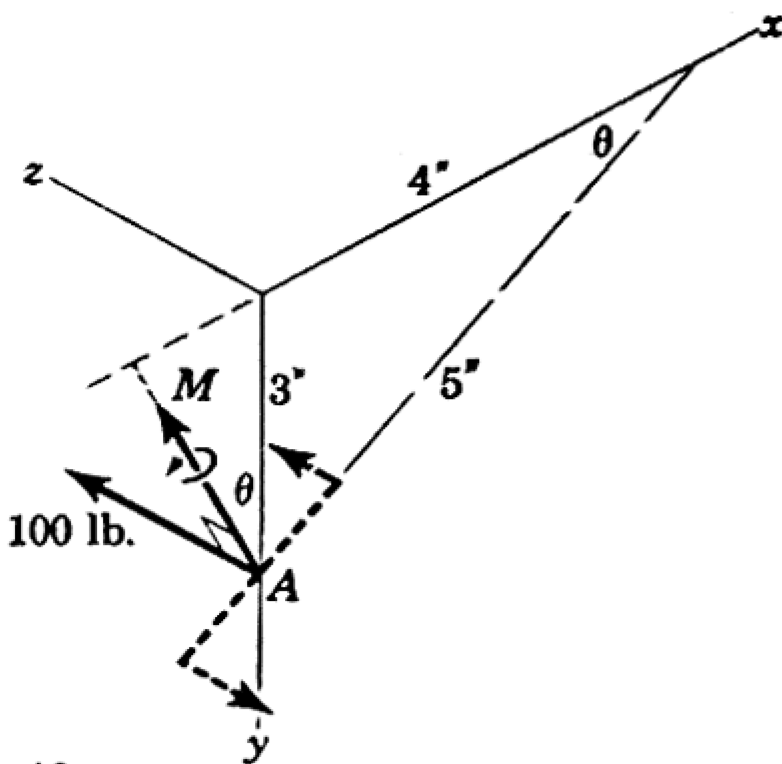
$$\theta = \tan^{-1}\frac{M_1}{M_2} = \tan^{-1}\frac{20}{12} = 59° 2'. \qquad \textit{Ans.}$$

The forces F lie in a plane normal to M and hence make an angle of $\theta = 59° 2'$ with the y-direction. The magnitude of the forces is

$$F = \frac{M}{d} = \frac{23.3}{4} = 5.83 \text{ lb.} \qquad \textit{Ans.}$$

40. A force of 100 lb. is applied to a lever which is attached to a fixed shaft as shown. In determining the effect of the force on the shaft at a section such as A it is convenient to replace the force by a force at A and a couple M. Find the magnitude and direction of M.





PROB. 40

Solution: Shifting the force a distance of $\sqrt{3^2 + 4^2} = 5$ in. to a parallel position at A requires the addition of a couple of magnitude

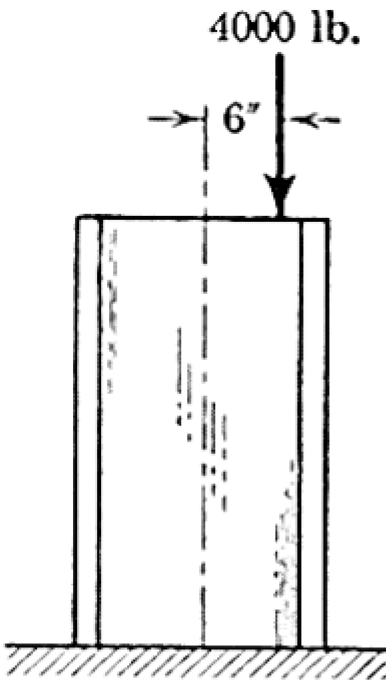
$$M = 100 \times 5 = 500 \text{ lb. in.} \qquad \textit{Ans.}$$

The vector M representing the couple is in the x–y plane normal to the force vector and makes an angle with the y-axis of

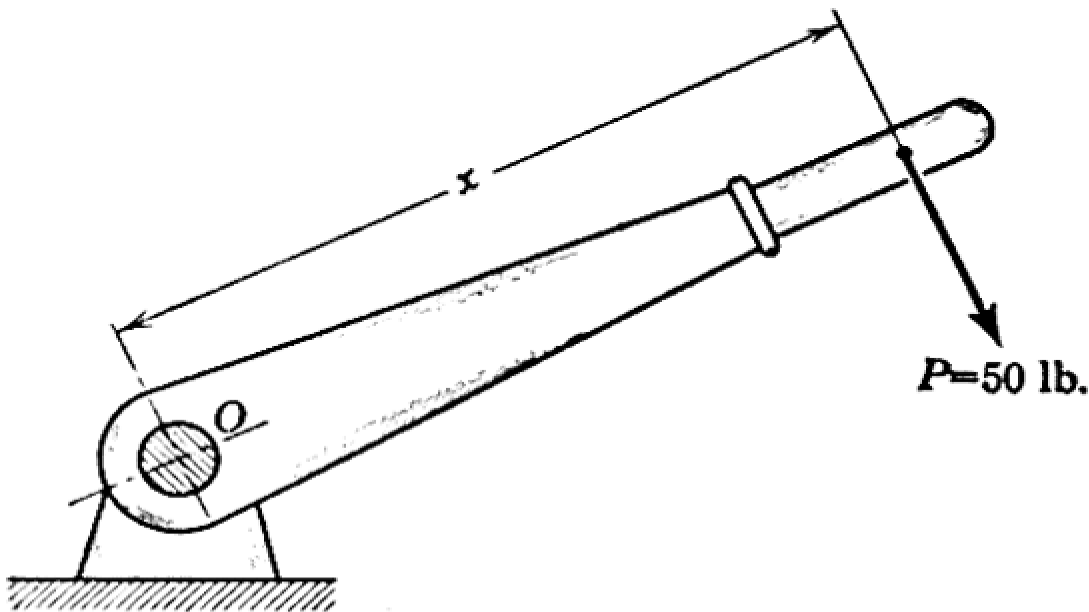
$$\theta = \cos^{-1} \tfrac{4}{5} = 36° \, 52'. \qquad \textit{Ans.}$$

PROBLEMS

41. A column is subjected to an eccentric load of 4000 lb. as shown. The effect of the applied load may be analyzed by considering it to produce a compression along the center line of the column and an accompanying twist or couple M. Determine M.



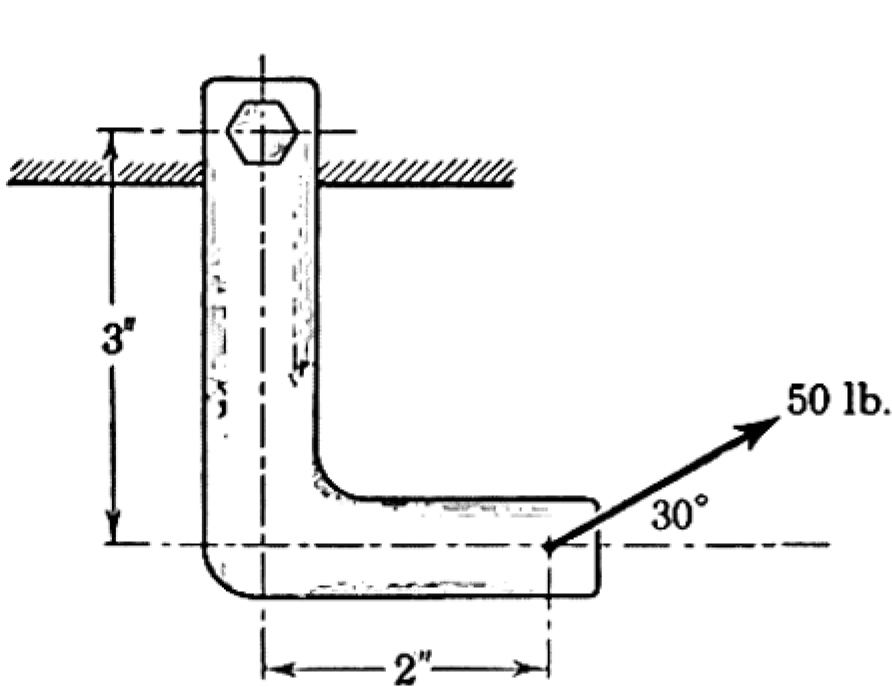
PROB. 41



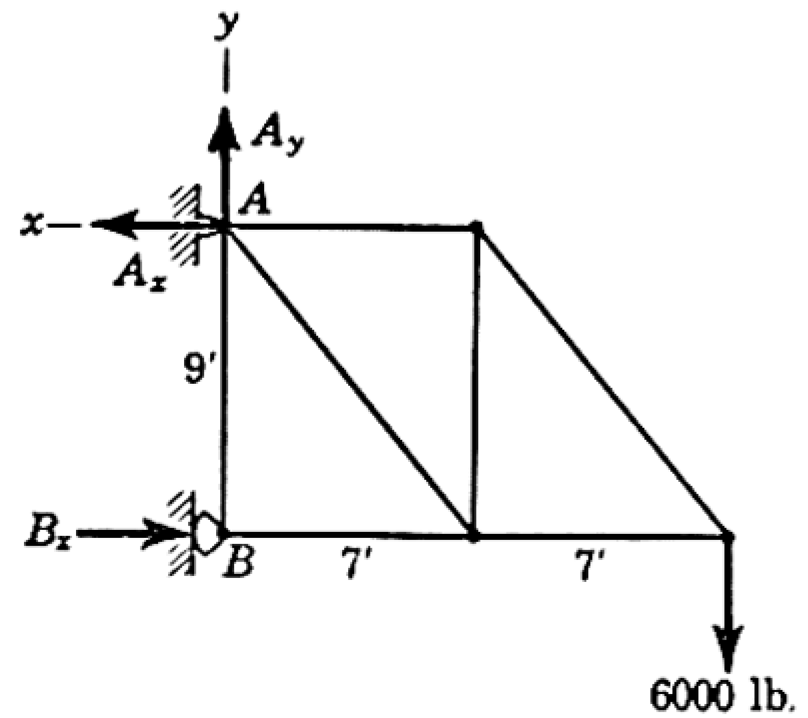
PROB. 42

42. The control lever is subjected to a counterclockwise couple of 90 lb. ft. exerted by its shaft and a force P of 50 lb. If the resultant of the couple and P passes through O, find the distance x. *Ans.* $x = 1.80$ ft.

43. Replace the 50 lb. force which acts on the angle bracket by a force at the center of the bolt and a couple M.



PROB. 43

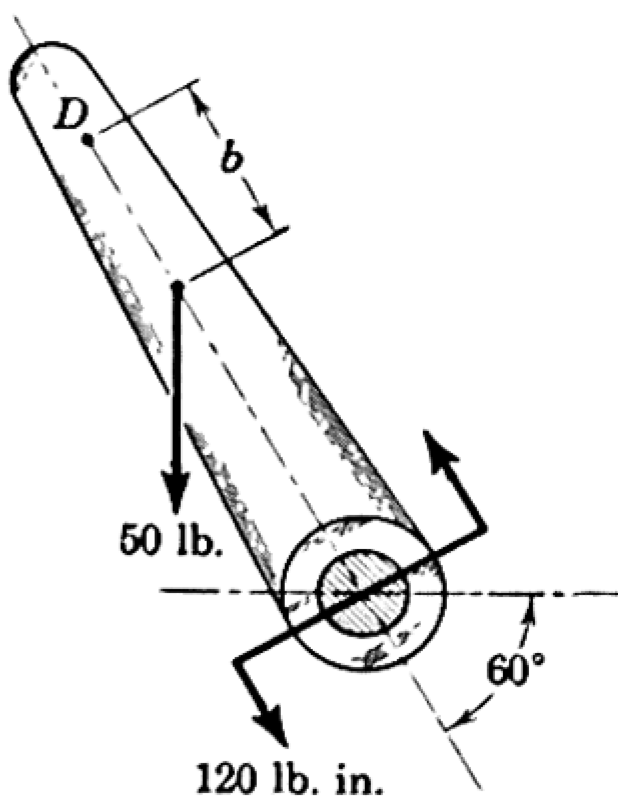


PROB. 44

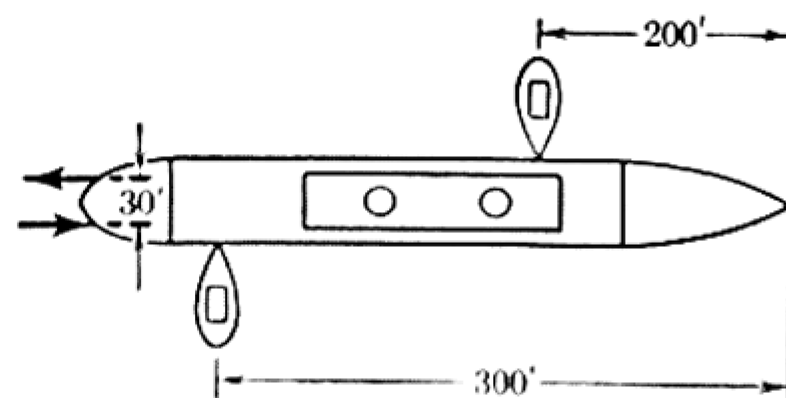
44. The truss shown supports the vertical load of 6000 lb. and is held in place by the reactive forces exerted on it by the supports as indicated in the figure. The force A_y and the applied load constitute a couple, and the remaining two forces constitute an equal and opposite couple. Find the reaction component B_x and the total force A on the supporting pin connection.

Ans. $B_x = 9330$ lb., $A = 11{,}090$ lb.

45. Replace the couple and force shown by a single force F applied at a point D. Locate D by determining the distance b.



PROB. 45

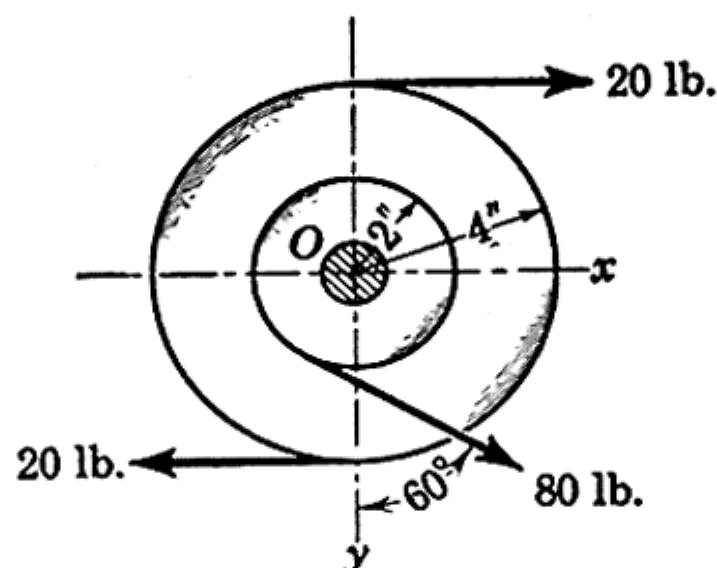


PROB. 46

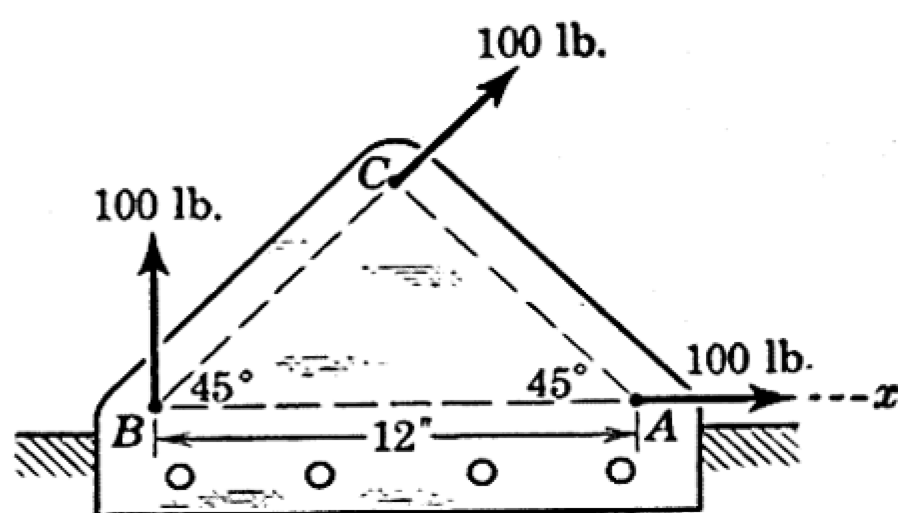
46. Each propeller of the twin-screw ship shown develops a full-speed thrust of 90,000 lb. In maneuvering the ship one propeller is turning full speed ahead and the other full speed in reverse. What force F must each tug exert on the ship to counteract the turning effect of the ship's propellers?

Ans. $F = 27{,}000$ lb.

47. Replace the three forces acting on the pulley shown by an equivalent single force R.



PROB. 47



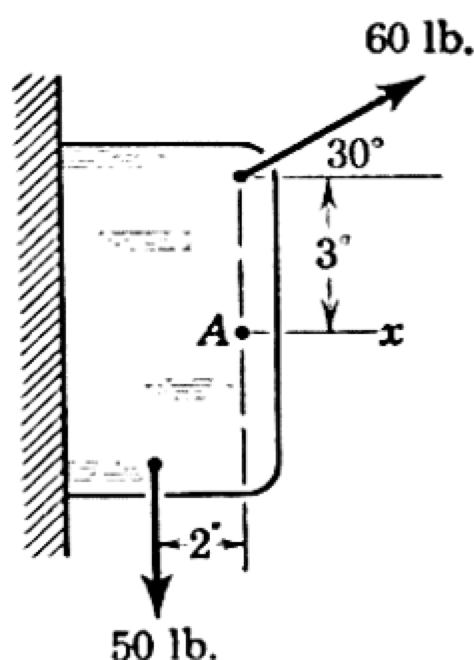
PROB. 48

48. The fixed plate is subjected to the three 100 lb. forces. Replace these forces by an equivalent system composed of a single force F at A and a couple M.

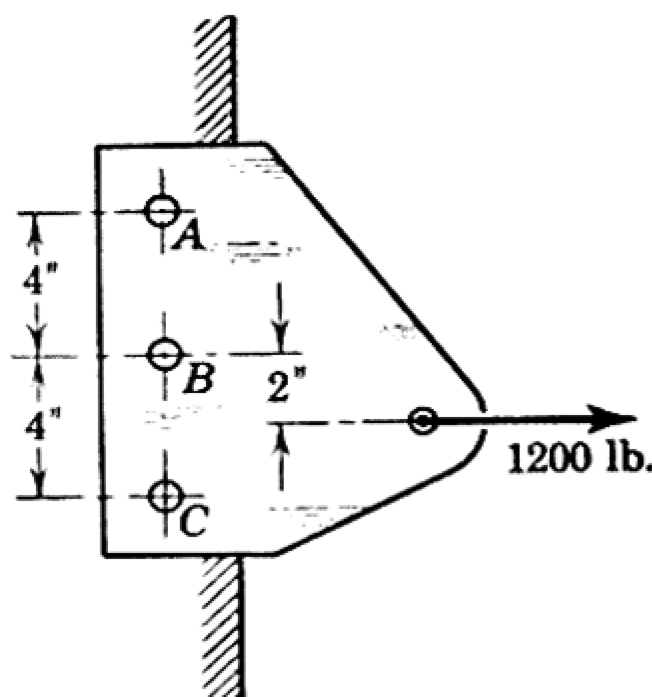
Ans. $F = 241$ lb., $\theta_x = 45°$; $M = 2045$ lb. in. clockwise

49. Replace the two forces shown by an equivalent system consisting of a force F acting at A and a couple M.

Ans. $F = 55.7$ lb., $\theta_x = 21° 5'$ clockwise; $M = 55.9$ lb. in. clockwise



PROB. 49



PROB. 50

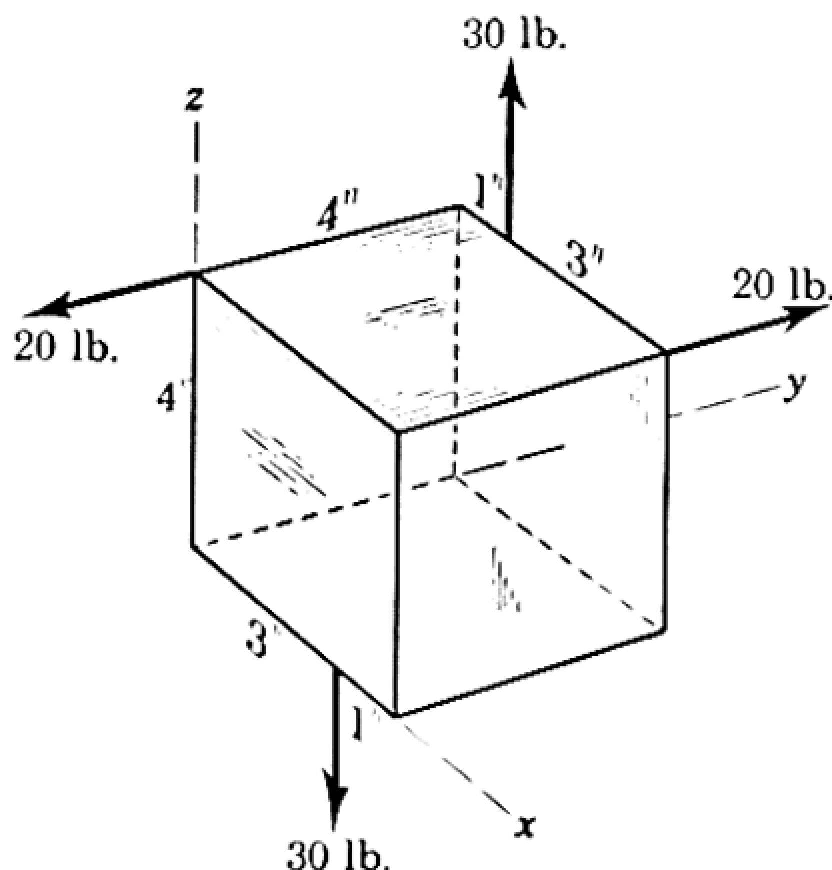
50. The bracket is fastened to the girder by means of the three rivets A, B, and C. For equilibrium the resultant of the forces exerted by the rivets on the bracket must be equal and opposite to and collinear with the 1200 lb. applied force. Assume the members to be perfectly rigid and find the force supported by each rivet by replacing the applied load by a force along the horizontal center line through B and a couple.

Ans. $A = 100$ lb., $B = 400$ lb., $C = 700$ lb.

51. If the bracket of Prob. 50 is secured to the girder by the two rivets A and B only, determine by the method outlined in Prob. 50 the forces exerted by these rivets on the bracket. Move the force to the horizontal center line midway between A and B.

52. Determine the magnitude and direction of the couple vector M which is equivalent to the two couples shown.

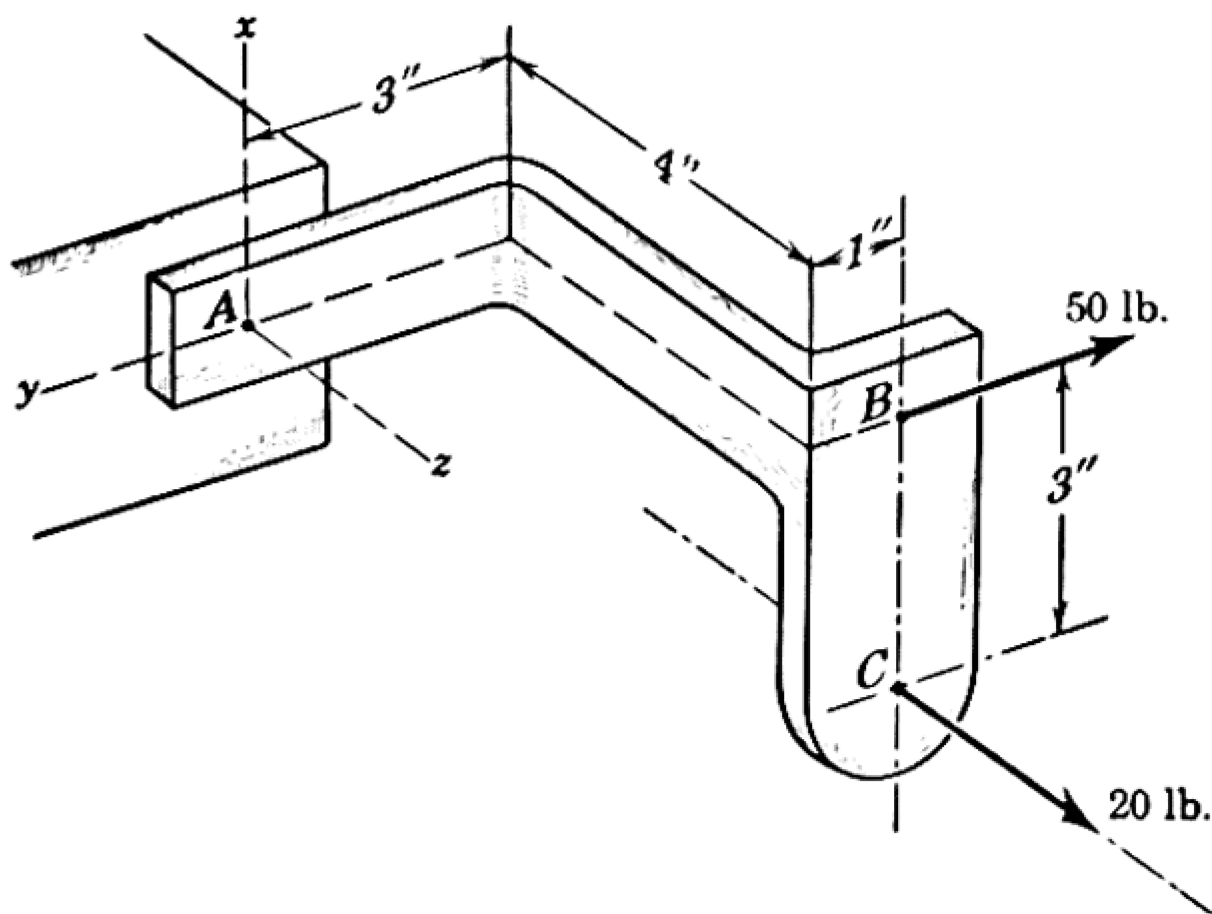
Ans. $M = 156$ lb. in., $\theta_x = \cos^{-1} 0.769$, $\theta_{xy} = \cos^{-1} 0.860$



PROB. 52

* **53.** Replace the two forces shown by a force F at A and a couple M.

Ans. $F = 53.9$ lb., $M = 134.2$ lb. in.



PROB. 53

* **54.** Work Prob. 53 if the 50 lb. force is applied in its same direction at C instead of B. Specify M by its x-, y-, and z-components.

17. Resultant of Coplanar Force Systems. The resultant of a system of forces is the simplest force system which can replace the original forces without altering their external effect on a rigid body. The equilibrium of a body is the condition wherein the resultant of all forces is zero, and the

acceleration of a body is described by equating the force resultant to the product of mass and acceleration. Thus the determination of resultants is basic to both statics and dynamics. The properties of force, moment, and couple discussed in the preceding articles will now be used to determine the resultants of coplanar force systems.

The resultant of a system of coplanar forces may be obtained by adding the forces two at a time and then combining their sums. The three forces F_1, F_2, F_3 of Fig. 19*a* may be combined by first adding any two, such as F_2 and F_3. By the principle of transmissibility they may be

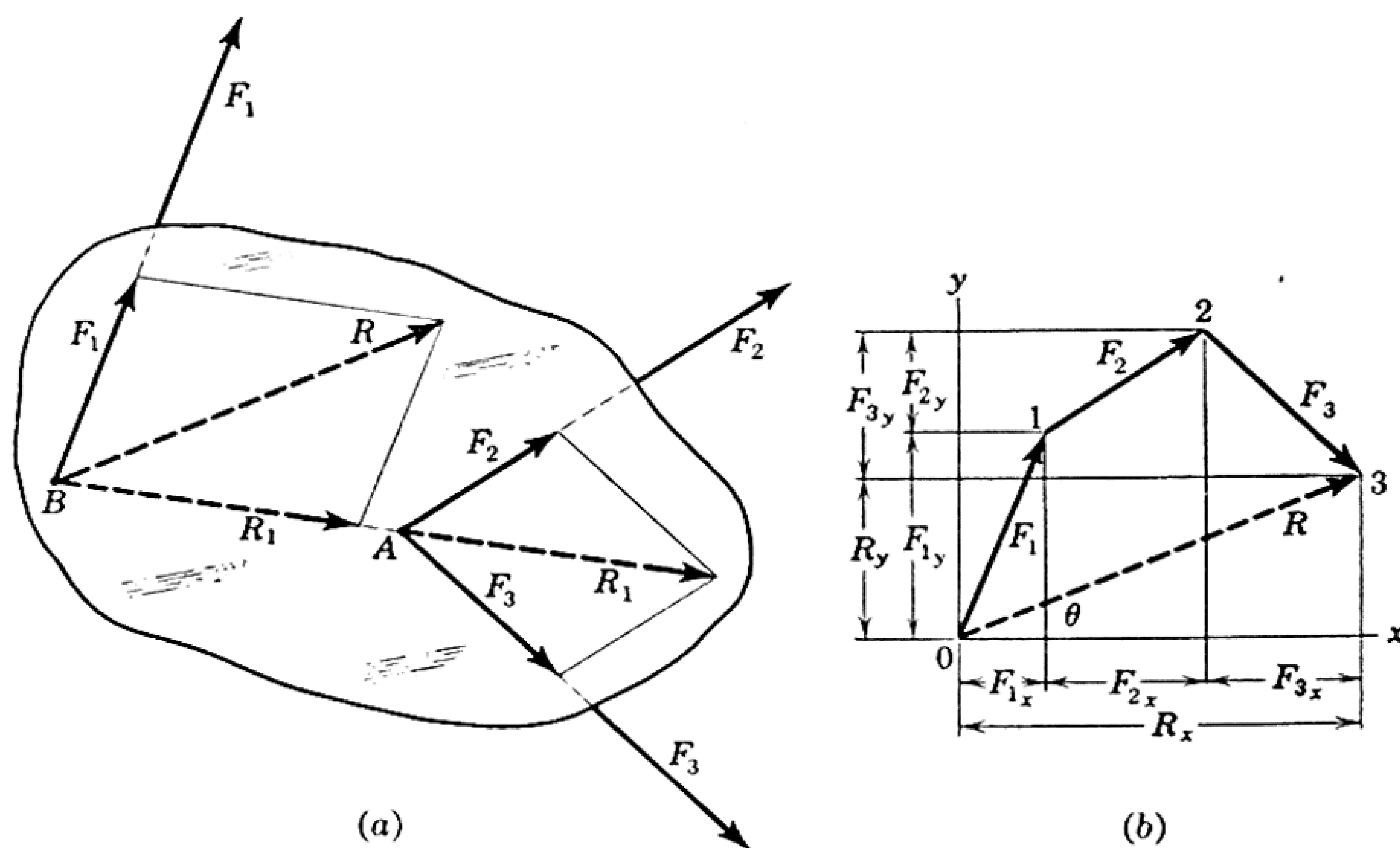


FIG. 19

moved along their lines of action to their point of concurrency A, and their sum R_1 formed by the parallelogram law. The force R_1 may then be combined with F_1 by the parallelogram law at their point of concurrency B to obtain the resultant R of the three given forces. The order of combination of the forces is immaterial as may be verified by combining them in a different sequence. The force R may be applied at any point on its established line of action.

The magnitude and direction of R may be obtained with addition by the triangle law as shown in Fig. 19*b*. Here the forces are treated as free vectors and added head-to-tail. The resultant of F_1 and F_2 is a vector directed from 0 to 2 and when combined with F_3 gives R correct in magnitude and direction. The polygon 0–1–2–3 is known as a *force polygon*. Algebraically these results may be obtained by forming the rectangular components of the forces in any two convenient perpendicular directions.

In Fig. 19*b* the x- and y-components of R are seen to be the algebraic sums of the respective components of the three forces. Thus, in general, the rectangular components of the resultant R of a coplanar system of forces may be expressed as

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y,$$

where

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}. \tag{7}$$

The angle made by R with the x-axis is clearly

$$\theta = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}. \tag{8}$$

The location of the line of action of R may be computed with the aid of Varignon's theorem. Although this theorem was proved for two concurrent components of a given force, it holds for any system of forces. The moment of R, Fig. 19*a*, about some point must equal the sum of the moments of its two components F_1 and R_1 about the same point. The moment of R_1, however, must equal the sum of the moments of its components F_2 and F_3 about the same point. It follows that the moment of R about any point equals the sum of the moments of F_1, F_2, and F_3 about this same point. Application of this *principle of moments* about the point O, Fig. 20, gives the equation

$$Rd = F_1d_1 - F_2d_2 + F_3d_3$$

for this particular configuration of forces where the clockwise direction is arbitrarily taken as positive. The distance d is computed from this relation, and R, whose magnitude and direction have been determined from Eqs. (7) and (8), may

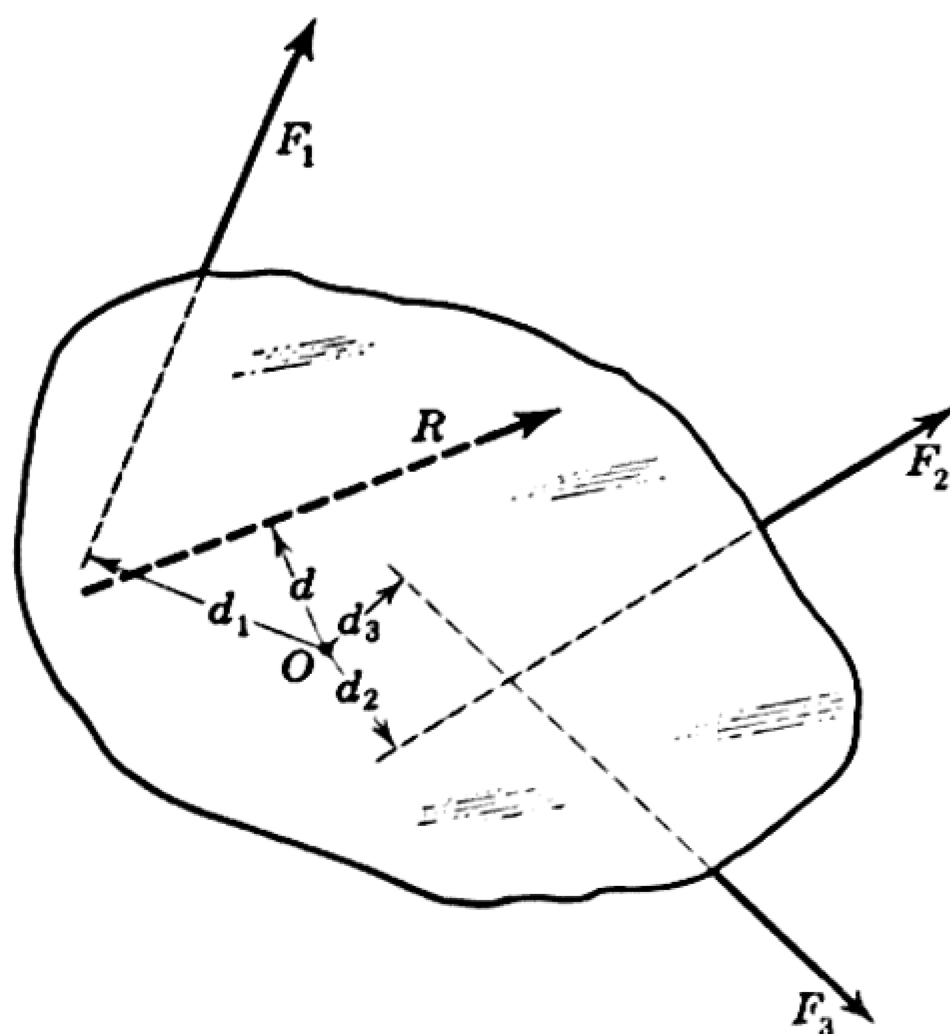


Fig. 20

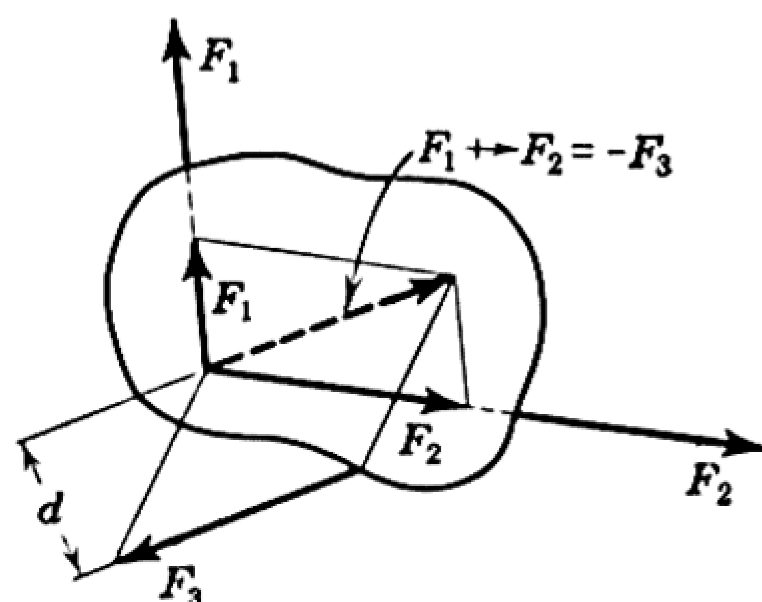


Fig. 21

now be completely located. In general, then, the moment arm d of the resultant R is given by

$$Rd = \Sigma M_O, \tag{9}$$

where ΣM_O stands for the algebraic sum of the moments of the forces of the system about any point O.

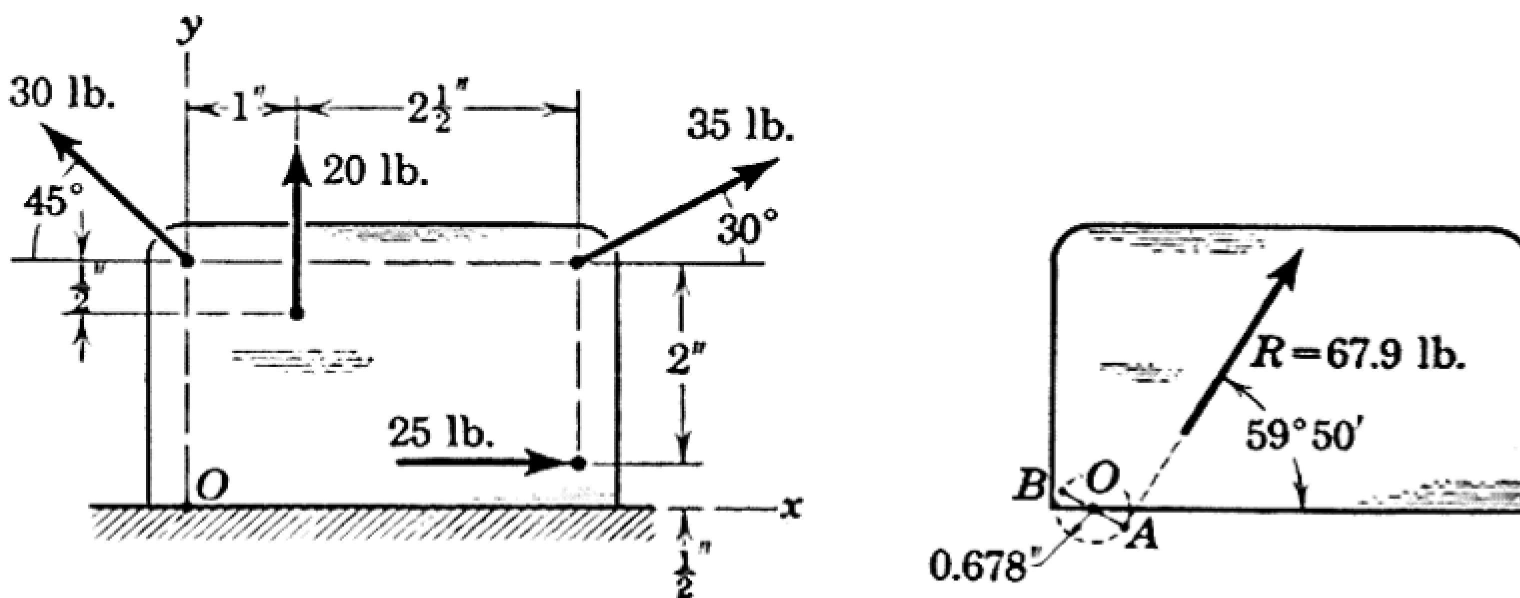
For a system of forces all concurrent at a given point the resultant passes through this point and may be determined graphically by parallelogram or triangle addition or may be computed from Eqs. (7) and (8).

For a system of parallel forces the magnitude of the resultant is the algebraic sum of the several forces, and the position of its line of action may be obtained from the principle of moments expressed by Eq. (9).

Consider now a force system such as shown in Fig. 21, where the polygon of forces closes and consequently there is no resultant force R. Direct combination by the parallelogram law shows that for the case illustrated the resultant is a couple of magnitude F_3d. The value of the couple is equal to the moment sum about any point. Thus it is seen that the *resultant of a coplanar system of forces may be either a force or a couple.*

SAMPLE PROBLEM

55. Determine the resultant of the four forces acting on the plate shown.



PROB. 55

Solution: Point O is selected arbitrarily as the origin of coordinates for an algebraic solution. The components R_x and R_y, the resultant R, and the angle θ_x become

$[R_x = \Sigma F_x]$ $\quad R_x = -30 \cos 45° + 25 + 35 \cos 30° = 34.1$ lb.,

$[R_y = \Sigma F_y]$ $\quad R_y = 30 \sin 45° + 20 + 35 \sin 30° = 58.7$ lb.,

$[R = \sqrt{R_x^2 + R_y^2}]$ $\quad R = \sqrt{(34.1)^2 + (58.7)^2} = 67.9$ lb., *Ans.*

$\left[\theta_x = \tan^{-1} \frac{R_y}{R_x}\right]$ $\quad \theta_x = \tan^{-1} \frac{58.7}{34.1} = 59° \, 50'$. *Ans.*

The resultant is now determined in magnitude and direction. The position of its line of action is obtained from the principle of moments. With the clockwise direction as positive and point O as the moment center this principle requires

$$[Rd = \Sigma M_O] \qquad 67.9d = -30 \times 2.5 \cos 45° - 20 \times 1 + 35 \times 2.5 \cos 30° - 35 \times 3.5 \sin 30° + 25 \times 0.5,$$

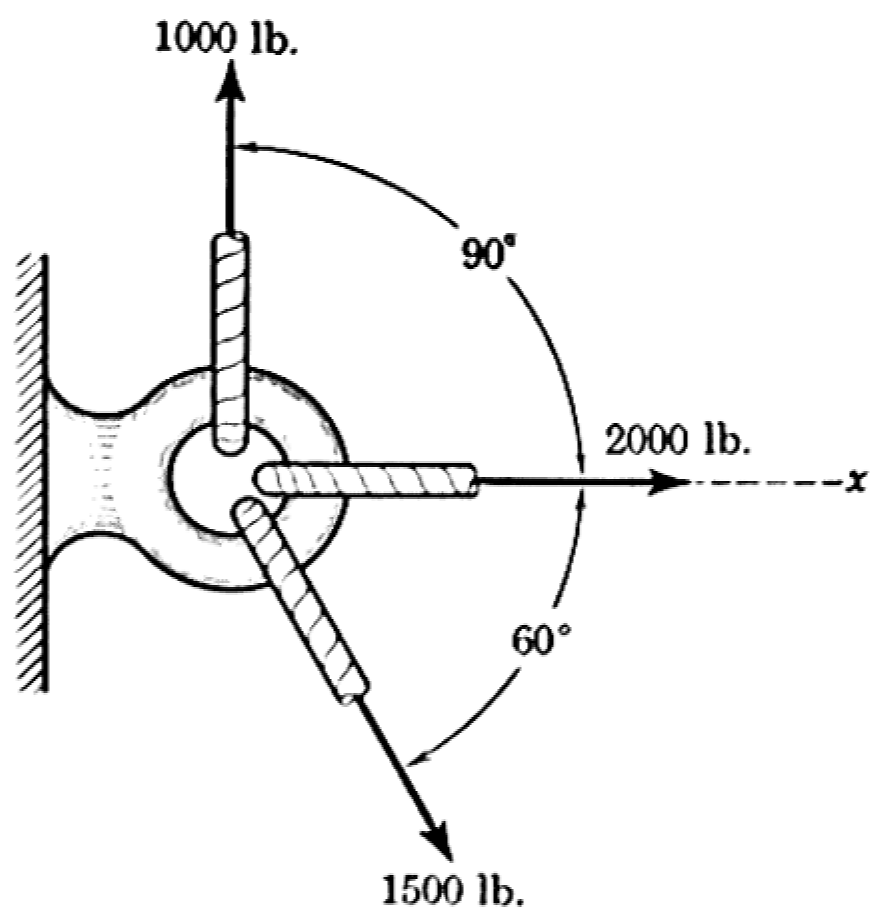
$$d = -\frac{46.0}{67.9} = -0.678 \text{ in.} \qquad \textit{Ans.}$$

The negative sign indicates that the moment of the sum is counterclockwise since the clockwise direction was taken as positive. Hence the resultant R may be applied at any point on a line making an angle of 59° 50′ with the x-axis and tangent to a circle of radius 0.678 in. about O as shown in the figure to the right of the drawing. The counterclockwise moment requires that the line of action of R be tangent at point A and not at point B as would have been the case if the moment had been positive in a clockwise sense.

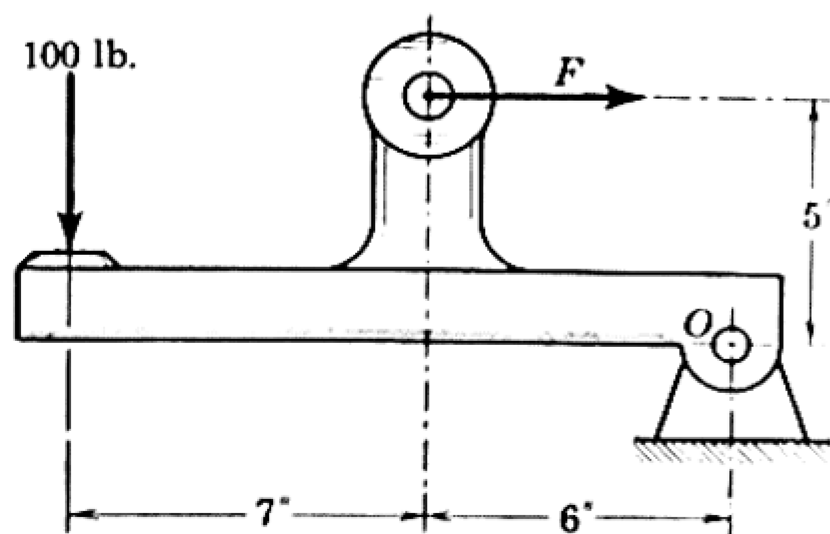
Direct combination of the forces by the parallelogram law will yield the same result as just obtained, and this may be verified graphically.

PROBLEMS

56. The fixed eye supports the tensions of three cables, indicated by the force vectors shown. If these three cables are removed and their combined effect on the eye is achieved by a single cable, determine its tension T and direction θ_x.
Ans. $T = 2770$ lb., $\theta_x = 6° 15'$ clockwise



Prob. 56

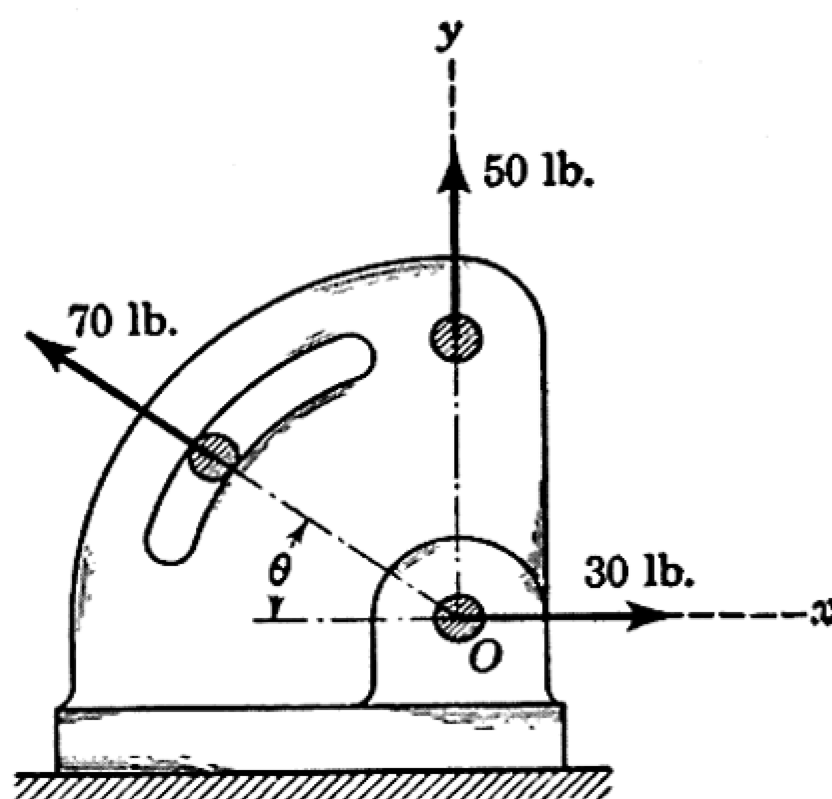


Prob. 57

57. Find graphically the magnitude of F such that the resultant of the two forces passes through the hinge at O.

58. Solve Prob. 57 algebraically.

59. Determine the resultant R of the three forces acting on the bracket if $\theta = 20$ deg.



PROB. 59

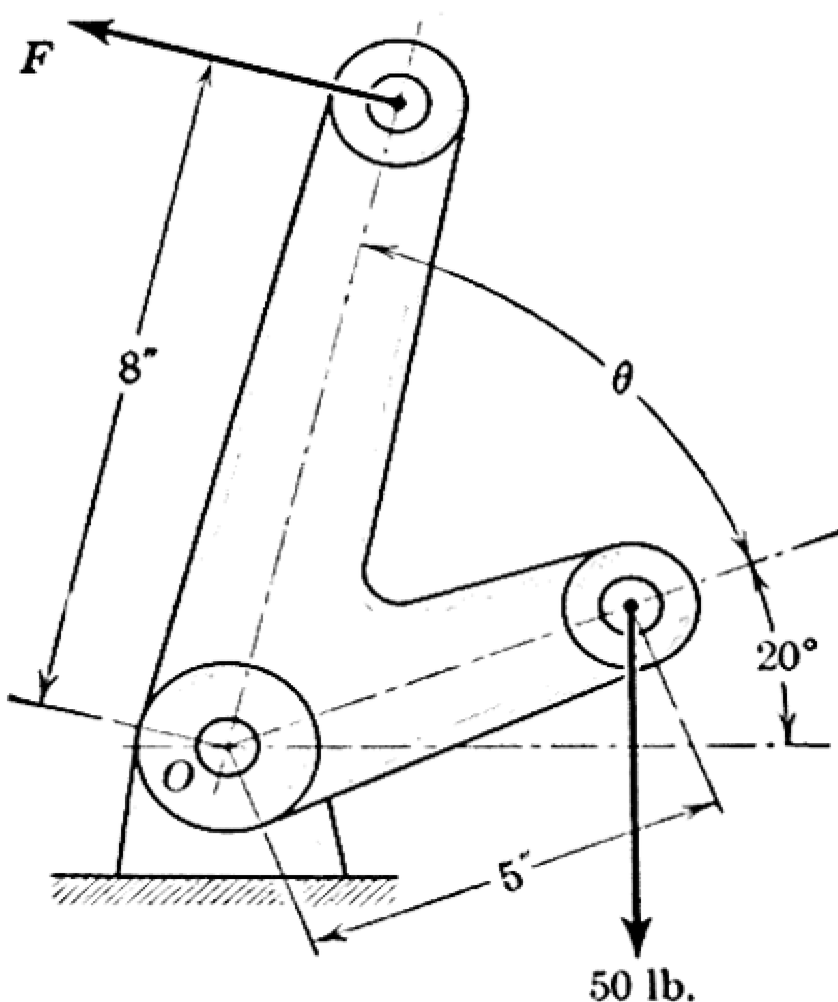
60. For the figure of Prob. 59 determine the angle θ so that the resultant of the three forces has no x-component.

61. If an additional cable is applied to the fixed eye in Prob. 56 such that the new resultant of the four forces is 3500 lb. in the x-direction, determine the cable tension T and its angle θ_x with the x-axis.

Ans. $T = 807$ lb., $\theta_x = 21° 45'$ counterclockwise

62. For the bell crank shown the resultant of F and the 50 lb. force passes through O. Compute F. Does the result depend upon θ?

Ans. $F = 29.4$ lb.

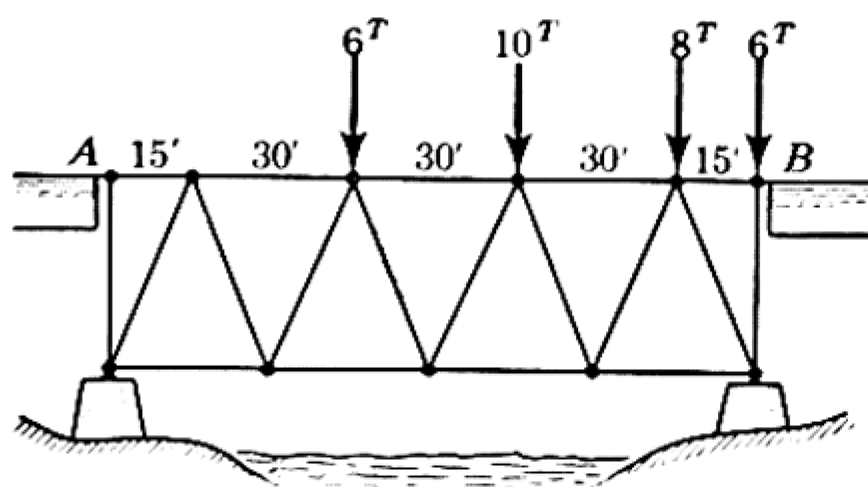


PROB. 62

63. By selecting two arbitrary values for θ in Prob. 62 find F graphically and show that F does not depend on θ.

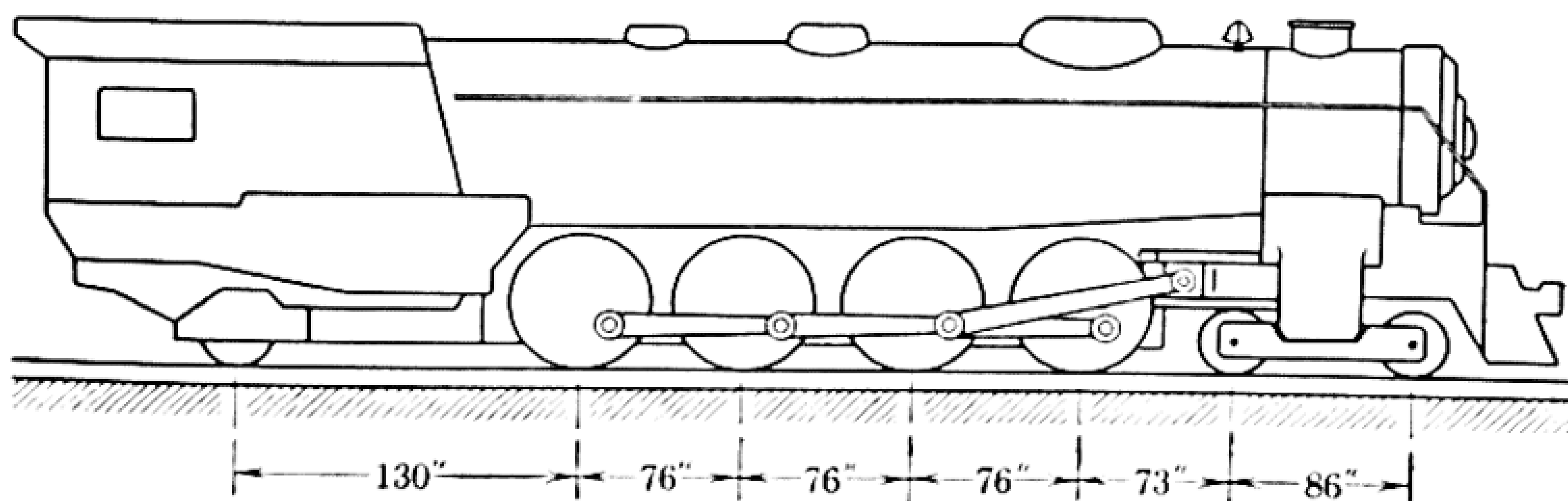
64. Determine the resultant R of the four loads acting on the bridge.

Ans. R = 30 tons 34 ft. to left of B



Prob. 64

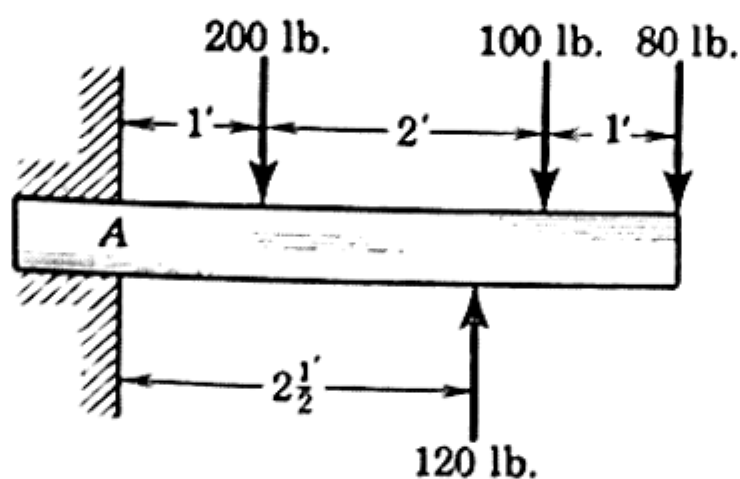
65. In the figure is shown a New York Central Mohawk type 4-8-2 locomotive weighing 401,000 lb. The front truck supports 74,900 lb. of the weight, which is evenly distributed over the four wheels. The eight driving wheels support 266,000 lb. of the weight, taken equally by each wheel. The trailing truck of two wheels supports the remainder. Locate the position of the resultant of the forces exerted by the wheels on each rail.



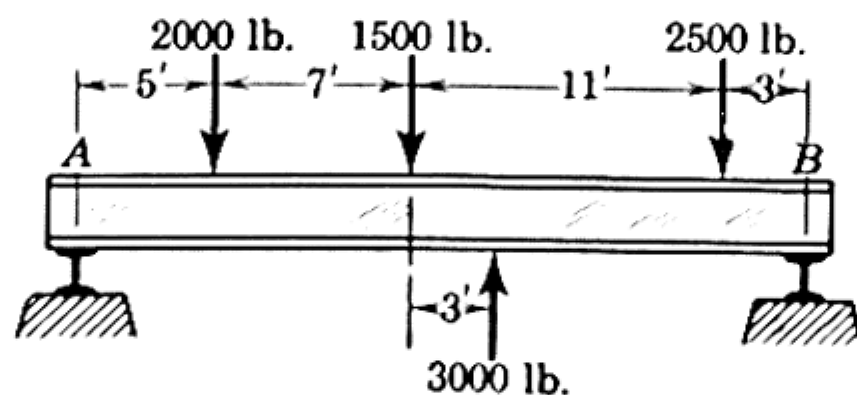
Prob. 65

66. Replace the four loads which act on the cantilever beam shown by a single force R which will leave the reactions at the support unaltered.

Ans. R = 260 lb. 2 ft. to right of A



Prob. 66

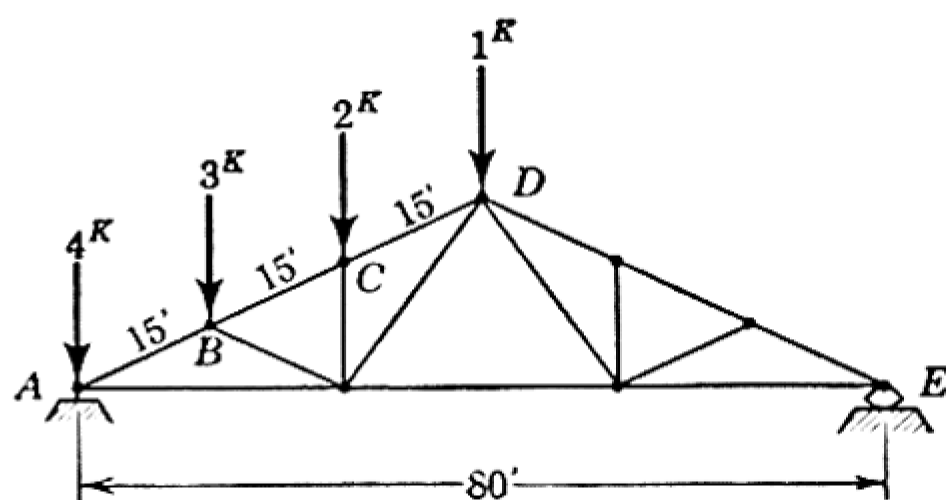


Prob. 67

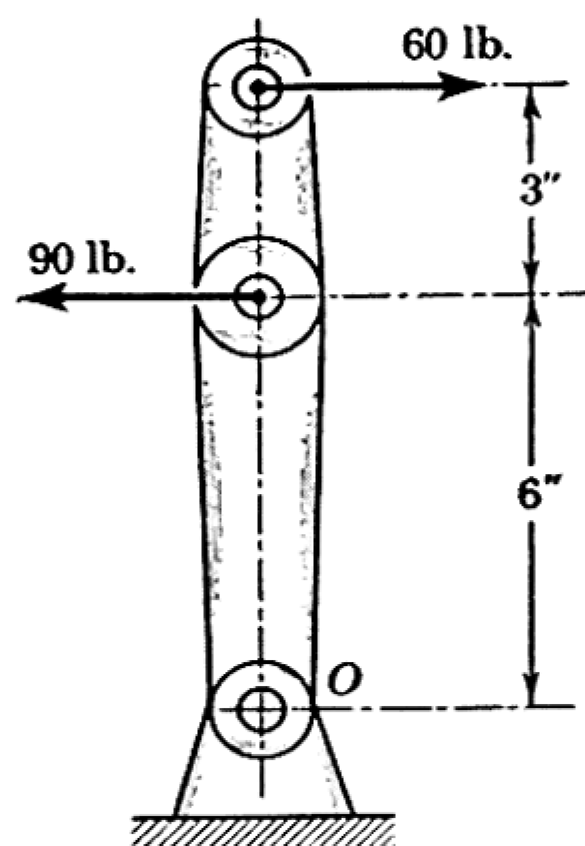
67. Determine the resultant R of the four forces acting on the beam as shown.

68. A roof truss is loaded as shown. Determine the resultant R of this system of forces. (The loads are expressed in 1000 lb. units or kips.)

Ans. $R = 10$ kips through B



PROB. 68



PROB. 69

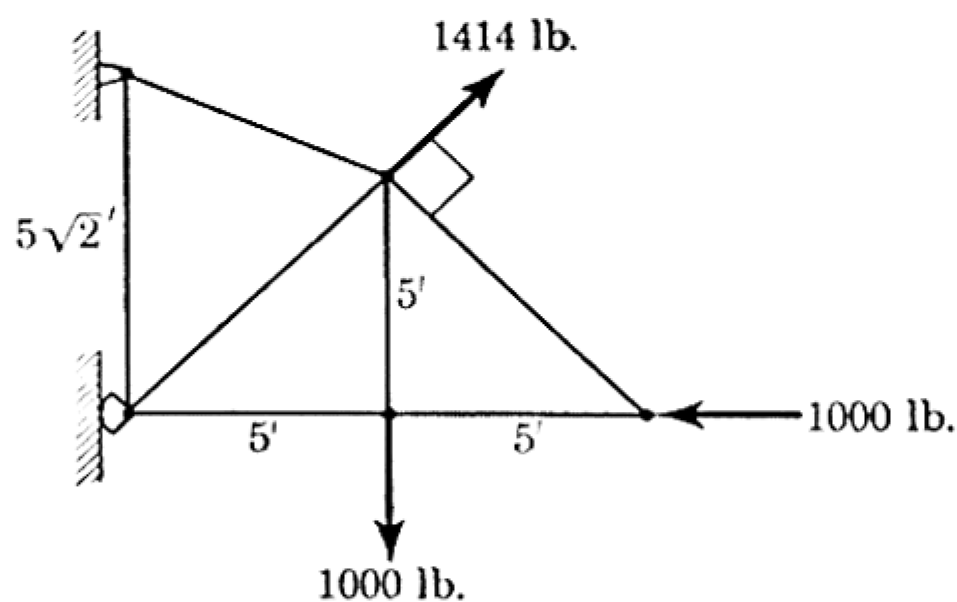
69. Replace the two parallel forces acting on the control lever by a single equivalent force R. *Ans.* $R = 30$ lb. to left through O

70. Solve Prob. 69 graphically.

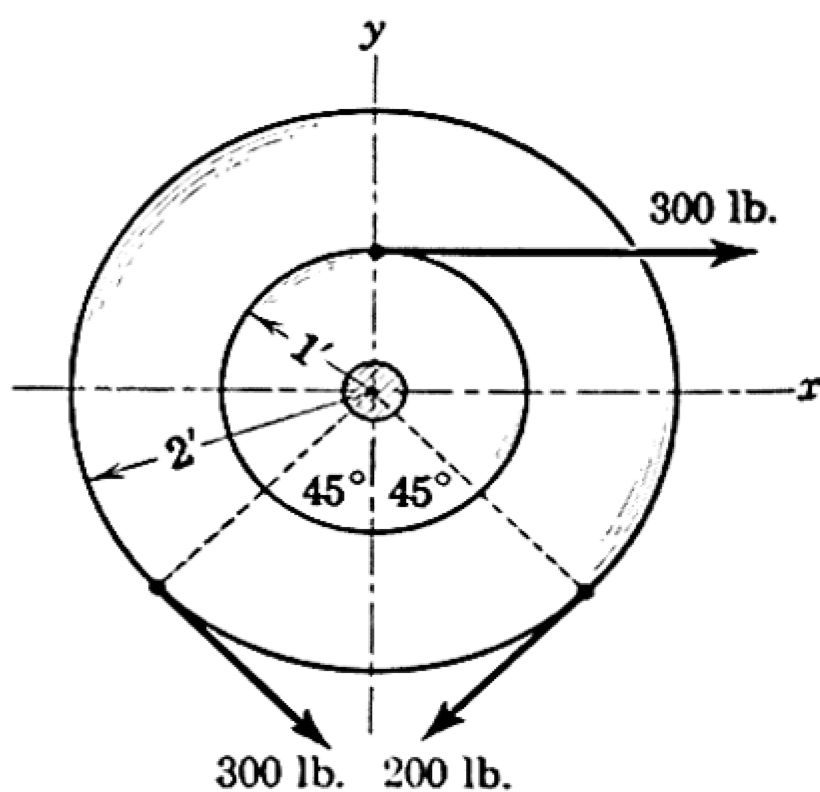
71. Replace the 90 lb. force on the lever of Prob. 69 by a force F acting horizontally to the left such that the resultant R of F and the 60 lb. force passes through a point 1 in. above O. Find F and R.

72. Determine the resultant of the three forces acting on the truss.

Ans. $M = 5000$ lb. ft., clockwise couple



PROB. 72

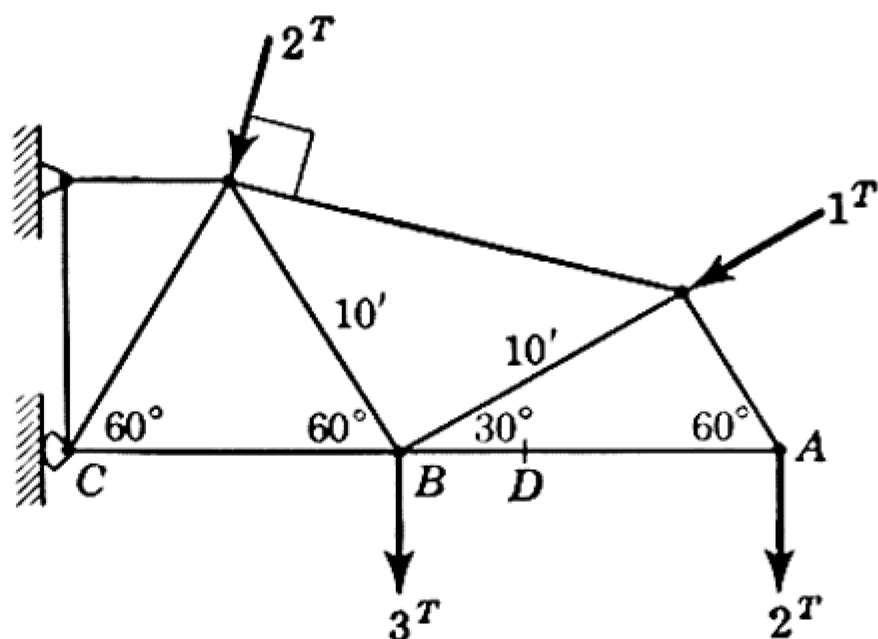


PROB. 73

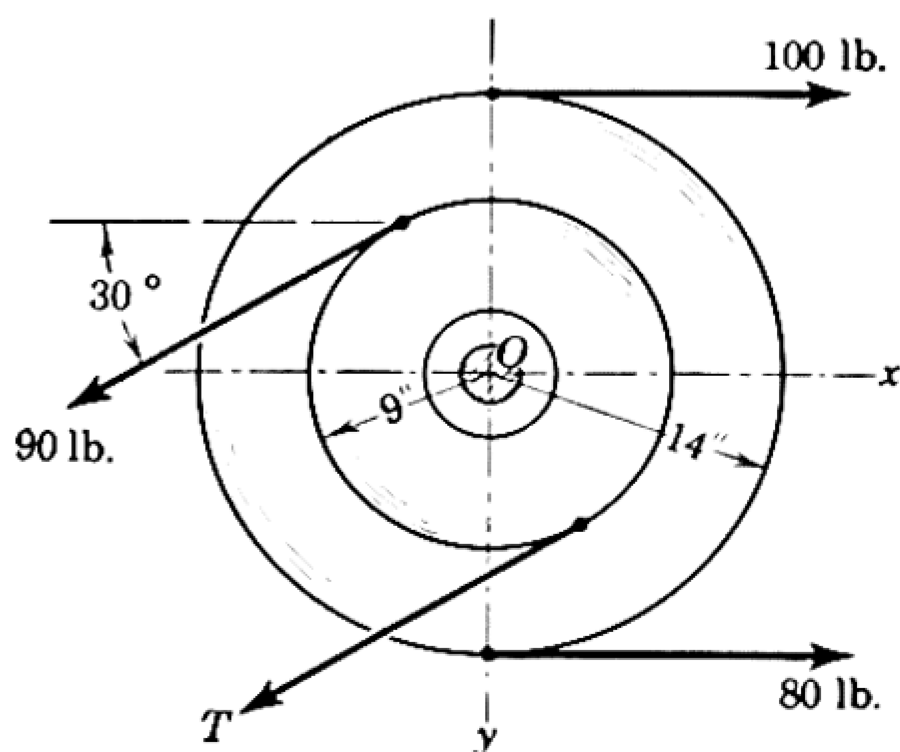
73. Determine graphically the resultant R of the three forces acting on the pulley.

74. Determine the resultant R of the four forces acting on the cantilever truss and locate the point D on line AB through which it must pass.

Ans. $R = 7.56$ tons through D 1.20 ft. to right of B



Prob. 74



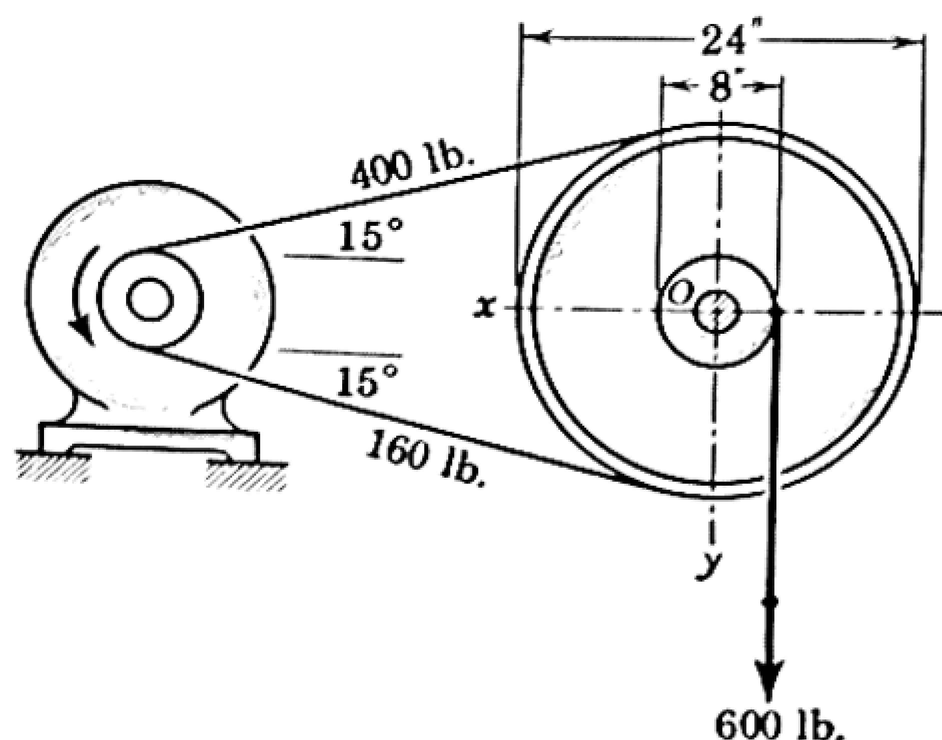
Prob. 76

75. Solve Prob. 74 graphically.

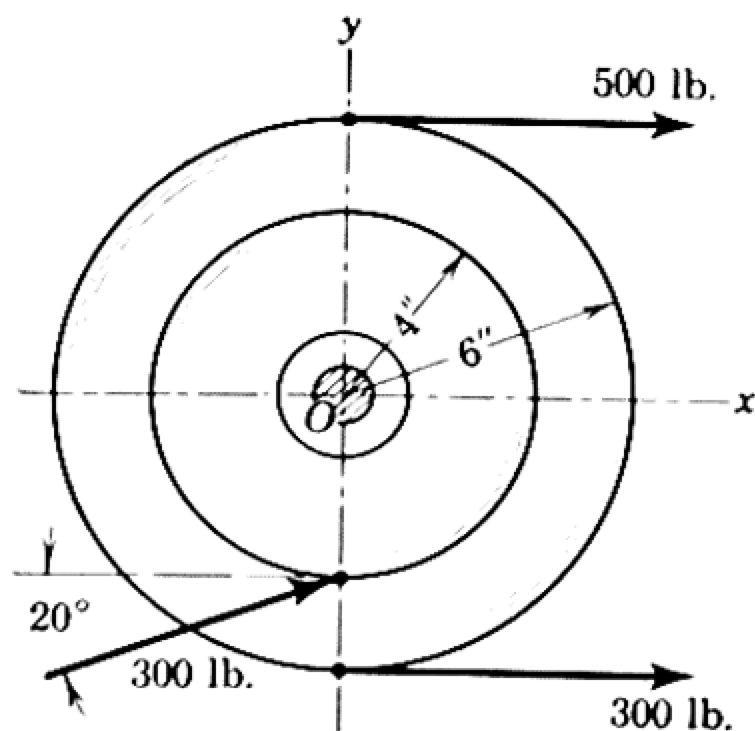
76. The two integral pulleys are subjected to the belt tensions shown. Determine the value of T if the resultant R of the four forces passes through O. Find the magnitude of R.

Ans. $T = 58.9$ lb., $R = 90.3$ lb.

77. Determine algebraically the resultant R of the two belt tensions and the 600 lb. load acting on the pulley as shown.



Prob. 77



Prob. 79

78. Solve Prob. 77 graphically.

79. Determine algebraically the magnitude, direction, and line of action of the resultant R of the forces acting on the pulley. If the resultant does not pass through O, what is happening to the pulley?

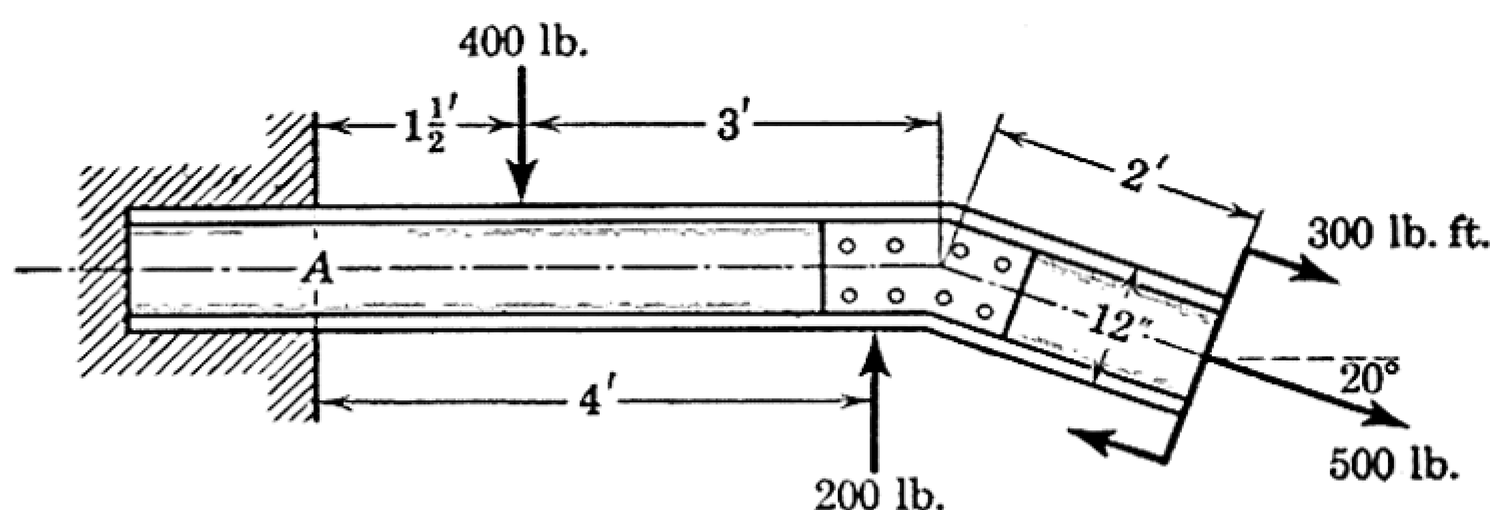
Ans. $R = 1088$ lb., $\theta_x = 5° 25'$ counterclockwise, 0.066 in. from O, clockwise moment. Pulley accelerates clockwise.

80. Approximate the answer to Prob. 79 graphically.

81. Solve Prob. 55 graphically.

* **82.** Find the magnitude of the resultant R of the forces acting on the cantilever support and the point on the beam center line through which it acts.

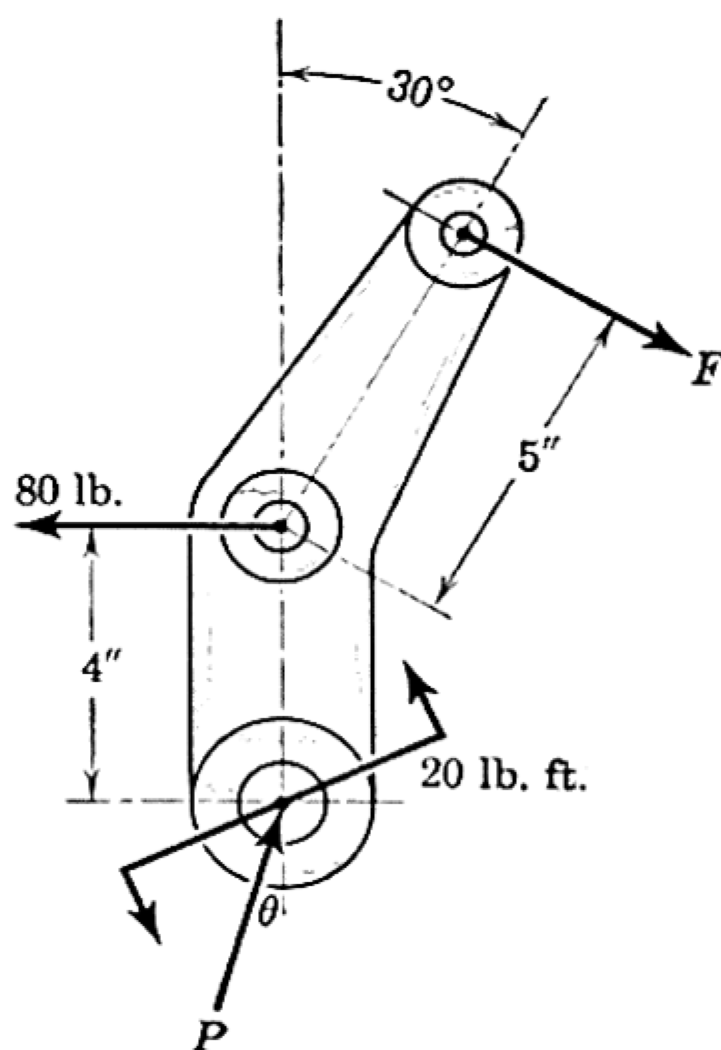
Ans. $R = 599$ lb. through a point 2.34 ft. to right of A



PROB. 82

* **83.** Determine the values of F, P, and θ such that the resultant of the three forces and couple shown is zero.

Ans. $F = 66.2$ lb., $P = 40.1$ lb., $\theta = 34° 25'$



PROB. 83

18. Funicular Polygon. A graphical procedure of wide application will now be developed for obtaining the resultants of coplanar force systems. The method is used primarily for dealing with parallel forces but will be demonstrated for a more general nonparallel coplanar system. The three forces F_1, F_2, and F_3 acting on the body in Fig. 22*a* constitute an example of such a system. A procedure using what is known as *Bow's notation* is convenient for labeling the various forces. The space between

each two forces is labeled by a lower-case letter so the line of action of each force is designated by two letters, such as *ab* for force F_1. The ends of the vectors which represent the forces are labeled with the corresponding capital letters. Thus force F_1 is labeled AB.

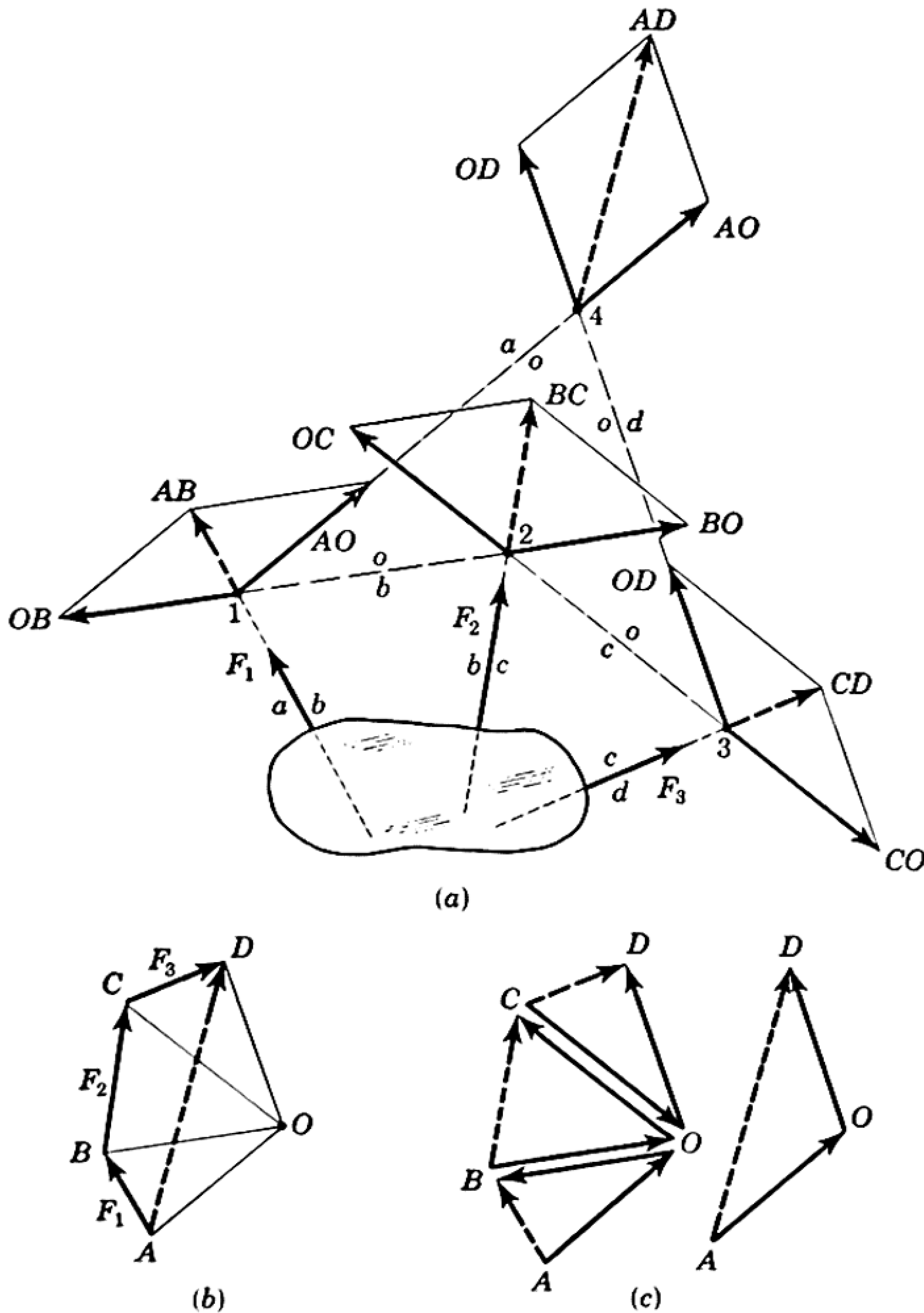


Fig. 22

The construction is begun with the force polygon $ABCD$, Fig. 22*b*, obtained by adding the forces head to tail. The magnitude and direction of their resultant are given by the vector AD, but its line of action remains to be determined. Next an arbitrary point O called the *pole* is chosen either inside or outside the polygon and is joined with points A, B, C, and D by straight lines called *rays*. These rays are components of

the given forces, as can be seen from the *c*-part of the illustration, where the figure is separated into its component triangles. It is seen that the force *AB* may be replaced by its components *AO* and *OB*. Likewise force *BC* may be replaced by its components *BO* and *OC*, and similarly for *CD*. Since forces *BO* and *OB* will cancel if they are applied collinearly and *CO* and *OC* will also cancel if collinear, the original three forces may be replaced by the two forces *AO* and *OD* whose sum is the resultant *AD*. It should be carefully observed that the sense of each force is designated by the sequence of the two letters which represent it.

These relations from the force polygon may now be used to obtain the correct line of action of the resultant. By starting at any convenient location, such as point 1 on the line of action of F_1, Fig. 22*a*, two lines *ao* and *bo* having the directions of rays *AO* and *OB* from Fig. 22*b* are constructed. At this point the forces *AO* and *OB* will replace *AB*. At the intersection of *bo* with the line of action of F_2 the line *co* is drawn parallel to ray *CO* of the force polygon. The components *BO* and *OC* applied at this point replace the force *BC*. The forces *OB* and *BO* cancel since they are collinear. Extending line *co* yields a point of intersection with force F_3. The line *do* is drawn through point 3 parallel to *DO* in the force polygon, and the components *CO* and *OD* applied at this point replace *CD*. Forces *OC* and *CO* cancel. The only forces left are *AO* and *OD*, which have now replaced the original system. The forces *AO* and *OD* may be combined at their point of intersection to give the resultant *AD*, which has the correct line of action.

The parallelograms in Fig. 22*a* were drawn to illustrate the force combinations but are superfluous once the method is understood. The points 1, 2, and 3 and the directions of the lines *ao* and *od* are sufficient to determine point 4 which is a desired point on the line of action of the resultant *AD*. With the magnitude and direction of the resultant known from the force polygon the correct line of action is established by drawing this vector through point 4.

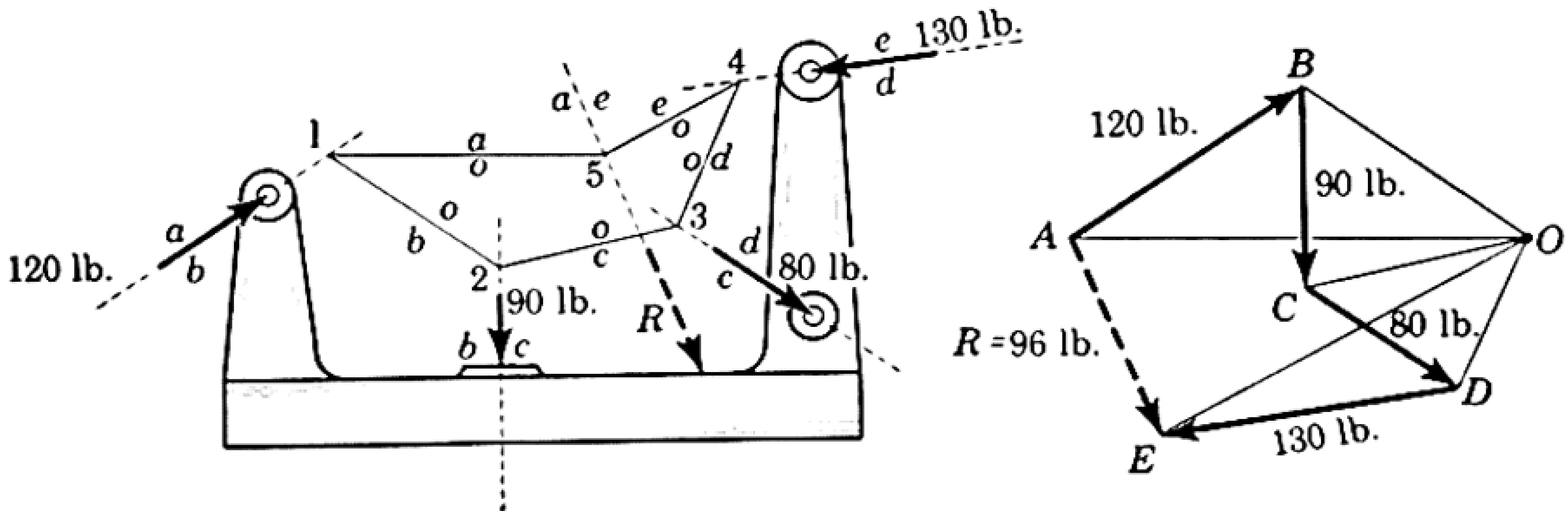
The lines *ao*, *bo*, *co*, and *do* are known as *strings*, and the polygon 1–2–3–4 is called a *string* or *funicular polygon*. The order in which the forces are labeled and added will not influence the result, but it is customary to use a clockwise or left-to-right alphabetical order as was done in this example. Different choices for initial starting points for the string polygon will result in polygons of differing size but all yielding a point on the same line of action of the resultant. For a system of parallel forces the force polygon becomes a straight line, and the string polygon will be the same size no matter where it is begun.

When the resultant of a coplanar force system is a couple, the polygon of the given forces will close, and the two strings whose intersection nor-

mally would give a point on the line of action of the resultant will be parallel. The magnitude of the couple will equal the value of the coincident rays on the force polygon multiplied by the perpendicular distance between their corresponding parallel strings in the string polygon.

SAMPLE PROBLEMS

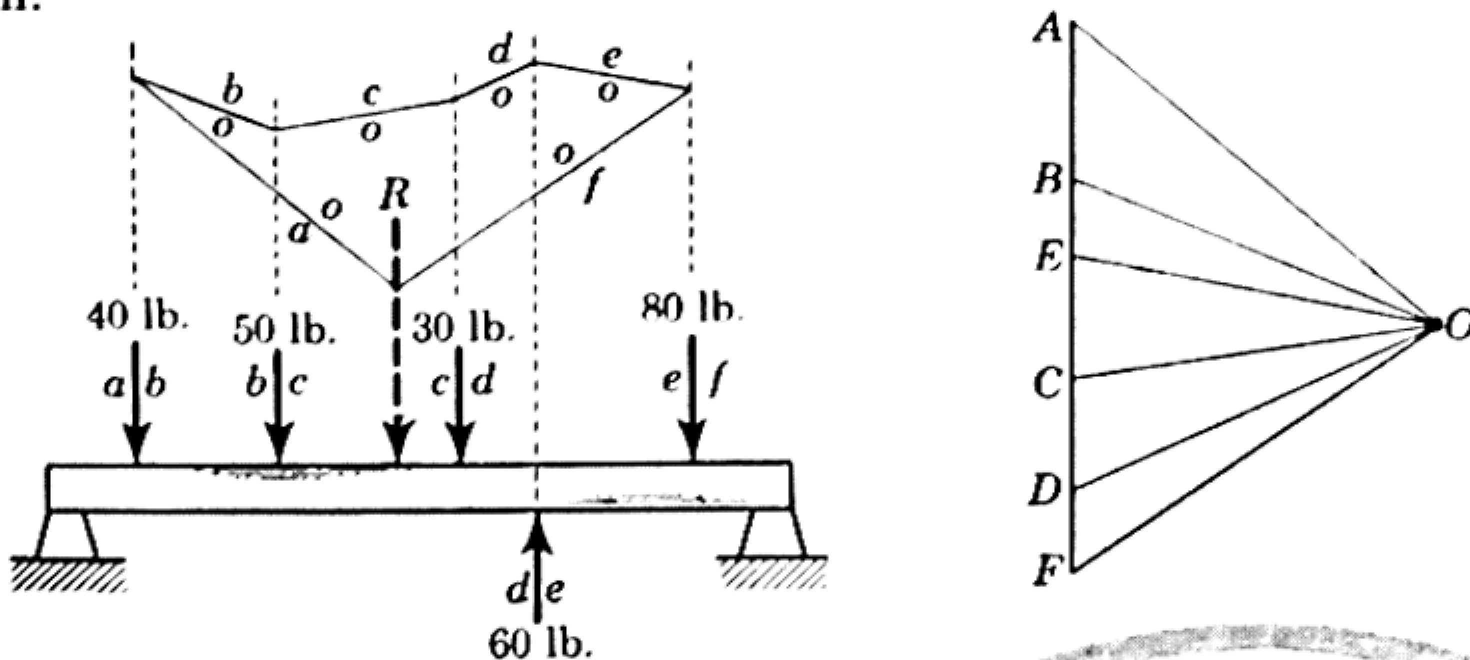
84. Determine by means of a force and string polygon the resultant of the four forces shown acting on the bracket.



PROB. 84

Solution: The four forces are first labeled with Bow's notation, starting with the 120 lb. force. Next the force polygon $ABCDE$ is constructed and an arbitrary selection of the pole O is made. The string polygon is begun at point 1, which is any convenient location on *ab*. Through point 1 the strings *ao* and *bo* are drawn parallel to the rays AO and BO in the force polygon. At point 2, the intersection of *bo* with *bc*, the next string *co* is drawn. Point 3 is now established at the intersection of *co* with *cd*. Point 4 is similarly obtained and the last string *eo* is drawn through this point. The intersection of *eo* and *ao* locates point 5, a point on the line of action of the resultant R which is labeled AE on the force polygon. The resultant, whose magnitude of 96 lb. and direction are determined from the force polygon, is now constructed through point 5 as indicated.

85. Determine the resultant R of the five vertical forces acting on the beam as shown.



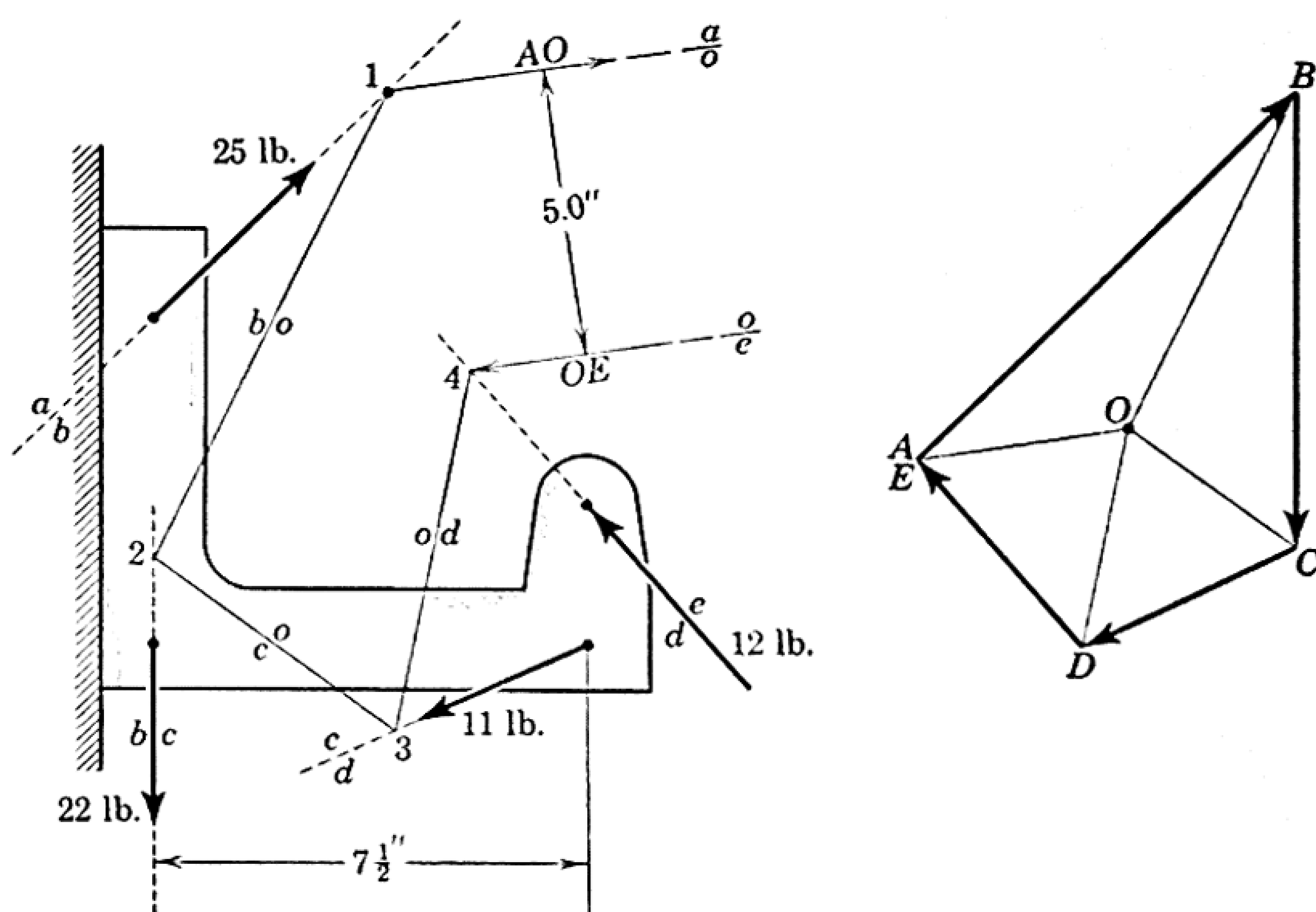
PROB. 85



Solution: The given forces are labeled with Bow's notation, beginning at the left. They are next added head-to-tail to produce the line $ABCDEF$ shown to the right of the beam. The arrowheads are omitted since the sense of the forces is given by the sequence of letters. The pole O is arbitrarily chosen, and the rays AO, BO, CO, etc., are drawn.

Beginning at any point on the line of action *ab*, the strings *ao* and *bo* are drawn parallel to the rays AO and BO in the force diagram. Where *bo* intersects the line of action *bc*, the string *co* is drawn, and so forth. Finally the intersection of the strings *fo* and *ao* locates a point on the line of action of the resultant R which is the algebraic sum of the given forces, or 140 lb. The position of R on the beam may be scaled from the drawing.

86. Determine the resultant of the four forces acting on the bracket shown.



PROB. 86

Solution: The forces are labeled in any convenient order, and the force polygon $ABCDE$ is then constructed. In this case it is seen that the polygon closes so that the resultant is not a force. The pole O is chosen at random and the rays are drawn. By starting at any position on the action line of force AB, such as point 1 in the figure, the strings are drawn parallel to their corresponding rays in the force polygon.

The string polygon 1–2–3–4 does not close in this case since the first string *ao* is parallel to the last string *eo*. The resultant of the given forces has been reduced to the two forces AO and OE, and, since the separation of their lines of action scales 5.0 in., the resultant of the given forces is a clockwise couple M of magnitude $AO \times 5.0$. From the force polygon AO scales 10.0 lb. so that

$$M = 10.0 \times 5.0 = 50.0 \text{ lb. in. clockwise.} \qquad \textit{Ans.}$$

PROBLEMS

Solve the following problems by means of a funicular polygon.

87. Prob. 64.
88. Prob. 65.
89. Prob. 66.
90. Prob. 67.
91. Prob. 68.
92. Prob. 69.
93. Prob. 72.
94. Prob. 73.
95. Prob. 74.

96. Prob. 76. (*Hint:* The string polygon should be started at point O, which is the only known point on the line of action of the resultant.)

97. In Prob. 74 replace the 3 ton load by a vertical force P of such a value as to make the resultant of the four forces pass through the midpoint of the member AB. Note the hint given in Prob. 96. *Ans.* $P = 2.88$ tons up

19. Resultants of Three-Dimensional Force Systems. There are many engineering problems which cannot be analyzed on the basis of a two-dimensional force system. Analysis in three dimensions requires representation of the system by a perspective drawing or by means of two or more orthographic projections. Facility to visualize and represent the third dimension is an absolute necessity for this type of analysis.

Consider first a system of concurrent forces acting on a body. An example of such a system is shown in Fig. 23a, where three forces not in the same plane act at O. The resultant R of this system may be obtained by adding the forces head to tail in space as shown in Fig. 23b. The resultant is the vector which closes the space polygon, and its three rectangular components are seen to be the algebraic sum of the corresponding components of the given forces. Thus, in general,

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y, \qquad R_z = \Sigma F_z,$$

and (10)

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}.$$

If the angles made by R with the x-, y-, and z-axes are θ_x, θ_y, and θ_z, respectively, then the direction of R is specified by its direction cosines

$$\cos \theta_x = \frac{\Sigma F_x}{R}, \qquad \cos \theta_y = \frac{\Sigma F_y}{R}, \qquad \cos \theta_z = \frac{\Sigma F_z}{R}. \qquad (11)$$

The space polygon in Fig. 23b may be projected onto the three coordinate planes, giving three related plane polygons as shown in Fig. 23c. Any two of these force polygons will involve all three of the first of Eqs.

(10), and hence two projections are sufficient for the full description of the force relations.

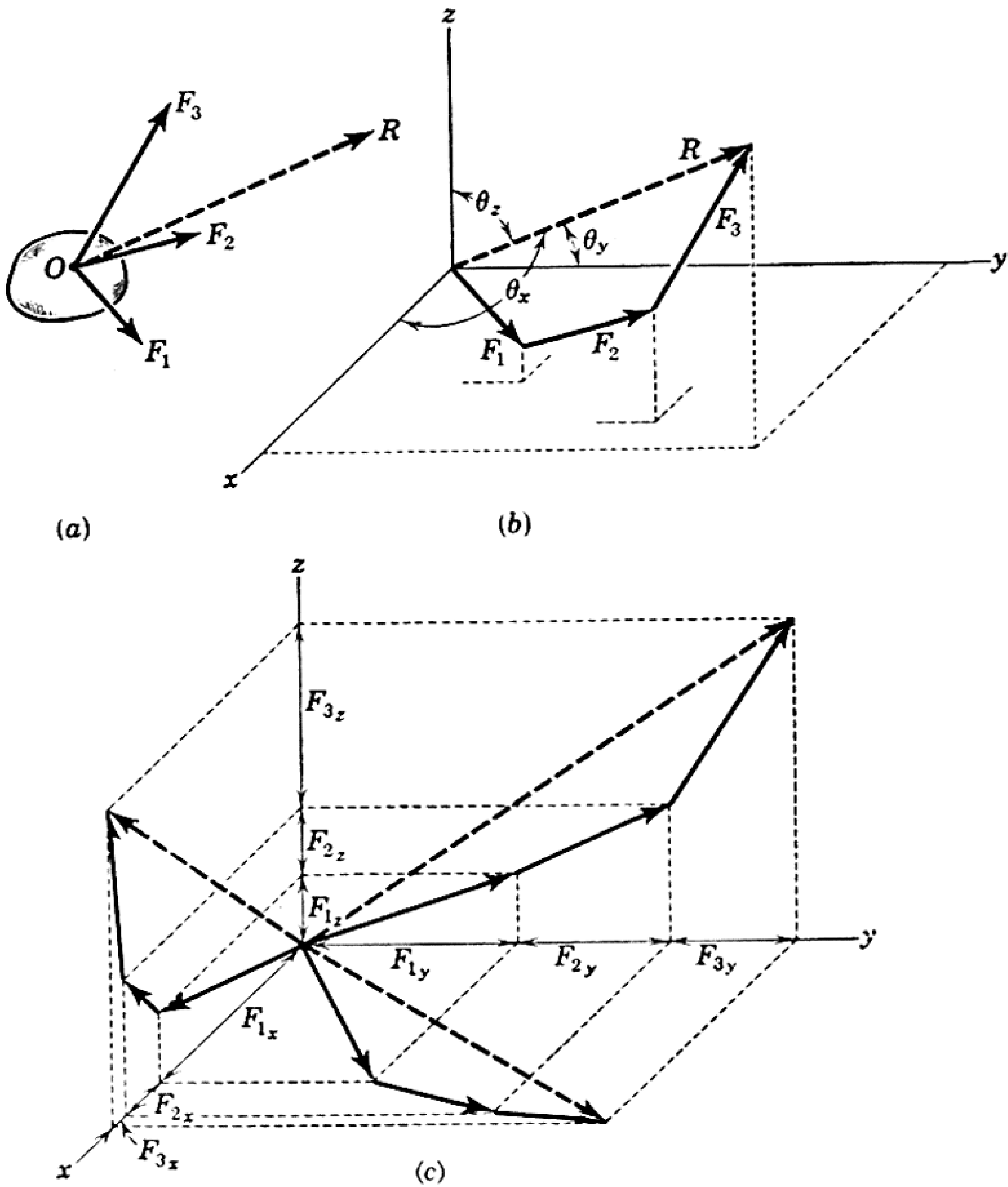


FIG. 23

Now consider a general system of nonconcurrent forces in space acting on a body. An example of such a system is represented in Fig. 24*a*, where three such forces are shown. By the principles of Art. 16 the external effect on the body will be unaltered by applying F_1 at some other point, such as O, and adding the necessary couple M_1. The magnitude of this couple is F_1d, where d is the perpendicular distance from O to the original line of action of F_1. This transfer for force F_1 is shown in Fig. 24*b*. The plane of the couple is the plane defined by the original line of action of F_1 and point O. The vector M_1 representing the couple is normal to this plane, and the sense is determined by the right-hand rule. The couple M_1 is a free vector but is shown at point O for convenience.

The remaining two forces are transferred to point O with the addition of their two corresponding couples as shown in Fig. 24c. The original force system has now been reduced to a system of concurrent forces at the arbitrary point O and a corresponding number of couples. The forces may be combined to get their resultant R acting at point O, and the couples are combined vectorially to get a resultant couple M as shown in

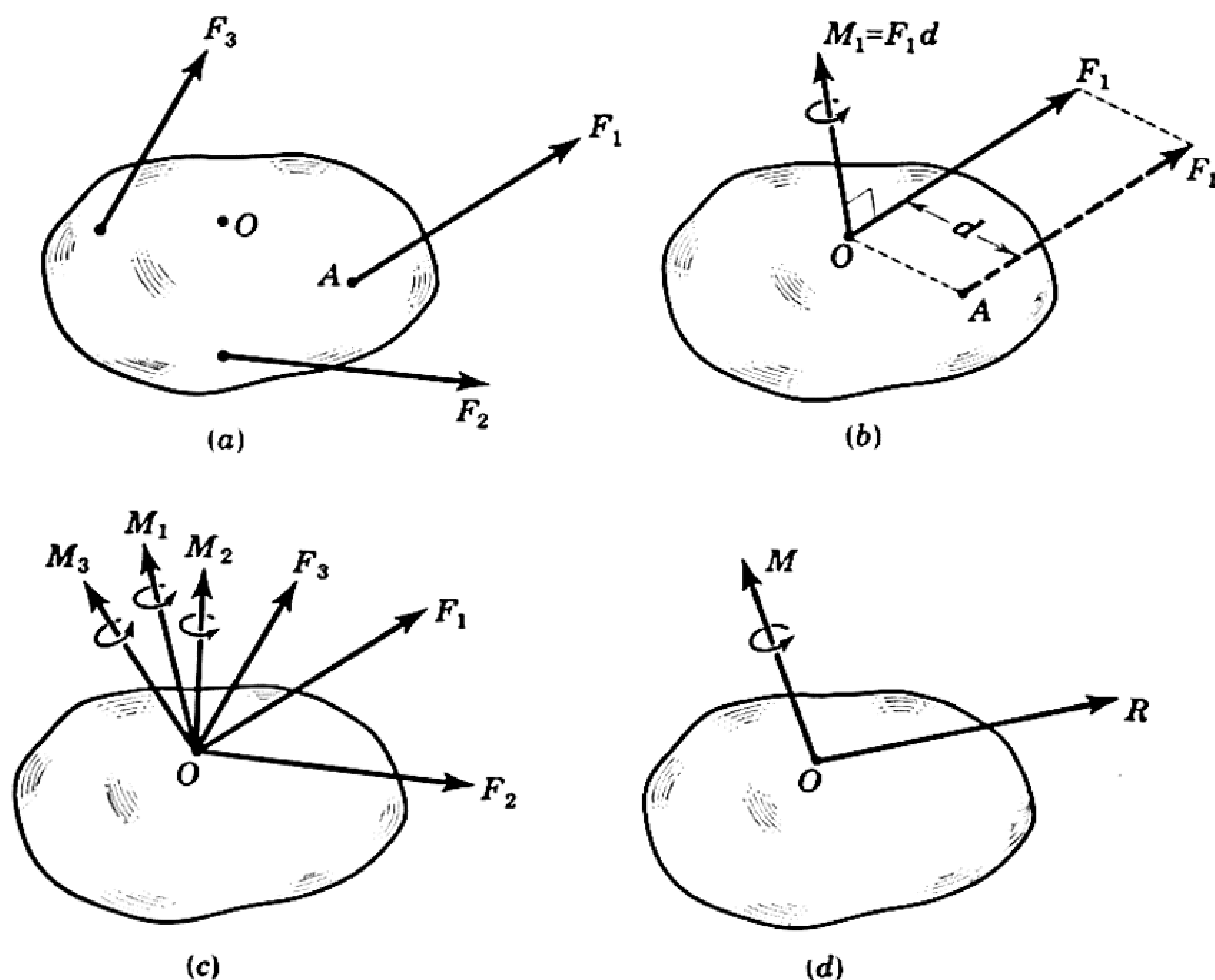


FIG. 24

Fig. 24*d*. For a given force system the couple M will vary in magnitude and direction, depending on the choice of the reference point O, although the resultant force R remains fixed in magnitude and direction regardless of the reference point.

It has now been shown that *the resultant of any general force system can be expressed in terms of a single resultant force and a single resultant couple.* Physically the effects may be described as a push (or pull) R and a twist M. This result has a very important bearing on both the statics problem and the dynamics problem. In statics, where the equilibrium condition obtains, both R and M are zero. In dynamics the linear and rotational accelerations will be dependent on R and M, respectively.

There are several special cases of general force system resultants which occur frequently. When there is no resultant force, then the given sys-

tem reduces to a couple or pure twist. When the couple vector M is perpendicular to the resultant force R, Fig. 25*a*, the two may be combined to form a single force R whose line of action is a distance $d = M/R$ from point O as shown in Fig. 25*b*. The new line of action of R must be on the side of O which makes the direction of the moment of R about O the same

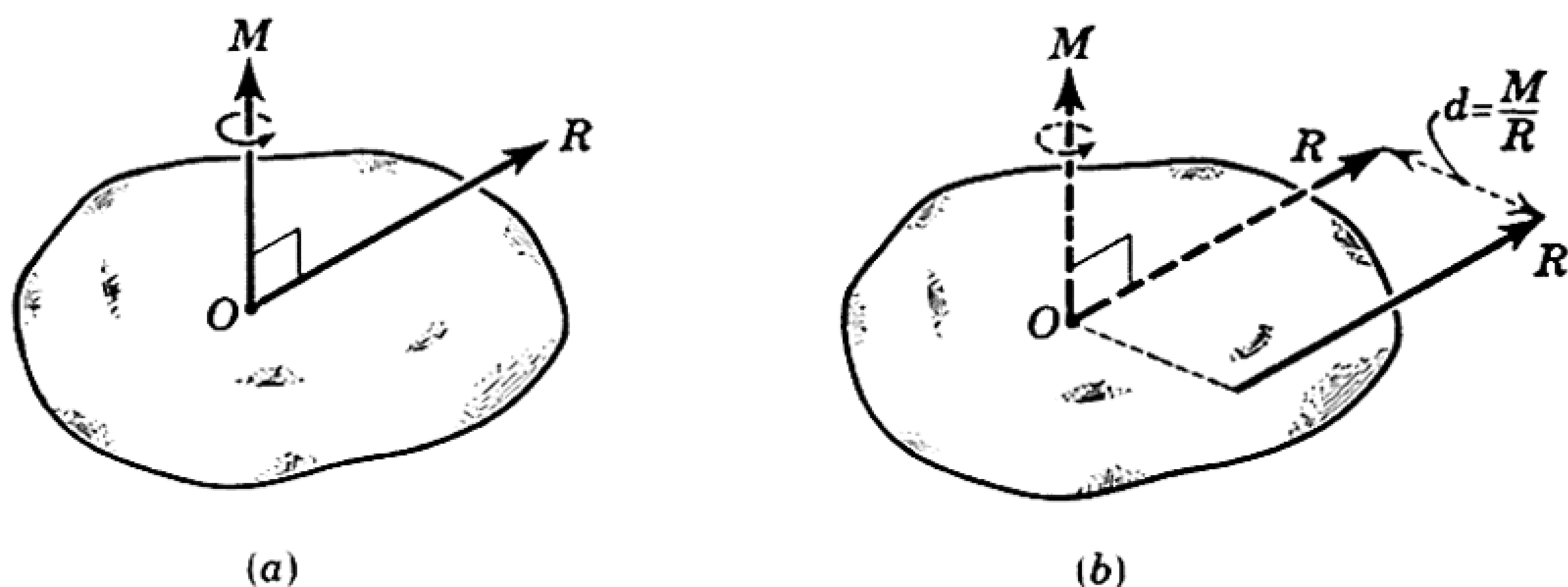


FIG. 25

as that of M. When the couple M is parallel to R, Fig. 26, the combination is known as a *wrench* or *screw*. The action may be described as a push (or pull) and a twist about an axis parallel to the push (or pull). When the force and moment vectors have the same sense, as in Fig. 26*a*, the wrench is said to be positive. When the vectors have the opposite sense, the wrench is negative, as indicated in Fig. 26*b*.

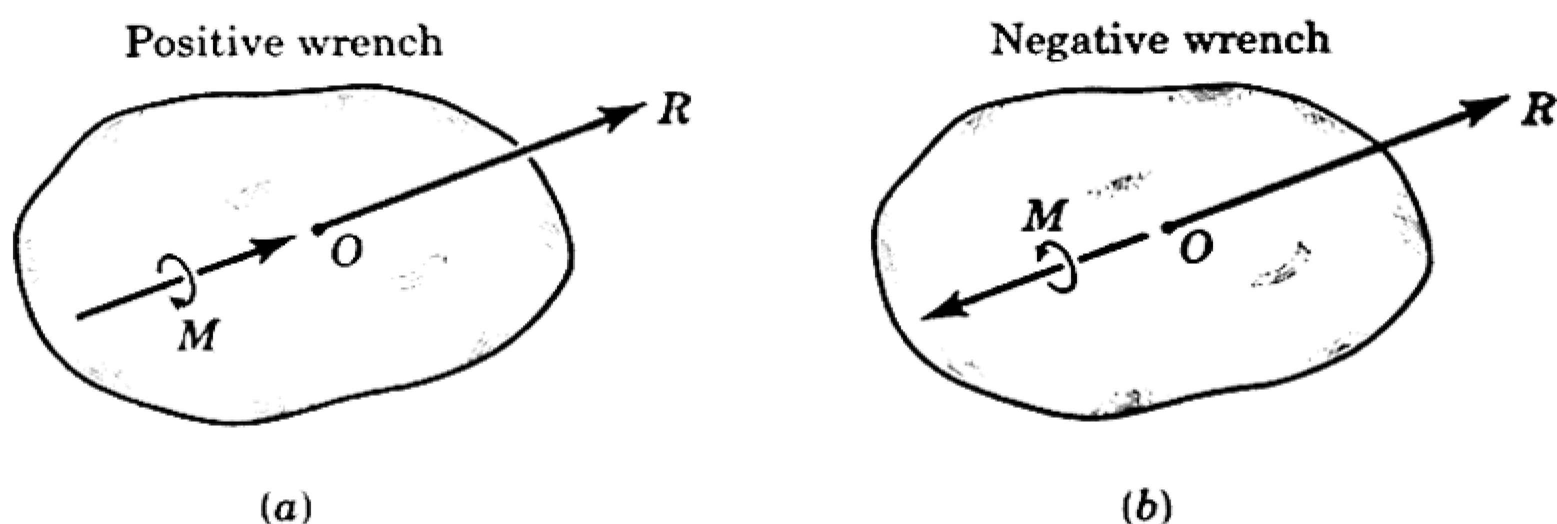


FIG. 26

If the given forces are all parallel, the resultant R, if a force, is the algebraic sum of the several forces, and its position may be determined by the principle of moments. Thus for the four parallel forces shown acting on the plate of Fig. 27 the resultant is

$$R = F_1 + F_2 + F_3 - F_4 = \Sigma F.$$

By the principle of moments the line of action of R is located by equating the moment of the sum to the sum of the moments about the y- and x-

axes. Thus

$$R\bar{x} = F_1x_1 + F_2x_2 + F_3x_3 - F_4x_4,$$

$$R\bar{y} = F_1y_1 + F_2y_2 + F_3y_3 - F_4y_4,$$

or in general

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}, \qquad \bar{y} = \frac{\Sigma Fy}{\Sigma F}.$$

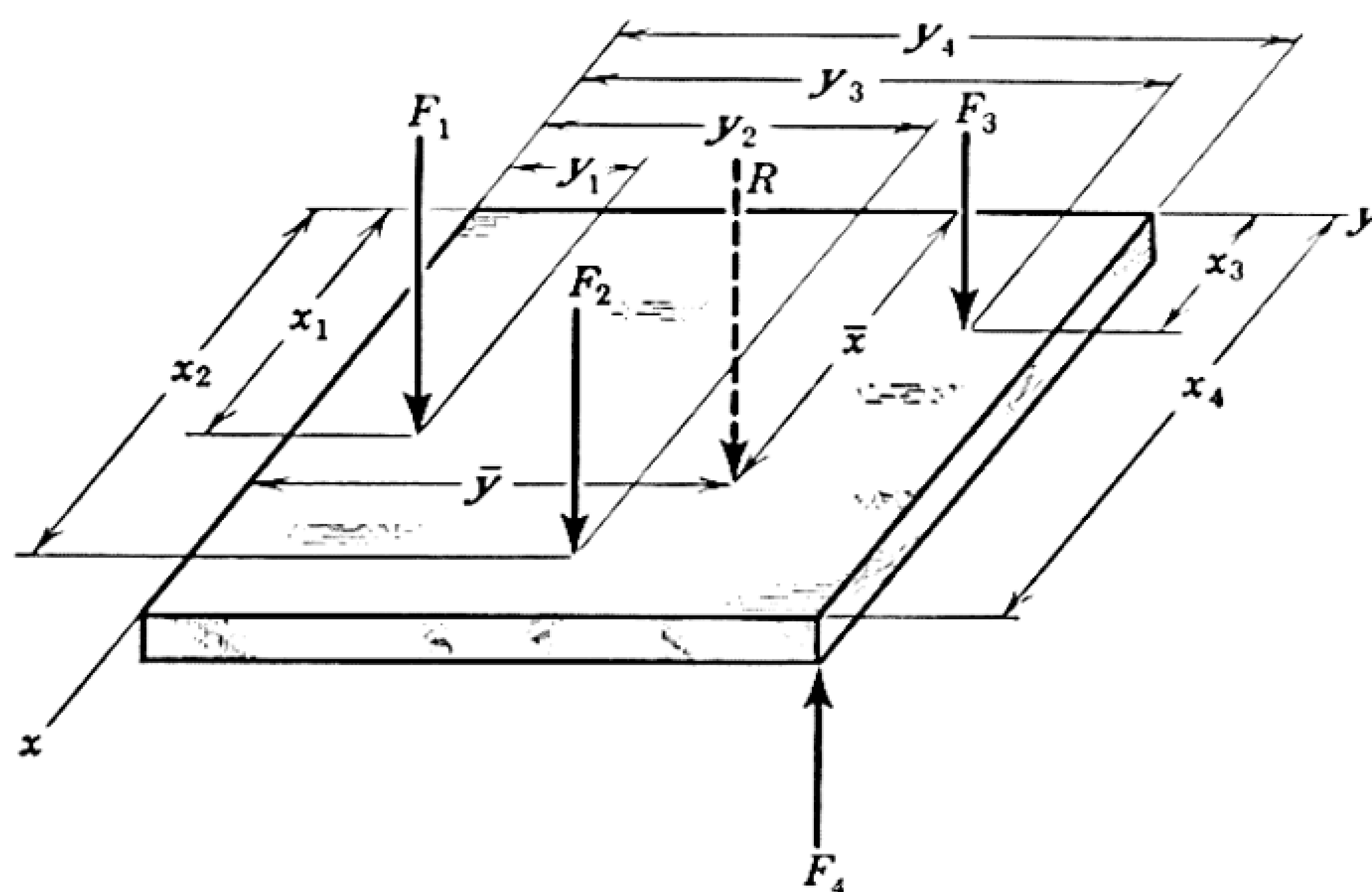
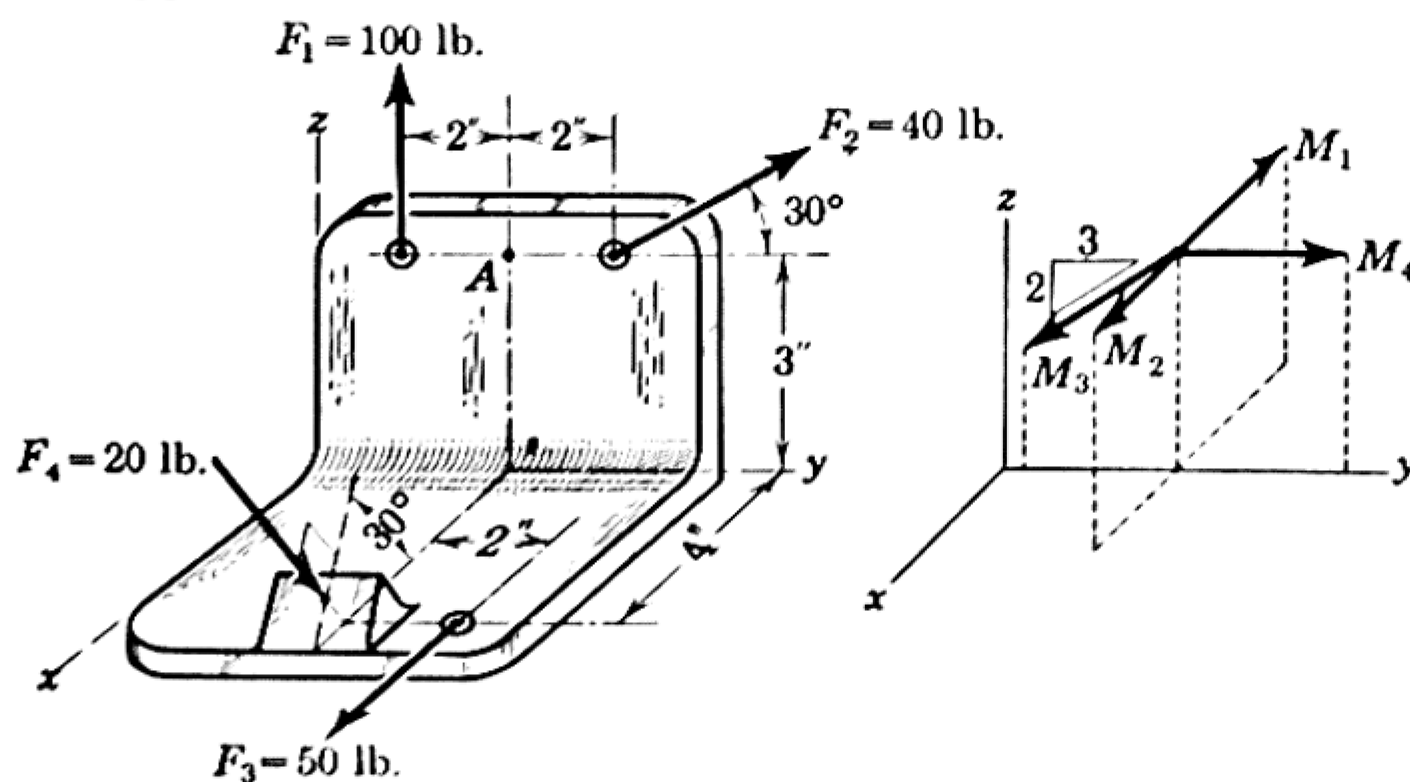


Fig. 27

In the event that a system of parallel forces reduces to two equal and opposite forces which are not collinear, the resultant is a couple whose vector lies in a plane perpendicular to the given forces.

SAMPLE PROBLEM

98. Replace the four forces acting on the bracket shown by a single equivalent force applied at point A and determine the corresponding couple which must also be applied.



Prob. 98

Solution: The three rectangular components of the resultant force are

$[R_x = \Sigma F_x]$ $\quad R_x = 50 - 20 \times 0.5 = 40$ lb.,

$[R_y = \Sigma F_y]$ $\quad R_y = 40 \times 0.866 = 34.6$ lb.,

$[R_z = \Sigma F_z]$ $\quad R_z = 100 + 40 \times 0.5 - 20 \times 0.866 = 102.7$ lb.

The resultant force is

$$\left[R = \sqrt{R_x^2 + R_y^2 + R_z^2}\,\right] \quad R = \sqrt{(40)^2 + (34.6)^2 + (102.7)^2} = 115.5 \text{ lb.} \quad \textit{Ans.}$$

The direction of R may be specified by its direction cosines:

$$\left[\cos\theta_x = \frac{R_x}{R}\right] \qquad \cos\theta_x = \frac{40}{115.5}, \qquad \theta_x = 69^\circ\, 45',$$

$$\left[\cos\theta_y = \frac{R_y}{R}\right] \qquad \cos\theta_y = \frac{34.6}{115.5}, \qquad \theta_y = 72^\circ\, 35',$$

$$\left[\cos\theta_z = \frac{R_z}{R}\right] \qquad \cos\theta_z = \frac{102.7}{115.5}, \qquad \theta_z = 27^\circ\, 15'. \qquad \textit{Ans.}$$

The couple due to the transfer of F_1 to point A is

$[M = Fd]$ $\quad M_1 = 100 \times 2 = 200$ lb. in. (minus x-direction).

Likewise the couples due to the transfer of F_2, F_3, and F_4 are

$$M_2 = 40 \times 2 \times 0.5 = 40 \text{ lb. in. (plus } x\text{-direction)},$$

$$M_3 = 50 \times \sqrt{2^2 + 3^2} = 180.3 \text{ lb. in. } (y\text{–}z \text{ plane}),$$

$$M_4 = 20 \times 4 \times 0.866 + 20 \times 3 \times 0.500 = 99.3 \text{ lb. in. (plus } y\text{-direction)}.$$

These four couple vectors are shown in the right-hand part of the figure. The three rectangular components of the resultant couple are seen from the figure to be

$$M_x = -200 + 40 = -160 \text{ lb. in.},$$

$$M_y = 99.3 - 180.3 \times \frac{3}{\sqrt{2^2 + 3^2}} = -50.7 \text{ lb. in.},$$

$$M_z = -180.3 \times \frac{2}{\sqrt{2^2 + 3^2}} = -100 \text{ lb. in.}$$

The resultant couple is

$$\left[M = \sqrt{M_x^2 + M_y^2 + M_z^2}\,\right] \qquad M = \sqrt{(160)^2 + (50.7)^2 + (100)^2}$$

$$= 195.3 \text{ lb. in.} \qquad \textit{Ans.}$$

The direction of the couple may be specified by its direction cosines, which are

$$\left[\cos\theta_x = \frac{M_x}{M}\right] \qquad \cos\theta_x = \frac{-160}{195.3}, \qquad \theta_x = 145° \, 0',$$

$$\left[\cos\theta_y = \frac{M_y}{M}\right] \qquad \cos\theta_y = \frac{-50.7}{195.3}, \qquad \theta_y = 105° \, 05',$$

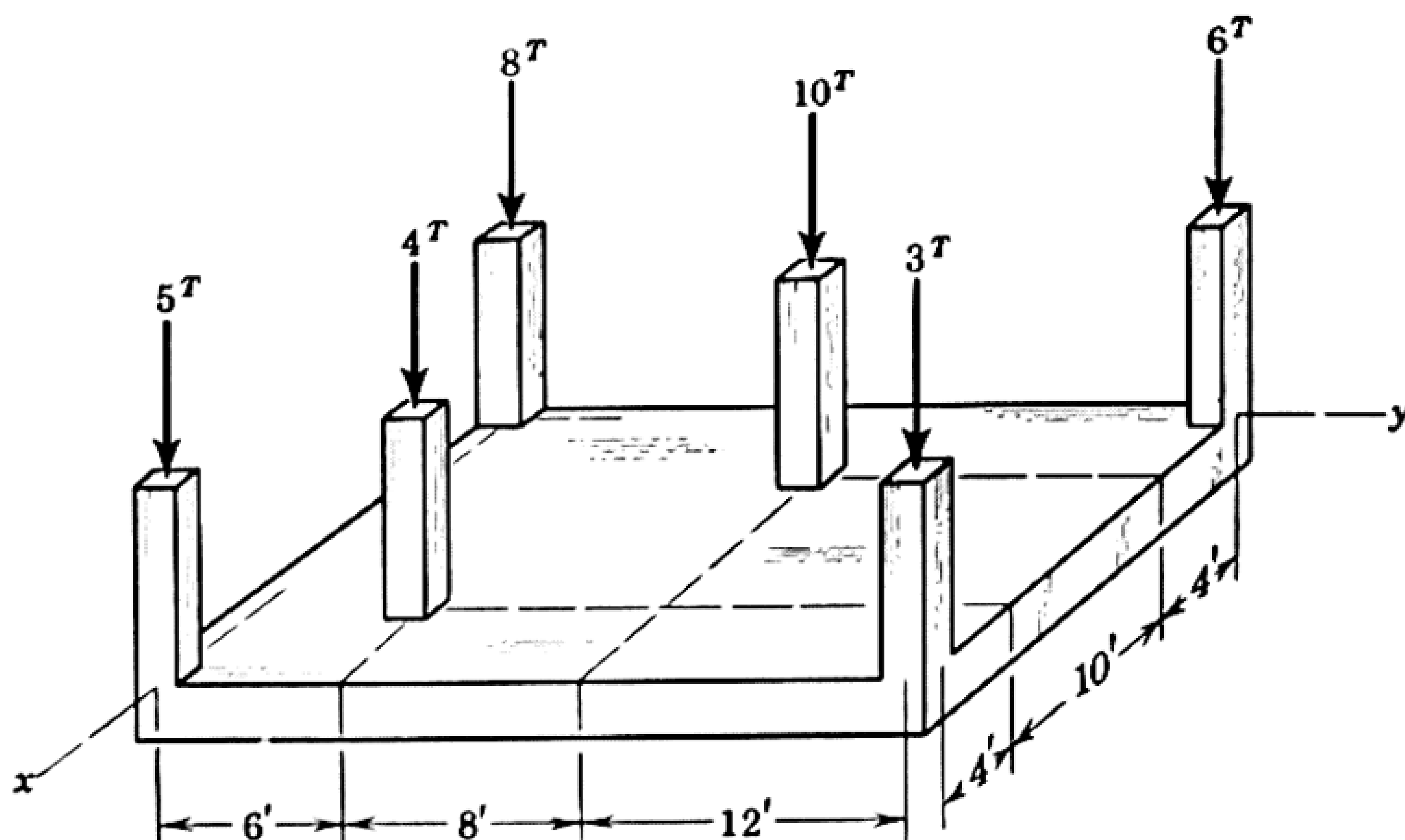
$$\left[\cos\theta_z = \frac{M_z}{M}\right] \qquad \cos\theta_z = \frac{-100}{195.3}, \qquad \theta_z = 120° \, 50'. \qquad \textit{Ans.}$$

The corresponding acute angles made by M with the negative directions of the x-, y-, and z-coordinate axes are 35° 0′, 74° 55′, and 59° 10′, respectively.

It should be noted that the moment M_3 due to F_3 could have been expressed in terms of its two components, $M_{3_y} = -150$ lb. in. and $M_{3_z} = -100$ lb. in., since M is most easily computed from the rectangular components of the moment vectors.

PROBLEMS

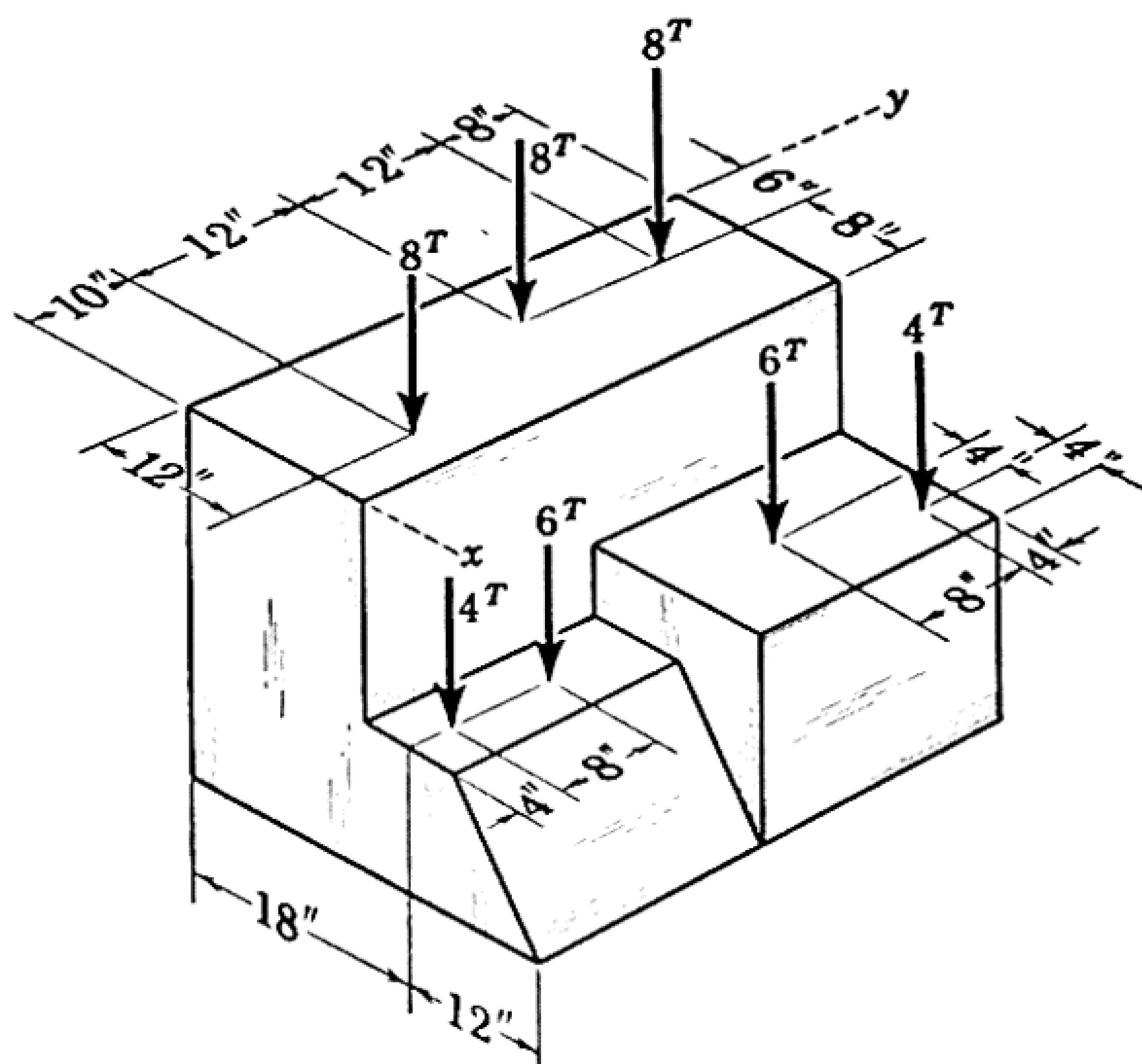
99. A concrete slab supports six columns loaded as shown. Determine the resultant of these forces and the x- and y-coordinates of a point through which it acts.



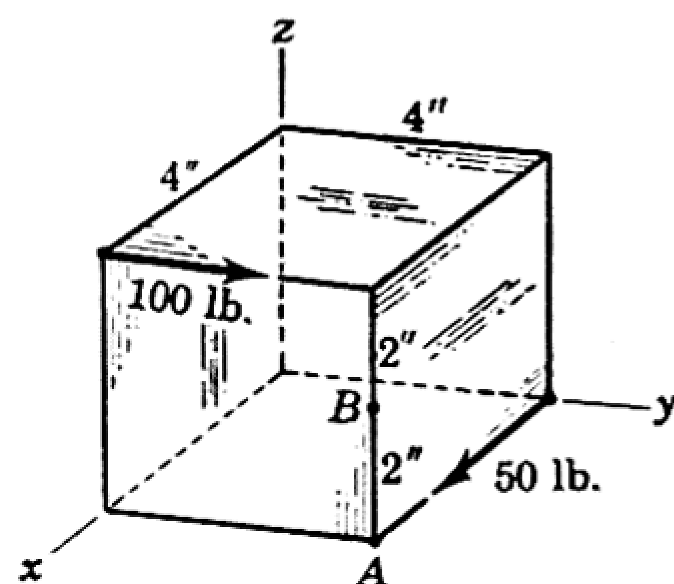
Prob. 99

100. Locate the position of the resultant of the forces shown in Prob. 99 by drawing two funicular polygons.

101. The concrete block supports the seven loads shown. Determine the resultant R of these forces and specify the coordinates of a point on its line of action. *Ans.* $R = 44$ tons, $x = 13.8$ in., $y = 21.5$ in.



Prob. 101



Prob. 103

102. Locate the position of the resultant of the loads shown in Prob. 101 by drawing two funicular polygons.

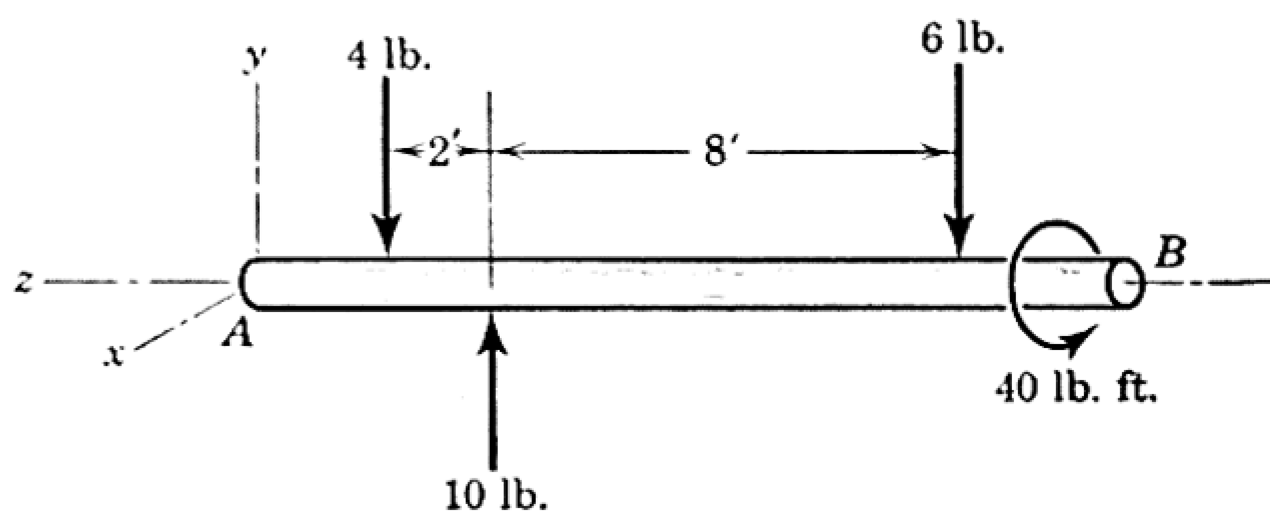
103. Find the resultant of the two forces shown, using point A as the point through which the resultant force shall pass.

Ans. $R = 112$ lb., $\theta_x = 63.5°$, $M = 400$ lb. in., minus x-direction

104. Solve Prob. 103, using point B instead of point A.

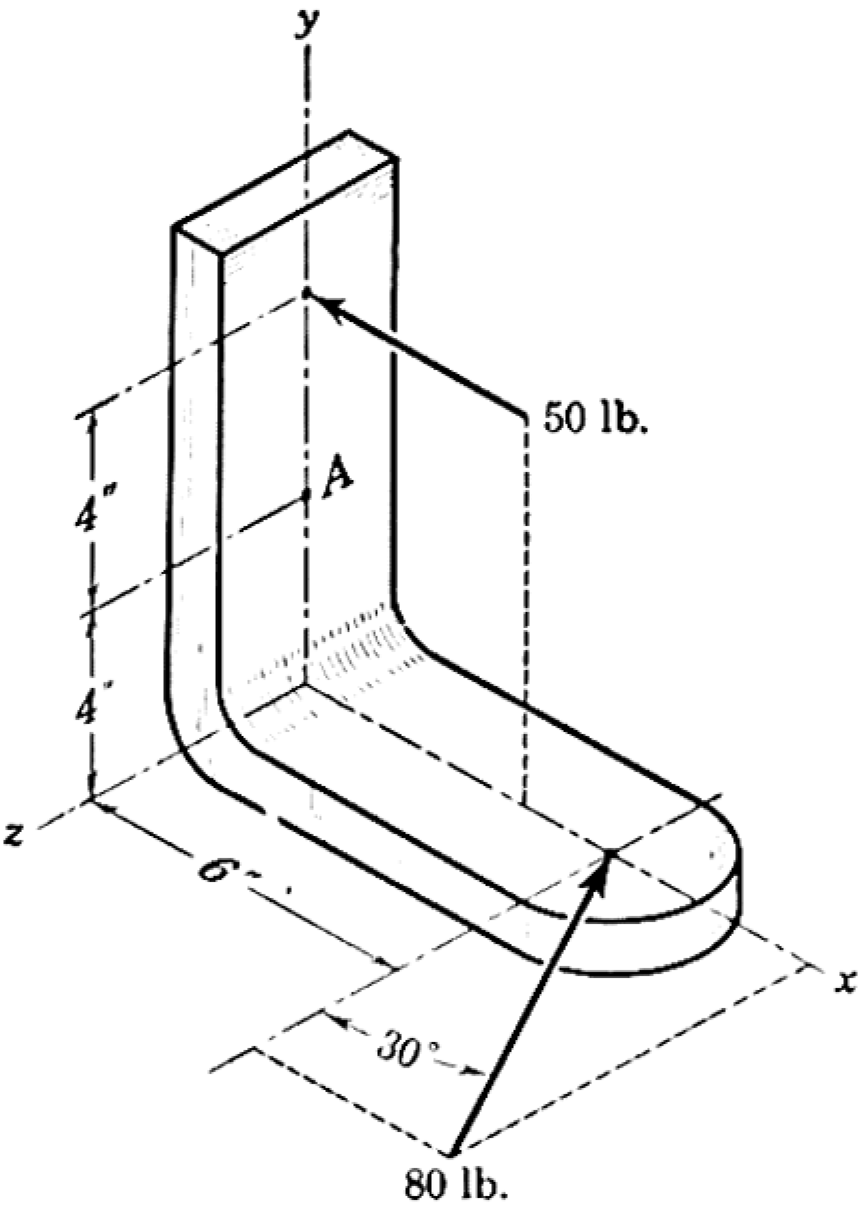
105. Three forces and a couple are acting on the shaft AB. Determine the resultant of this system.

Ans. Resultant is a couple $M = 56.6$ lb. ft., vector in x–z plane with $M_x = -40$ lb. ft.



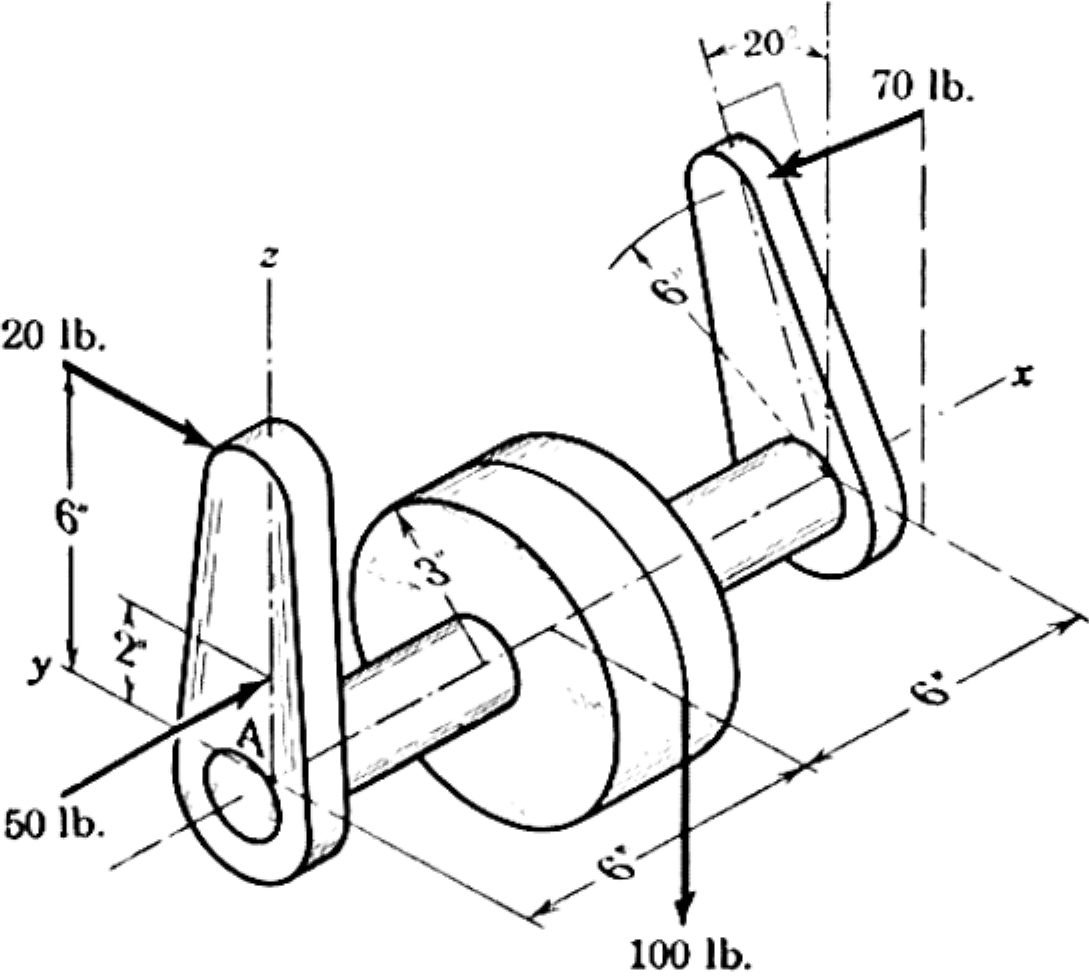
Prob. 105

106. Replace the two forces shown by a single force R acting at A and a couple M.



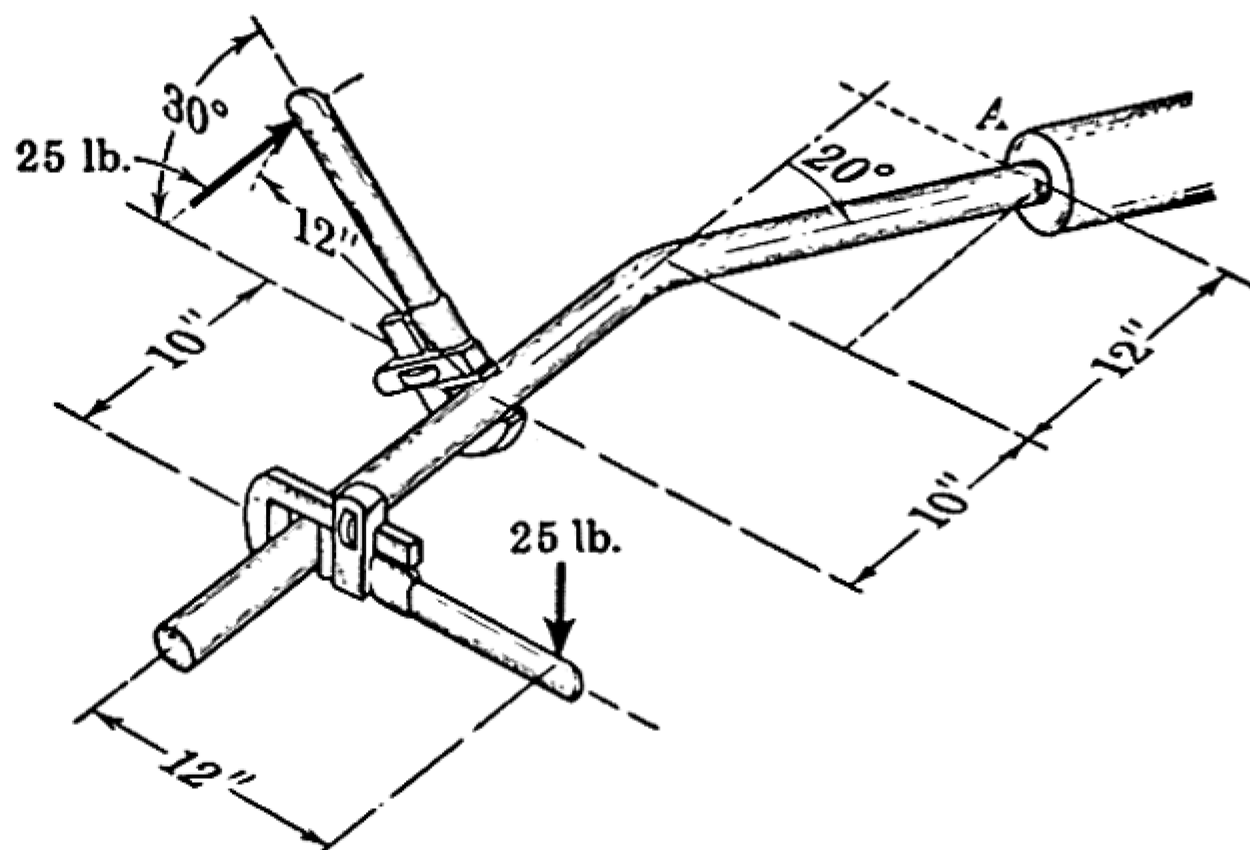
PROB. 106

* **107.** Replace the four forces shown by an equivalent system consisting of a force R at A and a couple M.



PROB. 107

* **108.** Two pipe wrenches are applied to the bent pipe as shown to screw it into the fitting at A. If a single pipe wrench applied to the pipe at A can accomplish the same job, what force F must be exerted on its handle at a moment arm of 12 in.?
Ans. $F = 55.1$ lb.



PROB. 108

109. Show that the most general force system may be expressed in terms of two nonintersecting forces.

110. Show that the most general force system, expressed as a resultant force R at a point O and a resultant couple M, is equivalent to a wrench applied along a unique line parallel to R.

CHAPTER III

Equilibrium

20. Equilibrium. The concept of equilibrium is derived from a balance of forces. More specifically equilibrium is the condition for which the resultant of all forces acting on a given body is zero. In Art. 19 it was shown that the most general force system may be expressed in terms of a resultant force R and a resultant couple M. Thus equilibrium requires that

$$\begin{aligned} R &= 0 \\ M &= 0. \end{aligned} \tag{12}$$

Equations (12) are the necessary conditions for equilibrium.* Physically these vector equations mean that for a body in equilibrium there is as much force acting on it in one direction as in the opposite direction and that there is as much twist or moment applied to it about an axis in one sense as in the opposite sense. Thus equilibrium implies a balance of forces and a balance of moments. Graphically Eqs. (12) require that the space polygon of forces and the space polygon of corresponding couple vectors shall both close.

The equations relating force and acceleration for rigid body motion are developed in *Mechanics, Part II: Dynamics*, from Newton's second law of motion. These equations show that, when the resultant of a force system is expressed by a single force R acting through the center of gravity and a corresponding couple M, the linear acceleration of the center of gravity of the body is proportional to R and the angular acceleration about the center of gravity is proportional to M. Hence the first of Eqs. (12) applies not only to bodies at rest but also to any body whose center of gravity moves with a constant velocity (no acceleration). Likewise the second of Eqs. (12) describes not only a body at rest but also one rotating about its center of gravity with a constant angular velocity. Although the subject of statics, as its name implies,

* When the forces acting on a body in equilibrium can be determined entirely from Eqs. (12), then these equations are also sufficient conditions. There is a certain class of problems explained in Art. 26 for which Eqs. (12), although necessary, are not sufficient.

concerns bodies which are static, i.e. have no motion or change in motion, the case of motion with constant linear velocity of the center of gravity and constant angular velocity about the center of gravity must be admitted when using the statical equations of equilibrium.

In order for a body to be in complete equilibrium it is necessary for both of Eqs. (12) to hold. These equations, however, represent two independent conditions, and either may hold without the other. If $R = 0$ and $M \neq 0$, then the center of gravity of the body either is at rest or is moving with a constant velocity, and the body has an angular acceleration. Under these conditions it may be said that the body is in equilibrium only in so far as its linear motion is concerned. On the other hand if $R \neq 0$ and $M = 0$, then the center of gravity of the body has a linear acceleration but there is no rotational acceleration. Such a body may be considered to be in rotational equilibrium. If the linear acceleration of a body is in the x-direction, then the body may be considered to be in equilibrium in the y- and z-directions since it has no acceleration in these directions. The term equilibrium, however, is most commonly used to describe a body which is completely at rest, as implied by the term statical equilibrium.

21. Free-Body Diagrams. In using Eqs. (12) it is necessary to account accurately and completely for *all* forces which are applied *on* the body in question. Error will result if one or more forces are not accounted for or if forces which do not act on the body in question are used. In analyzing the action of forces on a given body it is absolutely necessary to *isolate* the body in question by removing all contacting and attached bodies and replacing them by vectors representing the forces which they exert *on* the body isolated. Such a representation is called a *free-body diagram*. The free-body diagram is the means by which complete and accurate account of all forces acting *on* the body in question may be taken. Unless such a diagram is correctly drawn the effects of one or more forces will be easily omitted and error will result. The free-body diagram is a basic step in the solution of problems in mechanics and is *preliminary* to the application of the mathematical principles which govern the state of equilibrium or motion.

The subject of equilibrium involves application of the very basic physical laws expressed by Eqs. (12). In words these laws describe a condition wherein there is no resultant push (or pull) and no resultant twist on a body in equilibrium. The physical meaning of these laws of equilibrium is elementary, and the application of these laws for analytical solutions of equilibrium problems will also become elementary if accurate account of all forces is taken. The *free-body diagram* is the means by which this accurate account is made. An understanding of the

free-body diagram method is truly the *key* to the understanding of mechanics. This comprehension cannot be gained without ample practice, but a convincing initial impression of the importance of this method of attack is extremely valuable to the beginner. When the student encounters difficulty in the problem work, the correctness of the free-body diagram is the first step to check.

The actual procedure for drawing free-body diagrams will now be explained. There are two essential parts to this procedure. (1) A clear decision must be made as to exactly what body (or group of bodies considered as a single body) is to be isolated and analyzed. An outline representing the external boundary of the body selected is then drawn. (2) *All* forces, known and unknown, which are applied *externally* to the isolated body should be represented by vectors in their correct positions. Known forces should be labeled with their magnitudes, and unknown forces should be labeled with appropriate symbols. In many instances the correct sense of an unknown force is not obvious at the outset. In this event the sense may be arbitrarily assumed. The correctness or error of the assumption will become apparent when the algebraic sign of the force is determined upon calculation. A plus sign indicates the force is in the direction assumed, and a minus sign indicates that the force is in the direction opposite to that assumed.

Before free-body diagrams are further described, it is advisable to consider the representation of the several kinds of contact forces normally encountered. Figure 28 shows the most common contacts and connections together with the proper representation of the force. In each example the force exerted *on* the body to be isolated *by* the body removed is shown. The principle of action and reaction expressed by Newton's third law must be used with extreme care in all problems in order that there be no confusion between action and the corresponding reaction. Thus in example 1, Fig. 28, the *action* of the cable on the post is the force T shown acting *on* the post. The *reaction* of the post on the cable is the force T acting on the cable in the direction opposite to that shown acting on the post. Examples 2 through 6 show other types of forces applied by mechanical contact, and the characteristics of each contact are explained in the figure. Example 7 shows the force of attraction of the earth on a body. The force is actually distributed over all elements of the body but is shown as a concentrated resultant force or weight W acting through the center of gravity G. The weight of a body *always* acts *vertically down* toward the earth's center. In many instances the weight of a body is negligible compared with the other forces acting on it and is omitted from the free-body diagram and from the calculations. Example 8 shows the action of a force on an elastic spring. The

MECHANICAL ACTION OF FORCES	
Type of Contact or Connection	Action of Force on Body Isolated
1 Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable appreciable θ	θ, T Force exerted by a flexible cable is always a tension in the direction of the cable.
2. Smooth surfaces	N For smooth contacting surfaces the force is compressive and is normal to the surfaces.
3. Rough surfaces	R Rough surfaces may support force which is not normal to the contacting surfaces. (Discussed in Chapter VII)
4. Ball, roller, or rocker support	R For smooth ball, roller, or rocker support the force is compressive and is normal to the contacting surfaces.
5. Smooth pin, bolt, hinge, or bearing	R_x, R_y or θ A pin connection is capable of supporting force in any direction in a plane normal to the pin axis. The force may be represented by its components or by the force at an angle θ.
6. Ball and socket or point connection z, y, x	R_z, R_y, R_x A ball and socket or point connection is capable of supporting force in any direction. The force may be represented by its three components.
7. Gravity force or weight weight W	G, W The gravitational attraction or weight always acts down toward the center of the earth and acts through the center of gravity G of the body.
8. Elastic spring Stretched x Compressed x Unstretched position	F, $F=kx$ F, $F=kx$ For an elastic spring the force is tension or compression and is proportional to the deformation. The constant of proportionality k is the *stiffness* or *modulus* of the spring.

Fig. 28

force, either tensile or compressive, is proportional to the deflection of the spring. The constant of proportionality k is a property of the spring and is a measure of its stiffness. It should be made clear that the diagrams in Fig. 28, with the exception of example 8, are *not* free-body diagrams since they do not show *all* forces acting on the isolated body.

In Fig. 29 are shown five examples of complete and correct free-body diagrams. It should be carefully noted that only those forces acting externally on the body isolated are shown. Each of the forces indicated is in accordance with the representations of Fig. 28. The choice of reference axes should always be clearly indicated either on the sketch of the mechanism or on the free-body diagram. Many times it is convenient to indicate angles and dimensions on the free-body diagram to facilitate the resolution of forces and the computation of the moments of forces. It is well to remember, however, that the free-body diagram serves to focus accurate attention on the actual forces which act on the body, and thus the diagram should not be cluttered with extraneous information which diverts attention from the force vectors. The forces should be drawn and labeled without ambiguity. In the bottom example in Fig. 29 the horizontal component of the pin reaction at A is clearly to the left to balance P, which is to the right. The vertical component A_y, on the other hand, may be up or down, depending on the relative magnitudes of P and W. Therefore it should be arbitrarily drawn, say, down as shown, and the correctness or error of the assumed direction determined by the algebraic sign of the calculated value of A_y. It should be carefully noted that the entire frame of this example, including the pulley and cable between the pulley and the member BC, is taken as the isolated body. Only those forces applied externally to this body are shown on the free-body diagram. The force which the cable exerts on the member BC is equal and opposite to the force which BC exerts on the cable and consequently is not represented on the free-body diagram since it is *internal* to the body isolated. A force internal to a body is always accompanied by an equal and opposite internal reaction, and the net external effect is zero.

In addition to completing the vital step of drawing a correct free-body diagram it is extremely important to follow a clear, logical, and orderly procedure in executing the solution to the remainder of the problem. Such a procedure has been outlined in Art. 10, Chapter 1, and will be illustrated in the sample problems in the present chapter. Again it is emphatically stated that strict adherence to a well-formulated and disciplined method of attack will govern success to a large extent. The tendency of a large proportion of students in mechanics is to jump

SAMPLE FREE-BODY DIAGRAMS	
Mechanism	Free-body Diagram of Isolated Body
1. Ball of weight W resting on smooth surfaces.	G, W, R_1, R_2
2. Weight of truss assumed negligible compared with P.	A_x, A_y, B_x, P
3. Weight of uniform beam AB is W.	T, A_x, A_y, W, P
4. Weight of pulley is W	T_1, T_2, O_x, O_y, W, P
5. Weights of members are assumed negligible compared with applied loads.	P, A_x, A_y, W, C_y

FIG. 29

at conclusions before all factors have been accounted for. Careful attention to the drawing of a correct free-body diagram is the best insurance that account of all pertinent information has been taken. The student is *urged* to reread and study the first two articles of this chapter so that the problem work beginning with the next article will be started with an initial advantage of a proper method and procedure.

22. Equilibrium in Two Dimensions. When a body is in equilibrium under the action of forces all of which act in a single plane, say the x–y plane, Eqs. (12) become

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = 0,$$
$$M = M_z = 0.$$

These equations may be rewritten as

$$\begin{aligned} \Sigma F_x &= 0, \\ \Sigma F_y &= 0, \\ \Sigma M_O &= 0, \end{aligned} \tag{13}$$

where ΣM_O represents the algebraic sum of the moments of all forces acting on the body about an axis parallel to the z-direction and passing through any point O on the body or off the body but in the x–y plane. More frequently this summation is referred to as the sum of the moments about point O. Equations (13) are the three most commonly used equations in the subject of statics. Physically they express the fact that for any body in equilibrium under the action of a coplanar system of forces there is as much force acting in any one direction as in the opposite direction and there is as much twist or moment about any point in one sense as in the opposite sense. Graphically Eqs. (13) require that the polygon of forces must close (zero resultant force) and that the string polygon must also close (zero resultant couple). Use of the string polygon will be discussed in the next article. Equations (13) are the necessary conditions for two-dimensional equilibrium, and, as may be concluded from Art. 20, they are independent conditions. Any one of the relations may hold without the other when the equilibrium is not complete.

There are two additional ways of expressing the necessary conditions for the equilibrium of forces in two dimensions. For the body shown in Fig. 30a, if $\Sigma M_A = 0$, then the resultant R, if it still exists, cannot be a couple but must be a force R passing through A. If now the equation $\Sigma F_x = 0$ holds, where the x-direction is perfectly arbitrary, it follows from Fig. 30b that the resultant force R, if it still exists, must not only

pass through A but also must be perpendicular to the x-direction as shown. Now, if $\Sigma M_B = 0$, where B is any point such that the line AB is not perpendicular to the x-direction, it is clear that R must be

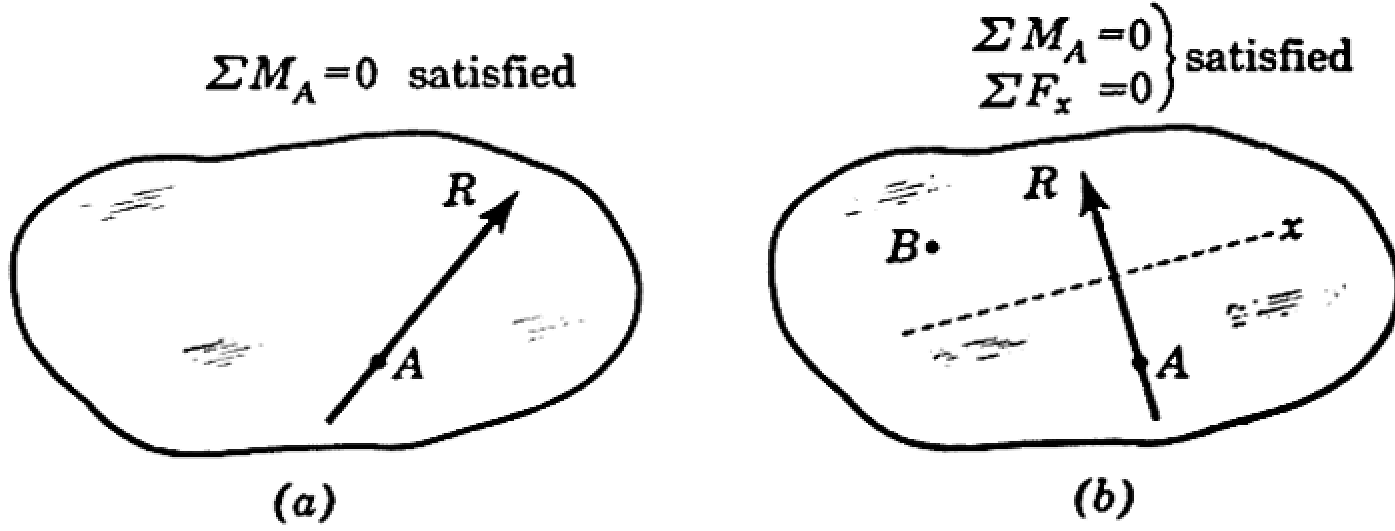


FIG. 30

zero, and hence the body is in equilibrium. Therefore an alternate set of equilibrium equations is

$$\Sigma F_x = 0,$$

$$\Sigma M_A = 0, \tag{14}$$

$$\Sigma M_B = 0,$$

where the two points A and B are *not* on a line perpendicular to the x-direction.

A third formulation of the conditions of equilibrium may be made. Again, if $\Sigma M_A = 0$ for any body such as shown in Fig. 31*a*, the resultant, if it exists, must be a force R through A. In addition if $\Sigma M_B = 0$,

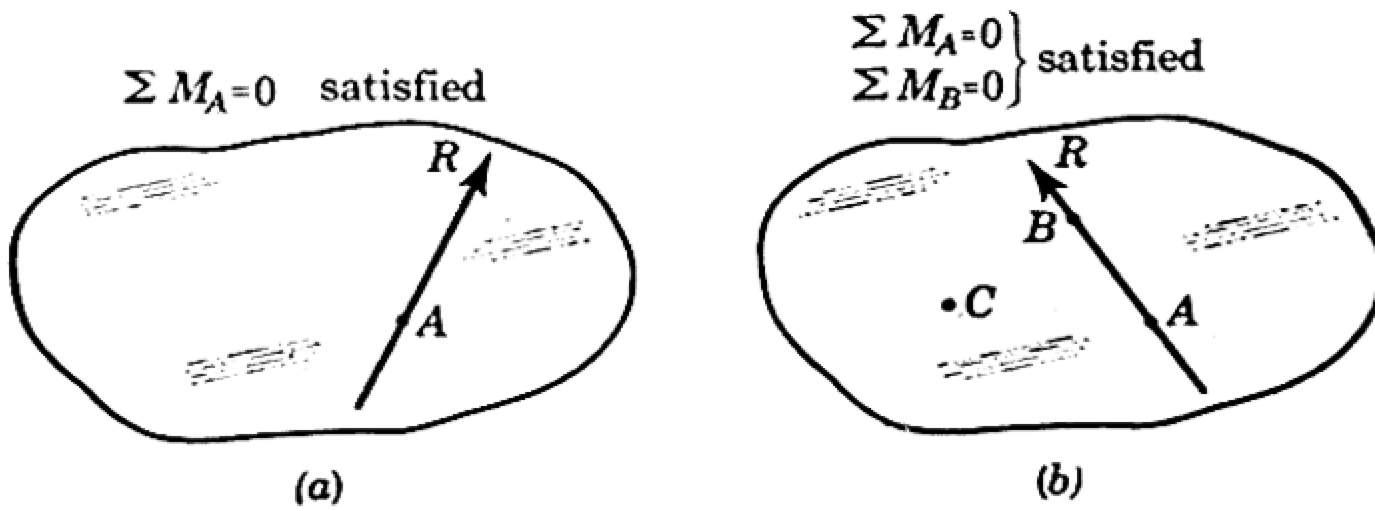


FIG. 31

the resultant, if one still exists, must pass through B as shown in Fig. 31*b*. Such a force cannot exist, however, if $\Sigma M_C = 0$, where C is not collinear with A and B. Hence the equations of equilibrium may be written

$$\Sigma M_A = 0,$$

$$\Sigma M_B = 0, \tag{15}$$

$$\Sigma M_C = 0,$$

where A, B, and C are any three points not on the same straight line.

When a body is in equilibrium under a system of concurrent forces, the moment sum about the point of concurrency is automatically satisfied, leaving from Eqs. (13) the relations

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0.$$

These two equations, or a corresponding pair of relations from either Eqs. (14) or (15), will insure the equilibrium of concurrent forces in all cases.

In the case of equilibrium produced by parallel forces, only two conditions need be specified, and from Eqs. (13) these may be taken to be

$$\Sigma F_x = 0, \qquad \Sigma M_O = 0,$$

where the x-direction is the direction of the forces and O is any point in their plane. As an alternative two moment equations may be used, provided the line joining the moment centers is not parallel to the forces.

When a body is in equilibrium under the action of three forces only, these forces must be concurrent. If they were not concurrent, as shown in Fig. 32, then one of the forces would exert a resultant moment about

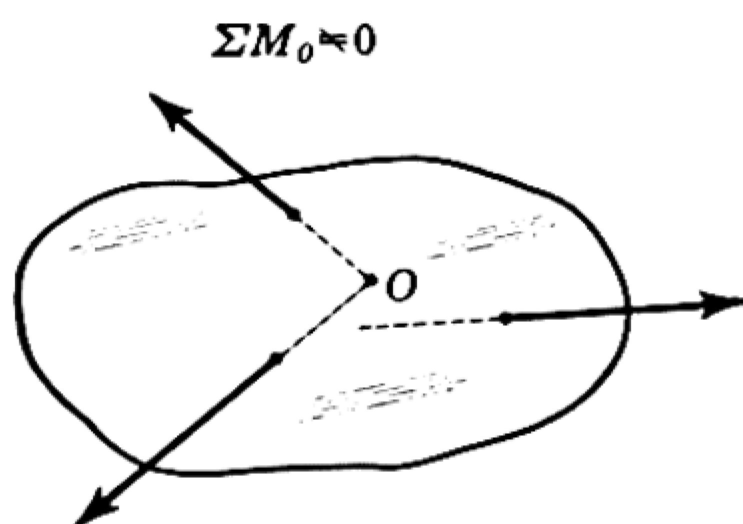


FIG. 32

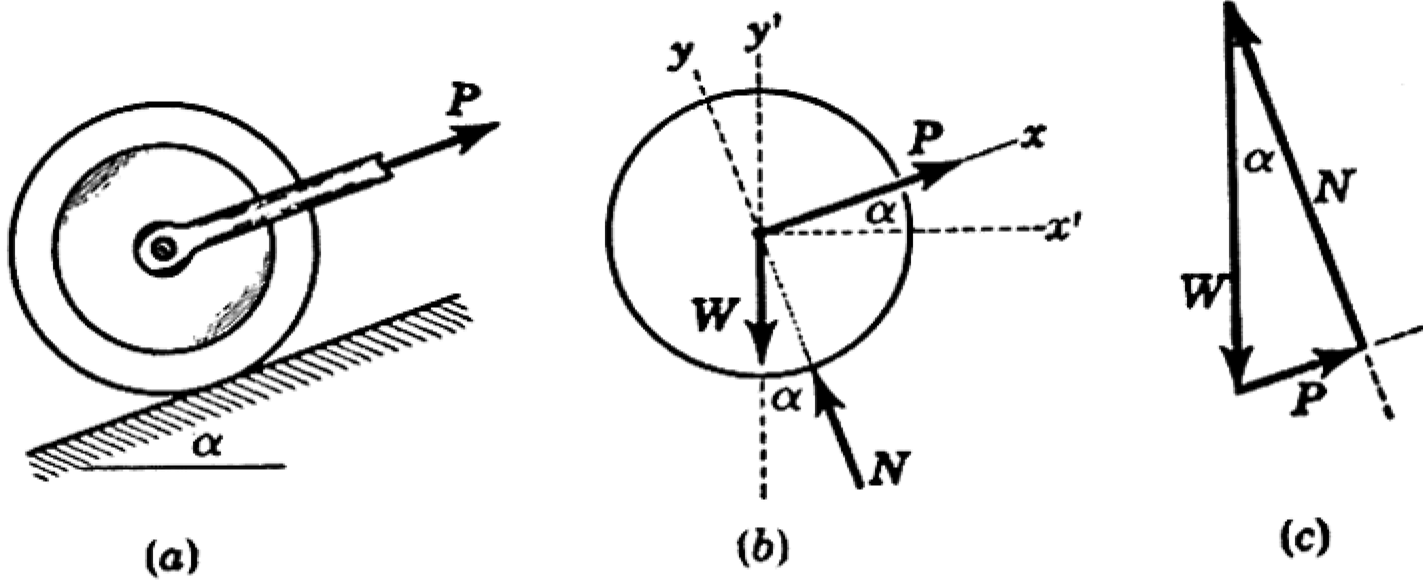
the point of concurrency of the other two which would violate the requirement of zero moment about every point. Any system of coplanar forces in equilibrium may always be reduced by direct combination of the forces to a system of three forces which must therefore be concurrent. This principle of concurrency is very useful for both algebraic and graphical solutions in determining the direction of an unknown force. The only exception to this principle exists for the case where three parallel forces are in equilibrium. In this event the point of concurrency may be said to be at infinity.

The equilibrium of collinear forces requires but one equation and is considered too trivial for further comment.

It is desirable that facility with both algebraic and graphical solutions be acquired in the problem work. The sample problems which follow illustrate both methods.

SAMPLE PROBLEMS

111. Determine the force P required to prevent the wheel of weight W from rolling down the incline. Also find the force N exerted by the plane on the wheel. Neglect the weight of the yoke.



PROB. 111

Algebraic Solution: The free-body diagram is first drawn as shown in the b-part of the figure. The weight W acts vertically down, and the force N is normal to the wheel surface. The diagram discloses the fact that the wheel is in equilibrium under the action of the three forces. Choosing the x- and y-directions along and normal to the plane, respectively, and applying the equations of equilibrium give

$[\Sigma F_x = 0]$ $\quad P - W \sin\alpha = 0, \quad P = W \sin\alpha,$ *Ans.*

$[\Sigma F_y = 0]$ $\quad N - W \cos\alpha = 0, \quad N = W \cos\alpha.$ *Ans.*

If the axes were chosen as with x' and y', then the equations of equilibrium give

$[\Sigma F_{x'} = 0]$ $\quad P \cos\alpha - N \sin\alpha = 0,$

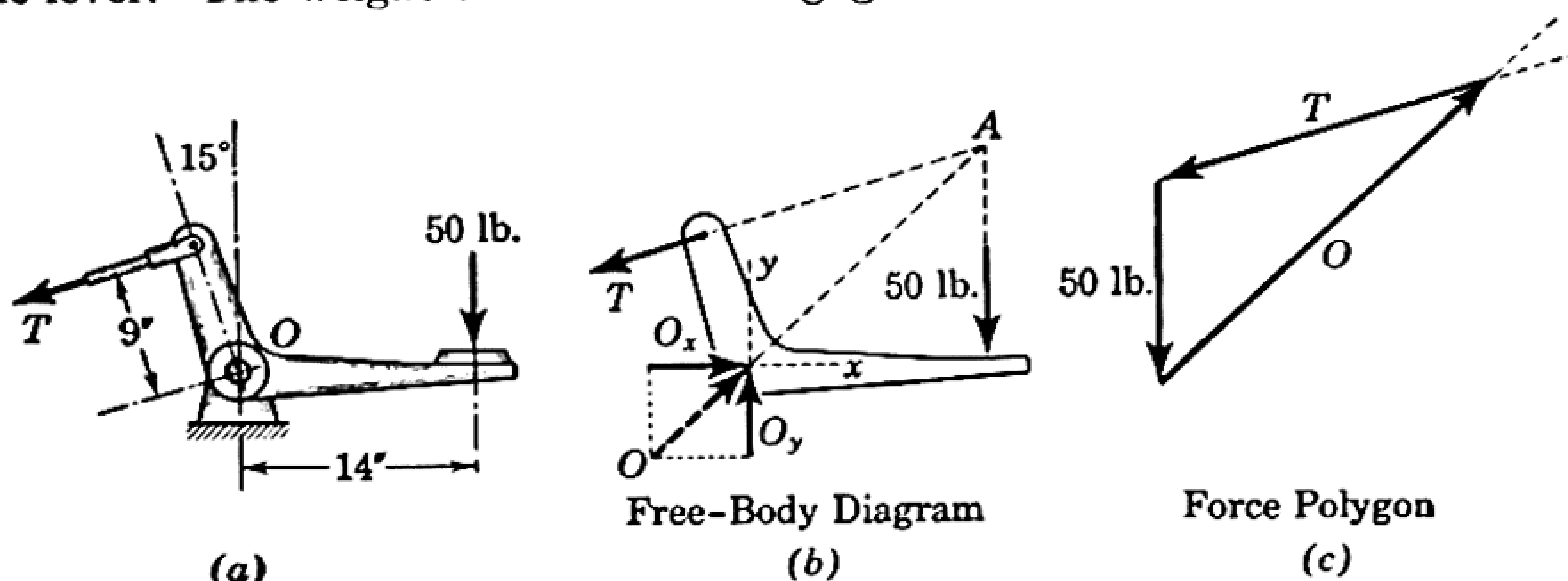
$[\Sigma F_{y'} = 0]$ $\quad P \sin\alpha + N \cos\alpha - W = 0.$

Eliminating N between the two equations and solving for P and then solving for N yield the answers just obtained, but the process was needlessly complicated by a less favorable choice of reference axes.

Graphical Solution: After the free-body diagram is completed the known force W is laid off to some convenient scale as shown in the c-part of the figure. Next lines with the known directions of P and N are constructed through the terminal points of W, and their intersection gives the solution and enables the correct magnitudes of P and N to be scaled from the drawing. It should be observed from this force polygon that the two answers may be obtained by inspection of the trigonometry of the triangle. Thus $P = W \sin\alpha$, $N = W \cos\alpha$, and also $P = N \tan\alpha$. In the case of concurrent forces a simple and approximate sketch of the force polygon will enable the required relations between the

forces to be written immediately by inspection. It should also be noted that the force polygon is a graphical statement of $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

112. A 50 lb. force is required to operate the foot pedal shown. Determine the tension T in the connecting link and the force exerted by the bearing O on the lever. The weight of the lever is negligible.



(a) (b) (c)

PROB. 112

Algebraic Solution: The free-body diagram is first drawn as shown in the *b*-part of the figure. The force exerted by the bearing on the lever at O is shown in terms of its x- and y-components. The solution is best started with the moment equation about O, which eliminates O_x and O_y from the relation. Thus

$[\Sigma M_O = 0]$ $\qquad 50 \times 14 - 9T = 0, \qquad T = 77.8$ lb. *Ans.*

The remaining two equations of equilibrium give

$[\Sigma F_x = 0]$ $\qquad O_x - 77.8 \cos 15° = 0, \qquad O_x = 75.1$ lb.,

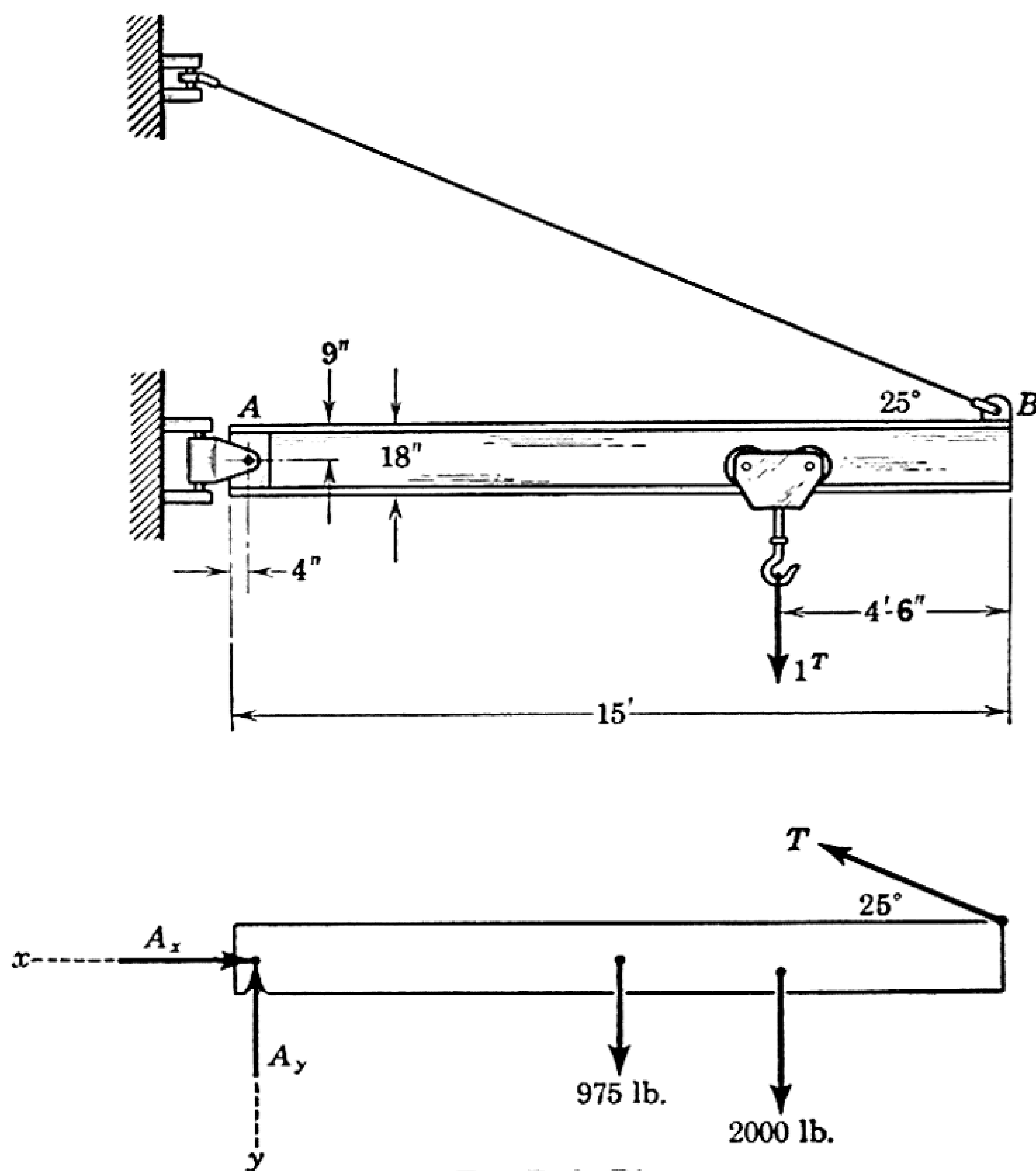
$[\Sigma F_y = 0]$ $\qquad O_y - 50 - 77.8 \sin 15° = 0, \qquad O_y = 70.1$ lb.,

$[O = \sqrt{O_x^2 + O_y^2}\,]$ $\qquad O = \sqrt{(75.1)^2 + (70.1)^2}, \qquad O = 102.7$ lb. *Ans.*

The direction of O can be specified by the angle determined by the ratio of O_x to O_y if desired.

Graphical Solution: The intersection of the 50 lb. force and the tension T defines the point A through which the reaction O must also pass, since three forces in equilibrium are concurrent. With the direction of O established, the force polygon is drawn in the c-part of the figure by constructing the directions of O and T through the ends of the 50 lb. vector. From the intersection of the two lines the magnitudes of T and O are scaled directly from the triangle, and the values are those obtained in the algebraic solution. In this problem the exact expressions for the unknown forces are more easily obtained from the algebraic solution than from the trigonometry of the force polygon.

113. Determine the tension T in the supporting cable and the force on the pin at A for the jib crane shown. The beam AB is a standard 18 in. I-beam weighing 65 lb./ft. of length.



Free-Body Diagram

PROB. 113

Solution: The free-body diagram of the beam is shown with the pin reaction at A separated in terms of its two rectangular components. The weight of the beam is $65 \times 15 = 975$ lb. and acts through its center. In applying the moment equation about A it is simpler to consider the moments of the x- and y-components of T than it is to compute the perpendicular distance from T to A. Thus

$$[\Sigma M_A = 0] \quad (T \cos 25°)\tfrac{9}{12} + (T \sin 25°)(15 - \tfrac{4}{12})$$

$$- 2000(15 - 4.5 - \tfrac{4}{12}) - 975(7.5 - \tfrac{4}{12}) = 0,$$

from which

$$T = 3980 \text{ lb.} \qquad \textit{Ans.}$$

Equating the sum of forces in the x- and y-directions to zero gives

$[\Sigma F_x = 0] \qquad A_x - 3980 \cos 25° = 0, \qquad A_x = 3610$ lb.,

$[\Sigma F_y = 0] \qquad A_y + 3980 \sin 25° - 975 - 2000 = 0, \qquad A_y = 1293$ lb.,

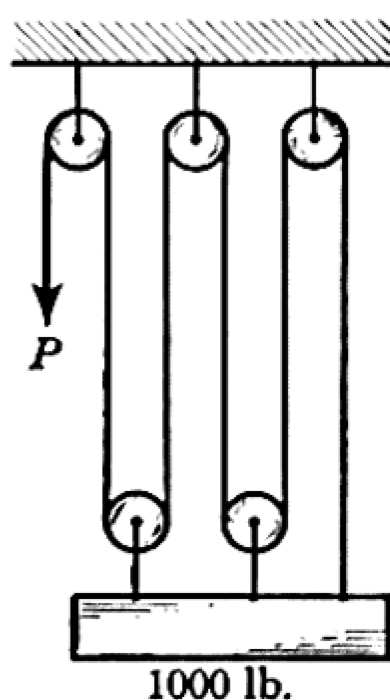
$[A = \sqrt{A_x^2 + A_y^2}\,] \qquad A = \sqrt{(3610)^2 + (1293)^2}, \qquad A = 3830$ lb. *Ans.*

A graphical solution could be obtained by combining the 2000 lb. and 975 lb. forces into a single force of 2975 lb. and establishing its intersection with T. The reaction at A would have to pass through this point, and the solution would proceed as in the previous sample problem.

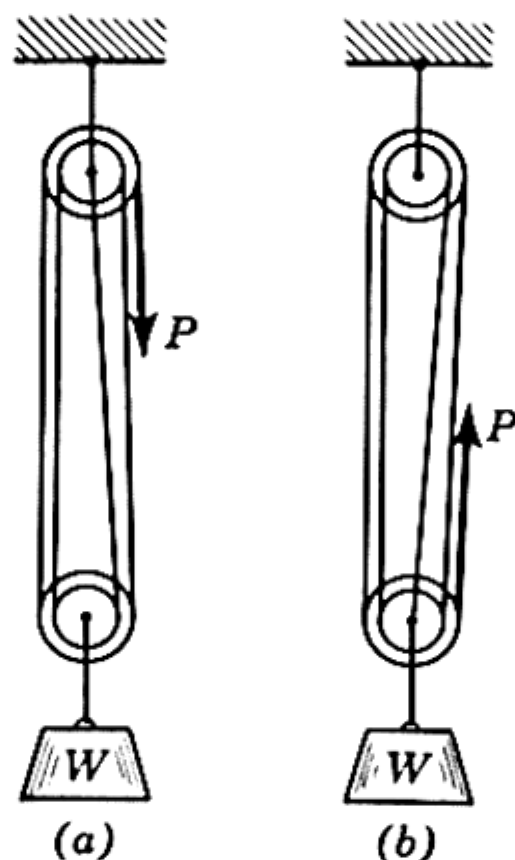
PROBLEMS

114. What force P is required to raise the 1000 lb. platform?

Ans. $P = 200$ lb.



PROB. 114

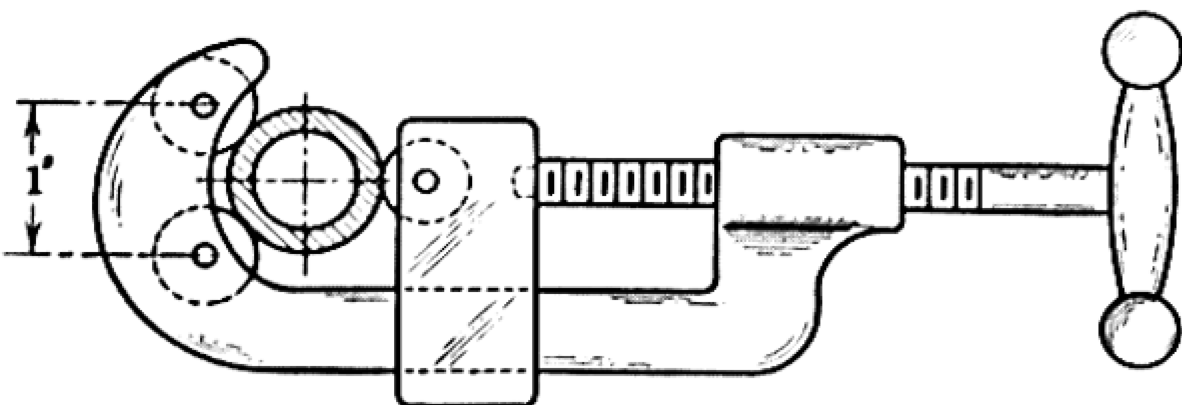


PROB. 115

115. Which combination of rope and pulleys requires the smaller force P to raise the load W? Each pulley is free to rotate independently.

116. In Prob. 31 determine the weight W of the boom, assuming its center of gravity is at midlength. *Ans.* $W = 1000$ lb.

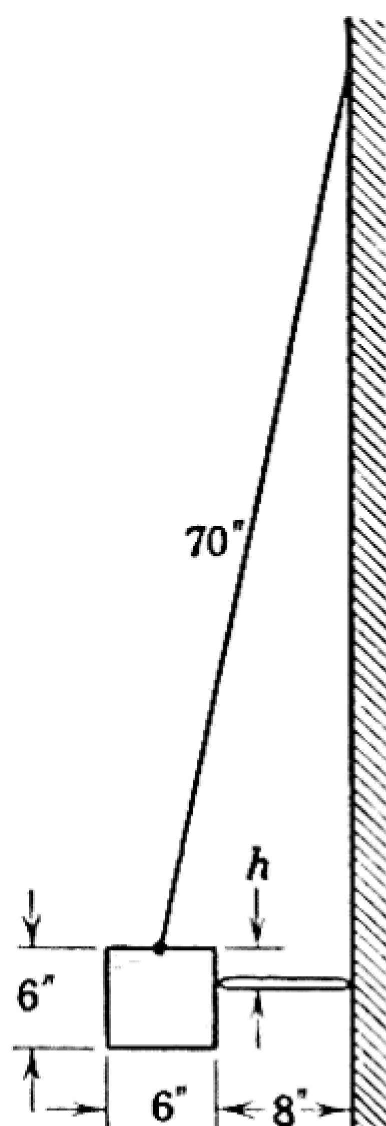
117. One type of pipe cutter, shown in the figure, consists of a clamp with three sharp-edged wheels which cut through the pipe as the screw is tightened and the device is rotated around the pipe. When cutting a ¾ in. pipe (outside diameter 1.050 in.) with a compression of 1000 lb. in the screw, determine the force R acting on each pin of the two ¾ in. diameter cutting wheels.



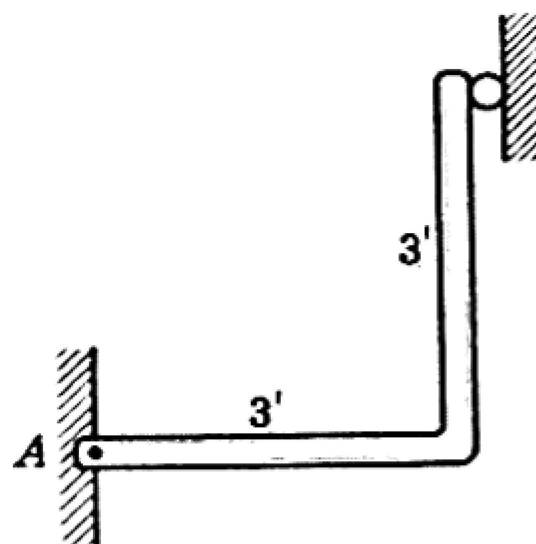
PROB. 117

118. The steel cube is 6 in. on a side and is propped away from the wall by a light slender bar 8 in. long. The supporting wire is attached to the center of the top face and has a length from this point to the wall of 70 in. Determine the distance h so that the sides of the cube will remain vertical, and find the tension T in the wire and the compression C in the bar.

Ans. $h = 0$, $T = 62.1$ lb., $C = 9.8$ lb.



Prob. 118



Prob. 119



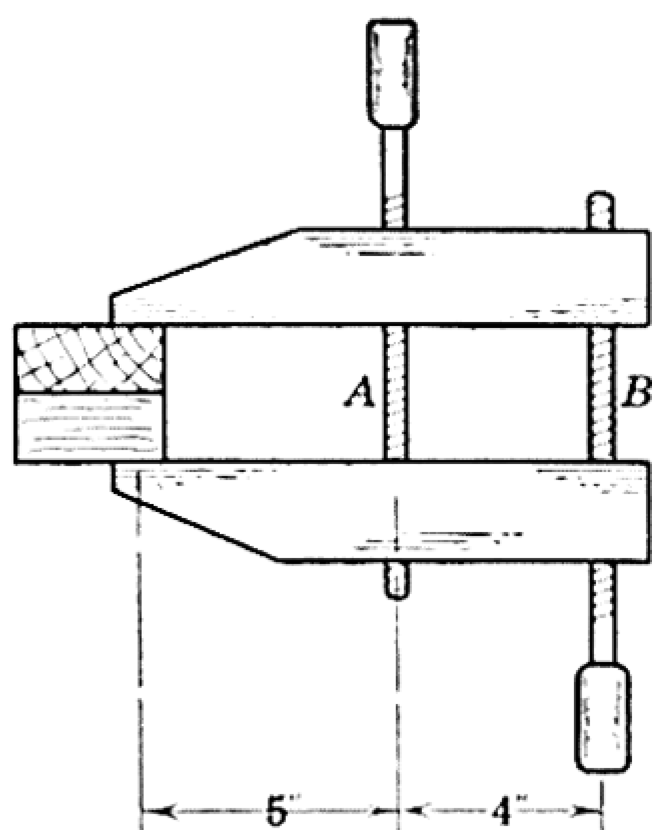
Prob. 120

119. A uniform angle with equal legs weighs 50 lb. and is supported as shown. Determine the pin reaction at A.

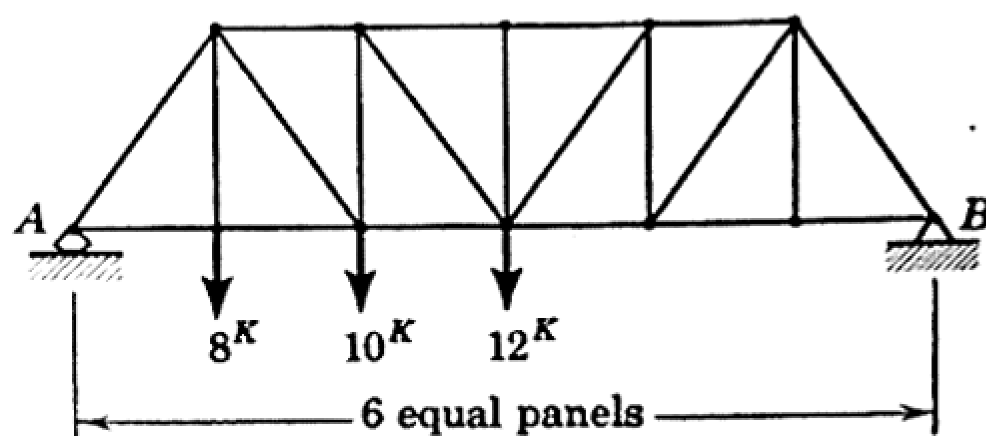
120. The 50 lb. force exerts a torque about the axis of the bolt of 60 lb. ft. Find the forces between the smooth jaws of the wrench and the bolt if contact is at the corners A and B of the hexagon. *Ans.* $A = 599$ lb., $B = 649$ lb.

121. If the clamp holds the two parts together with a force of 100 lb., what are the forces F_A and F_B in the two screws?

122. Determine the reactions at A and B for the truss shown.

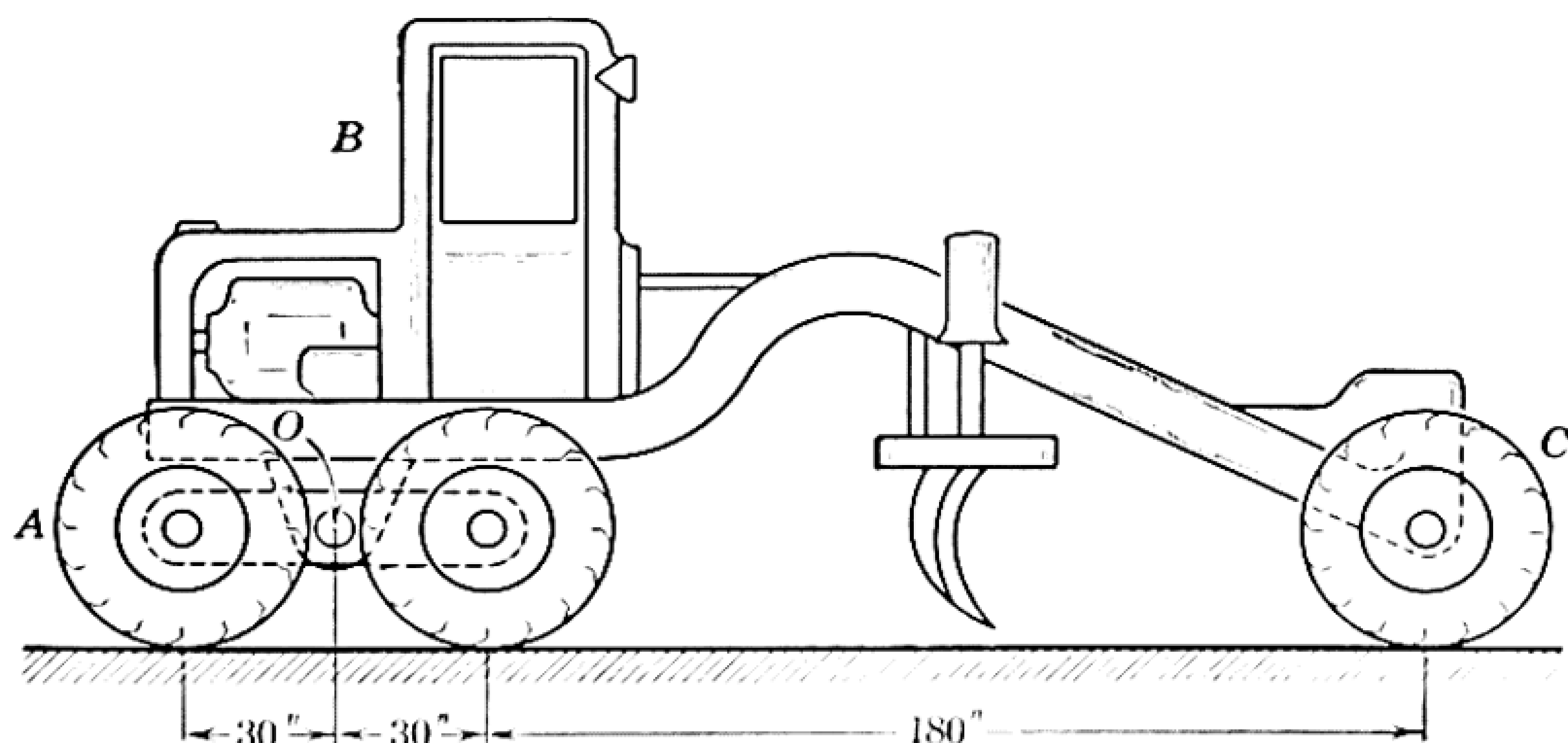


Prob. 121



Prob. 122

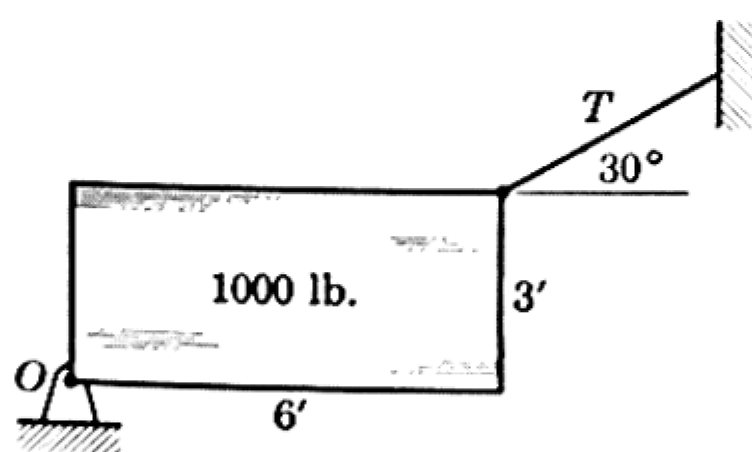
123. A road grader consists of the four-wheeled tandem unit A and the motive unit B with attached scraper frame and leading guide wheels C. The two units are freely pivoted at O, which is midway between the axles of the tandem unit. When the machine is weighed with wheels C off the scales, the total load on the four wheels of unit A is 20,000 lb. With unit A off the scales and wheels C on, a load of 8500 lb. is registered. Unit A separately is known to weigh 6500 lb. Locate the horizontal position of the center of gravity of unit B by finding its distance a to the right of O. *Ans.* $a = 81.1$ in.



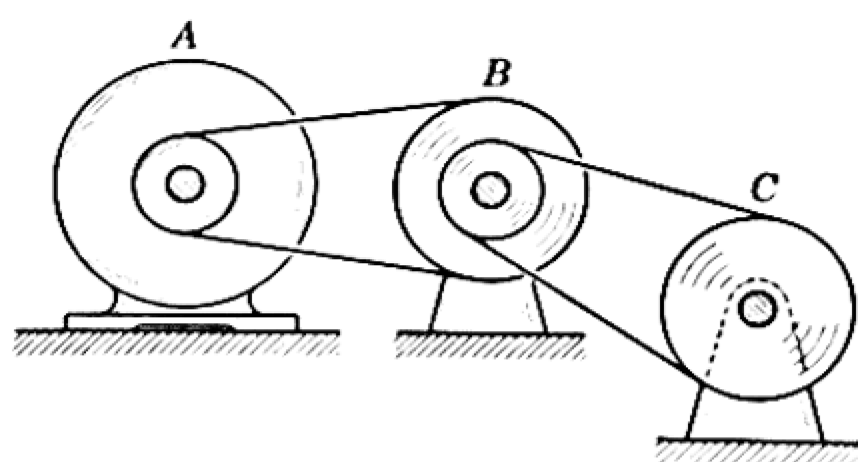
Prob. 123

124. Determine graphically the tension T in the cable and the reaction R at the pivot O. The 1000 lb. block is homogeneous.

Ans. $T = 7460$ lb., $R = 7010$ lb.



Prob. 124

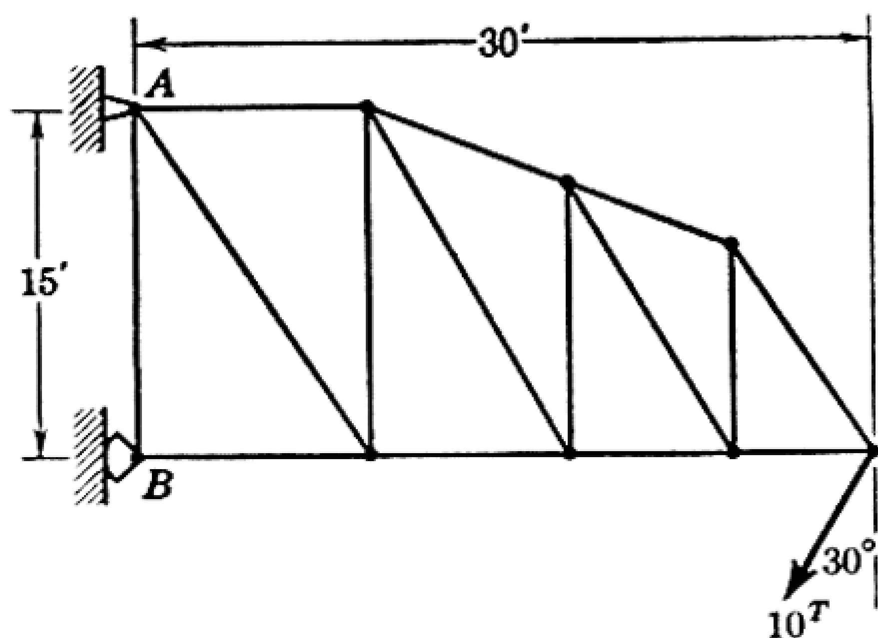


Prob. 126

125. If the block in Prob. 25 is uniform and is in equilibrium, determine its weight W.

126. The motor A supplies a torque (moment) of 100 lb. in. to its shaft and drives pulley C through the intermediate pulley B. Pulley C in turn drives the spindle of a lathe. What torque M is supplied by the shaft of C if the speeds of the pulleys remain constant and both speed reductions are 2:1?

Ans. $M = 400$ lb. in.

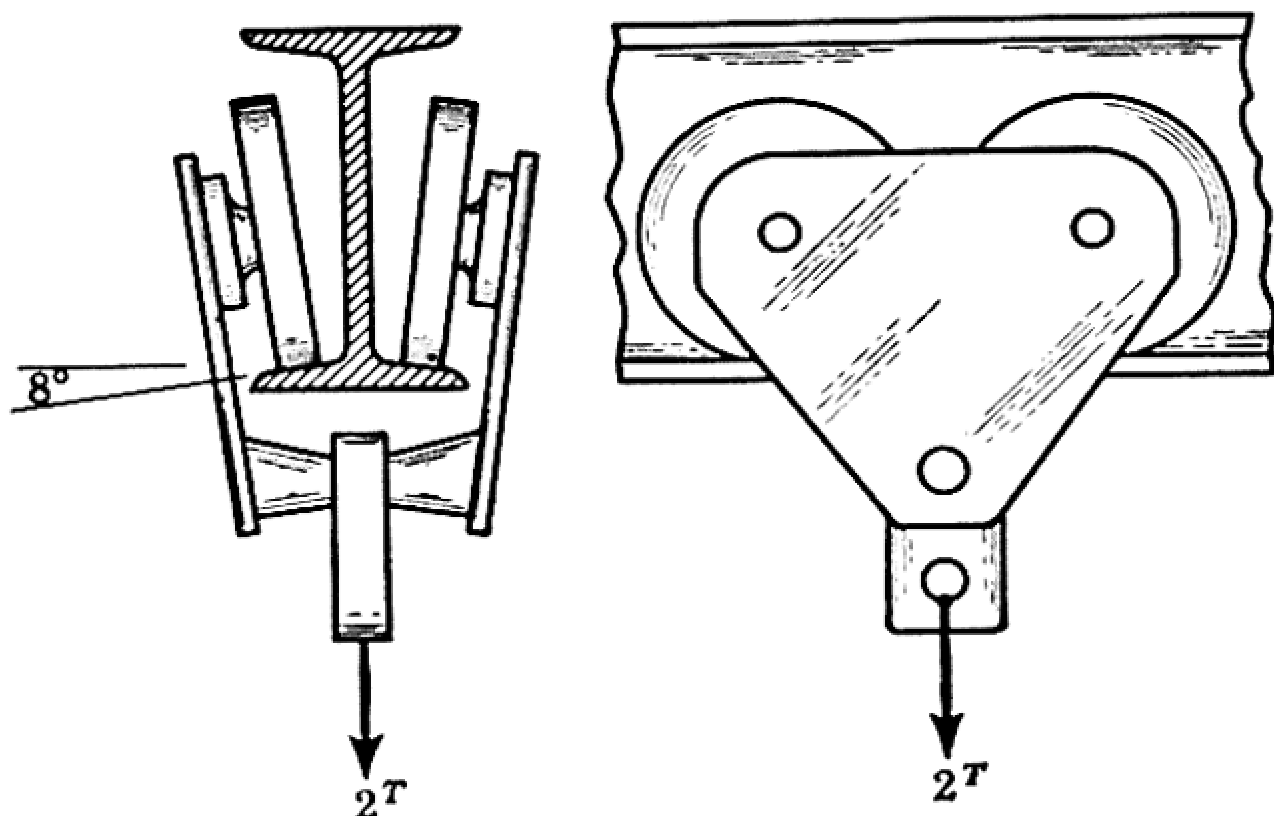


PROB. 127

127. Determine graphically the force on pin A and the force on the rocker B due to the 10 ton load.

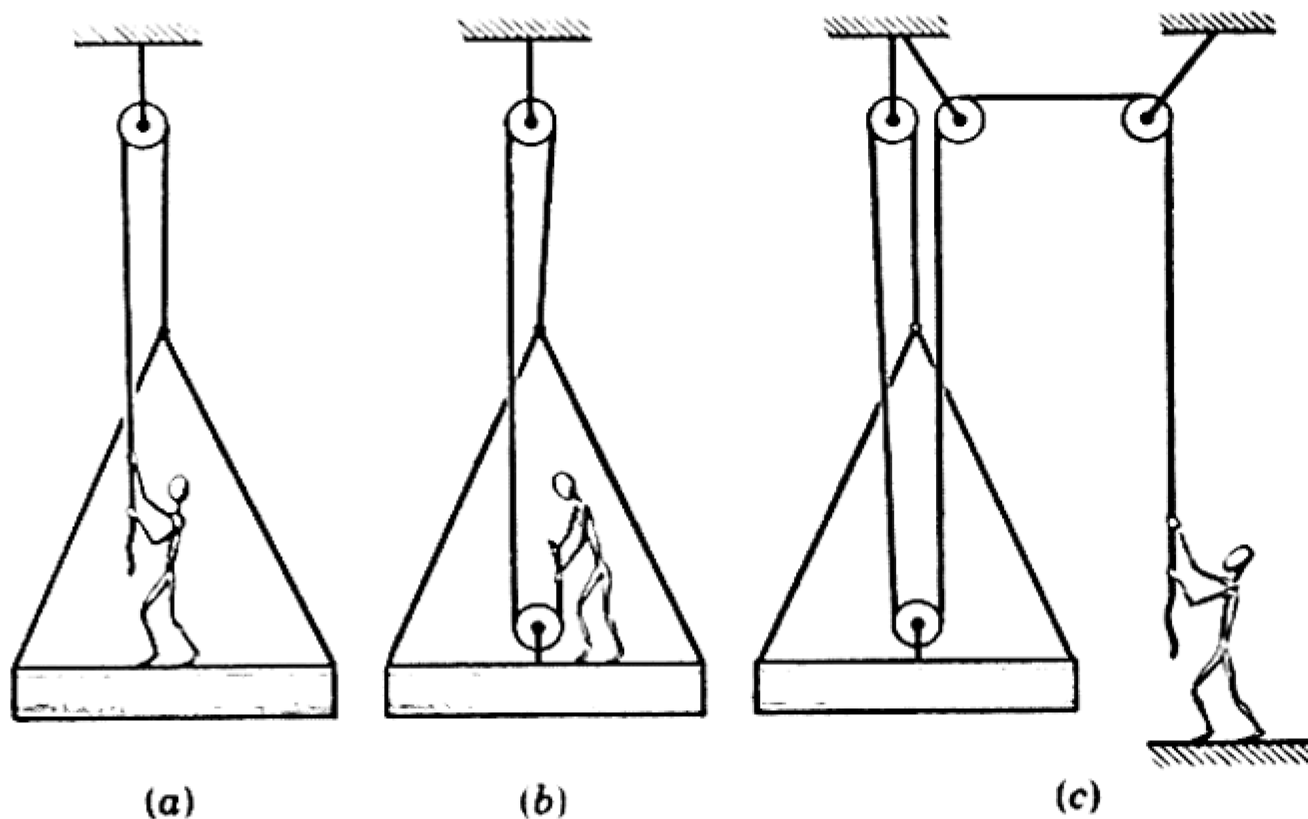
Ans. $A = 19.4$ tons, $B = 22.3$ tons

128. The trolley for an I-beam hoist runs on the flanges of the beam as shown. If the hoist supports a 2 ton load, what force R is exerted by the flange on each of the four wheels?



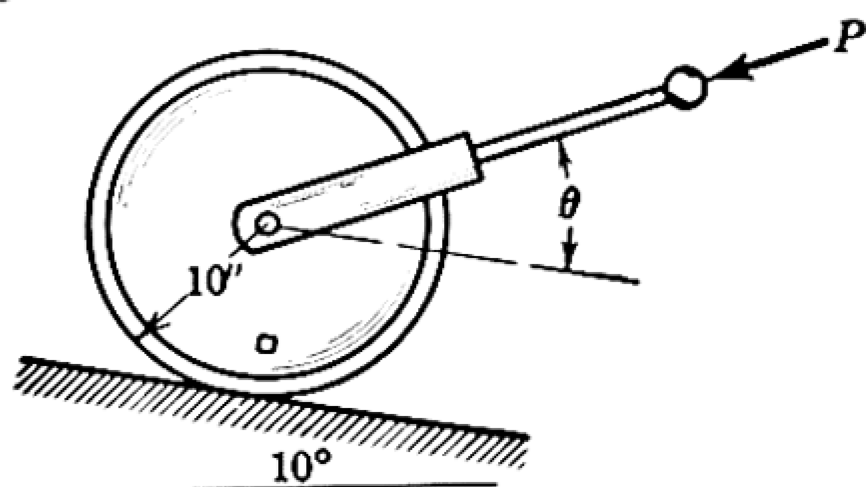
PROB. 128

129. A man weighing 150 lb. endeavors to elevate the 50 lb. platform in three separate ways as shown. Find the force P which he must exert on the rope in each case. *Ans.* (*a*) $P = 100$ lb., (*b*) $P = 100$ lb., (*c*) $P = 16.7$ lb.



PROB. 129

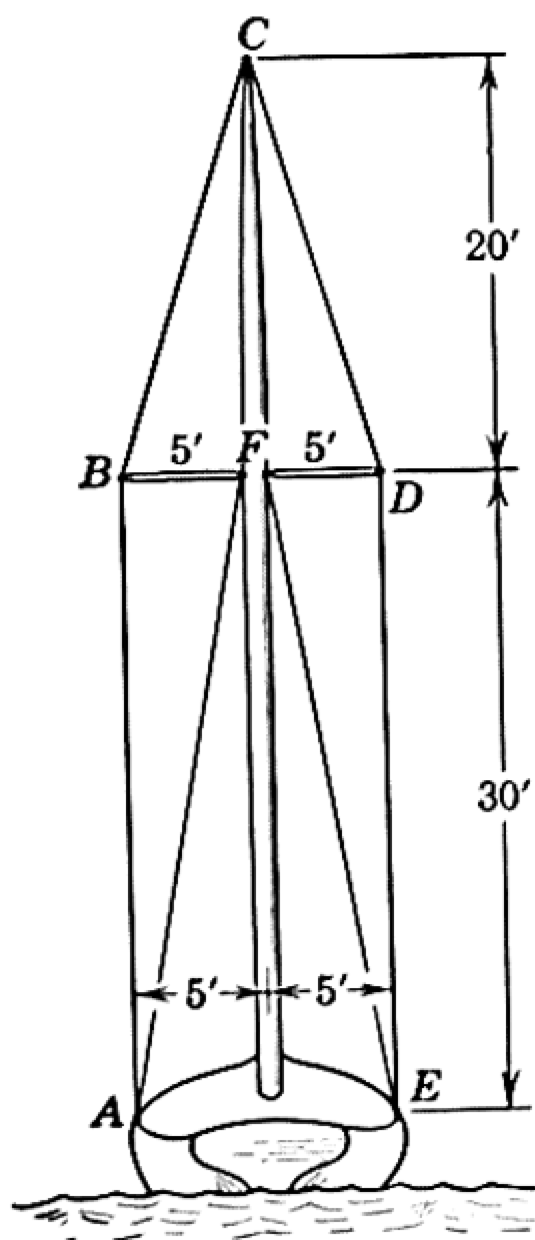
130. What force P is required to push the 200 lb. lawn roller up a 10 deg. incline for $\theta = 30$ deg.? *Ans.* $P = 40.1$ lb.



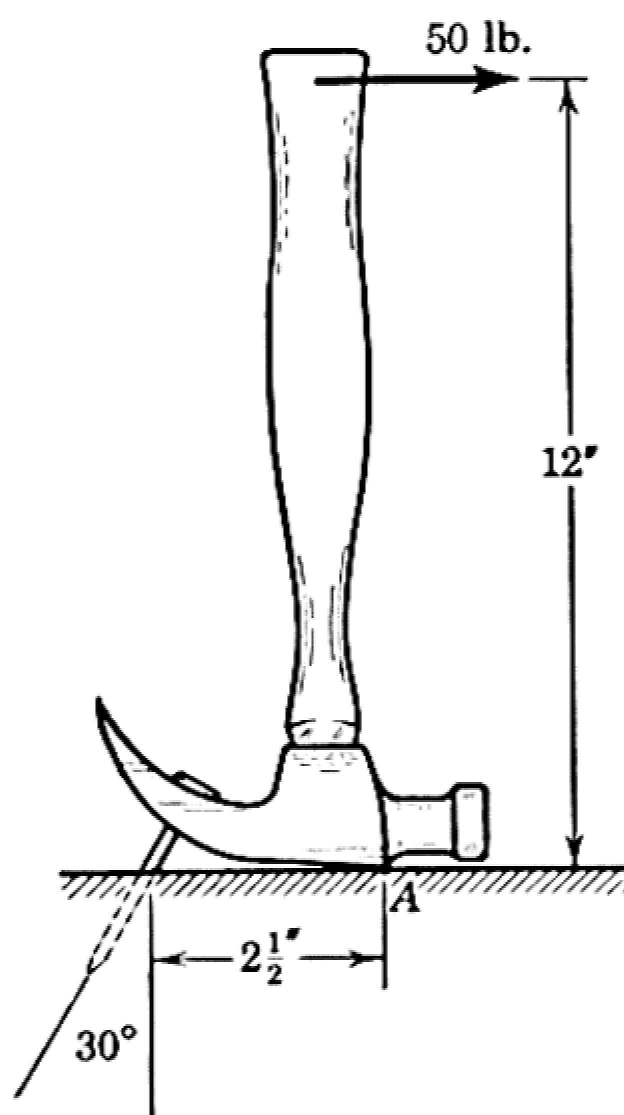
Prob. 130

131. In the sailboat the spreader is a compression member consisting of two parts BF and FD hinged at F. If turnbuckles at A and E are tightened until the tension in each of the four cables reaching the deck is 300 lb., find the compression in the spreader and the tension in BC and CD.

Ans. $BF = DF = 74.9$ lb., $BC = DC = 309$ lb.



Prob. 131



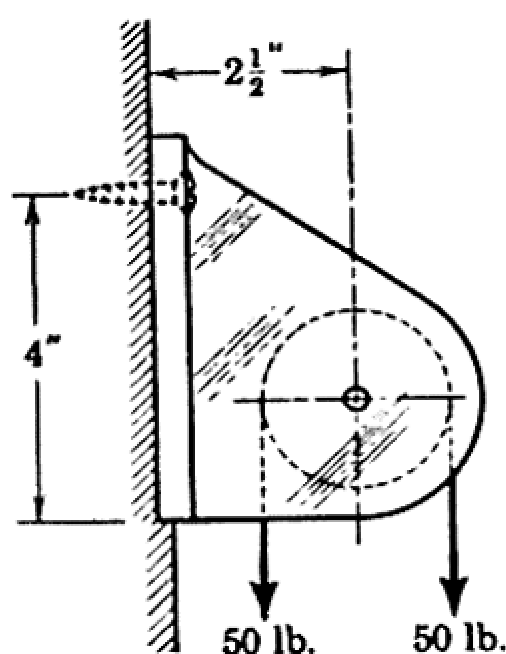
Prob. 133

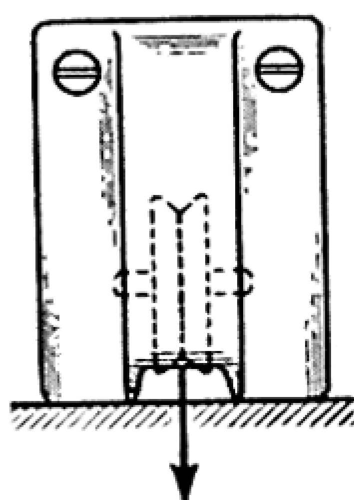
132. Use the results of Prob. 131 and determine the compression in the mast above and below point F.

133. Find the pull P exerted on the nail by the claws of the hammer and the total force R exerted by the board on the hammer head at the contact point A. The surfaces are not smooth, and it may be assumed that the force exerted on the nail is in the direction of the nail. *Ans.* $P = 277$ lb., $R = 256$ lb.

134. The pulley bracket rests on a ledge and is held to the wall by two screws. Find the tension T in each screw and the reaction R at the ledge.

Ans. $T = 31.3$ lb., $R = 118$ lb.





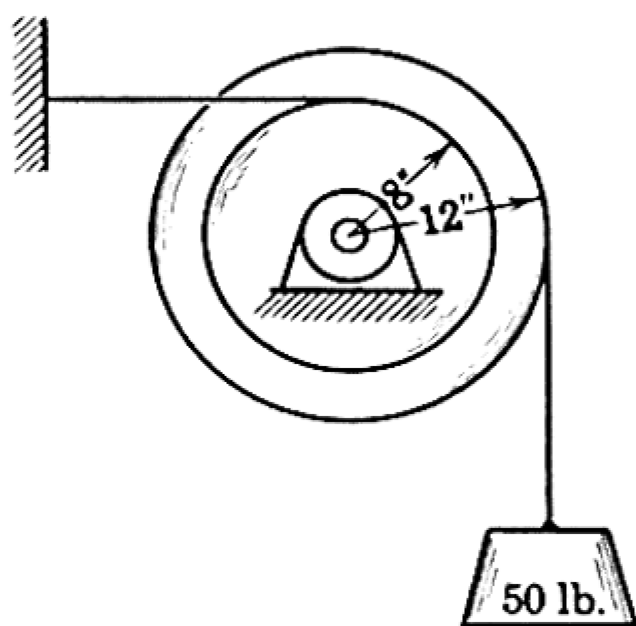
PROB. 134

PROB. 135

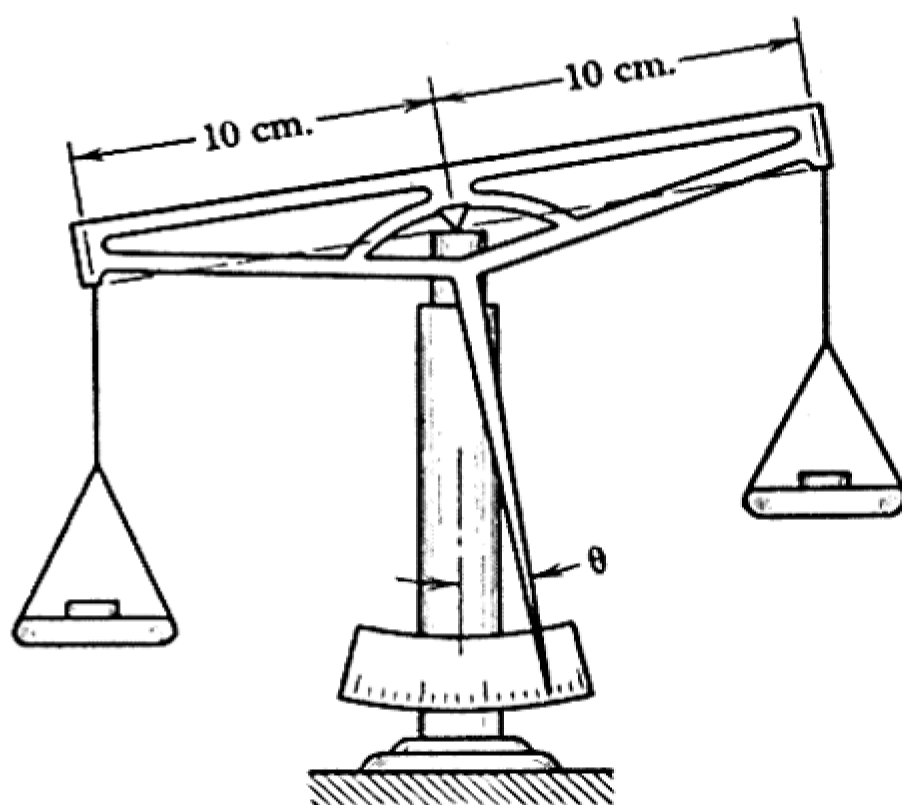
135. Determine the force on the pin at A due to the 4 kip load.

Ans. $A = 7.21$ kips

136. Determine the force R exerted by the bearing on the pulley. The pulley weighs 20 lb.



PROB. 136



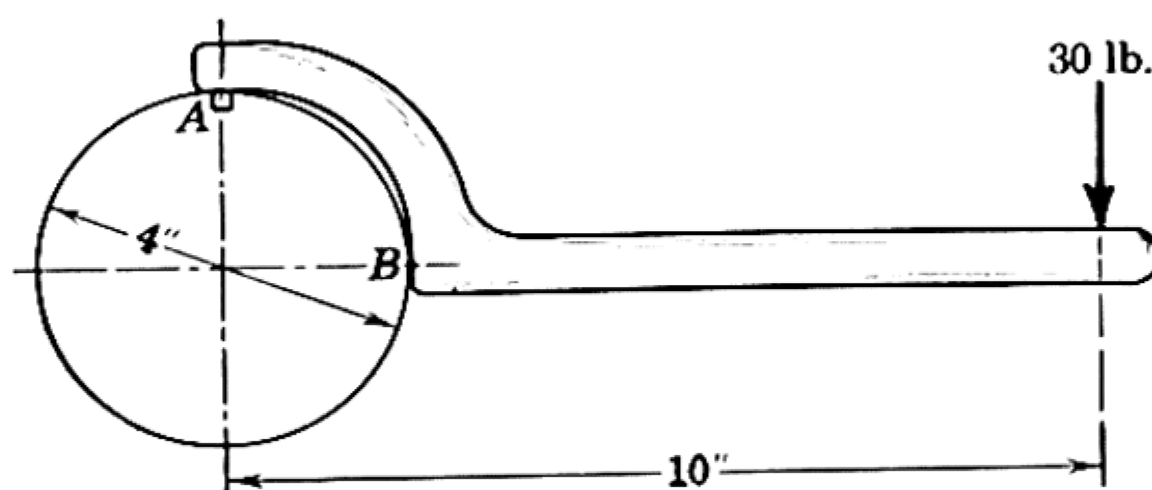
PROB. 138

137. Find the force exerted by the truss in Prob. 26 against the vertical support at A.

138. In the analytical balance shown the difference between the two weights in the pans is 50 mg. If the arm and pointer together weigh 180 gm. and the balance comes to rest when $\theta = 10$ deg., find the distance d of the center of gravity of the arm and pointer from the knife edge. When the weights in the pans are equal, the pointer comes to rest at $\theta = 0$.

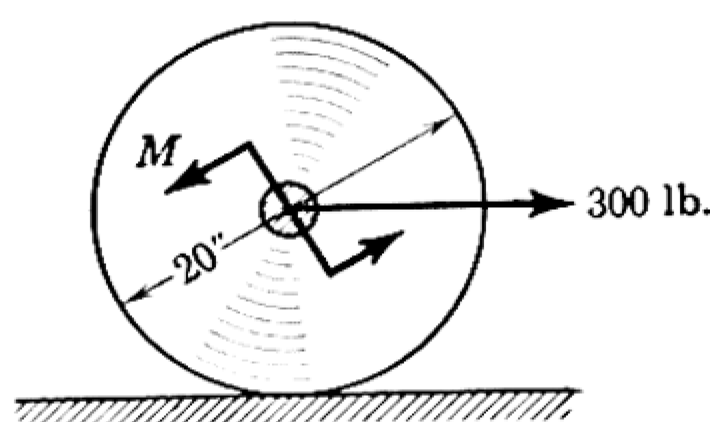
Ans. $d = 0.158$ mm.

139. A type of hook wrench known as a pin spanner is used for turning collars and shafts. For the applied load of 30 lb. determine the forces exerted at the contact points A and B. *Ans.* $A = 153$ lb., $B = 150$ lb.



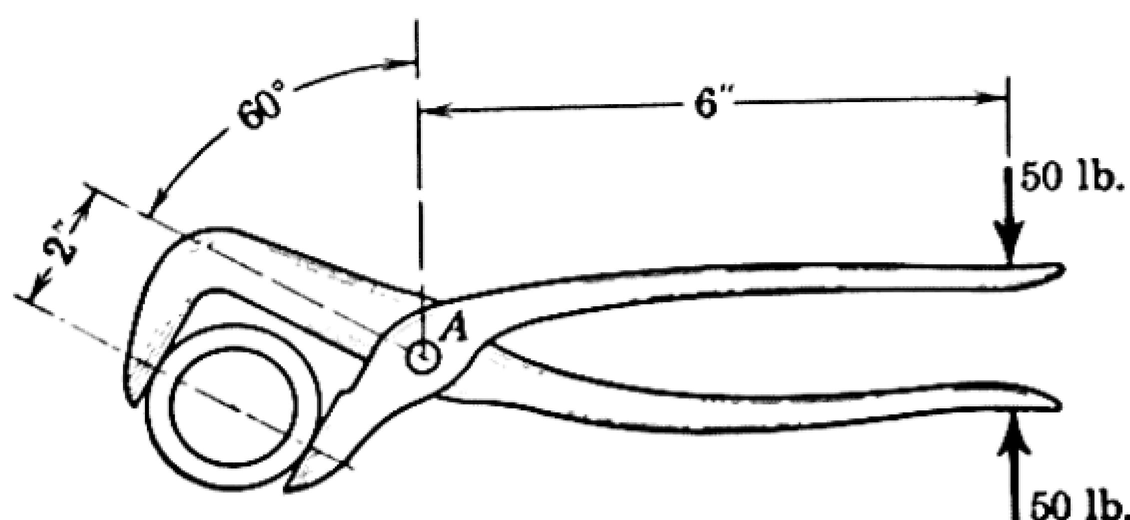
PROB. 139

140. The 20 in. diameter wheel has a combined weight and vertical axle loading of 1000 lb. and is subjected to a horizontal load of 300 lb. applied at the axle as indicated. Compute the counterclockwise torque M applied to the axle necessary to move the load, assuming that the wheel does not slip.



PROB. 140

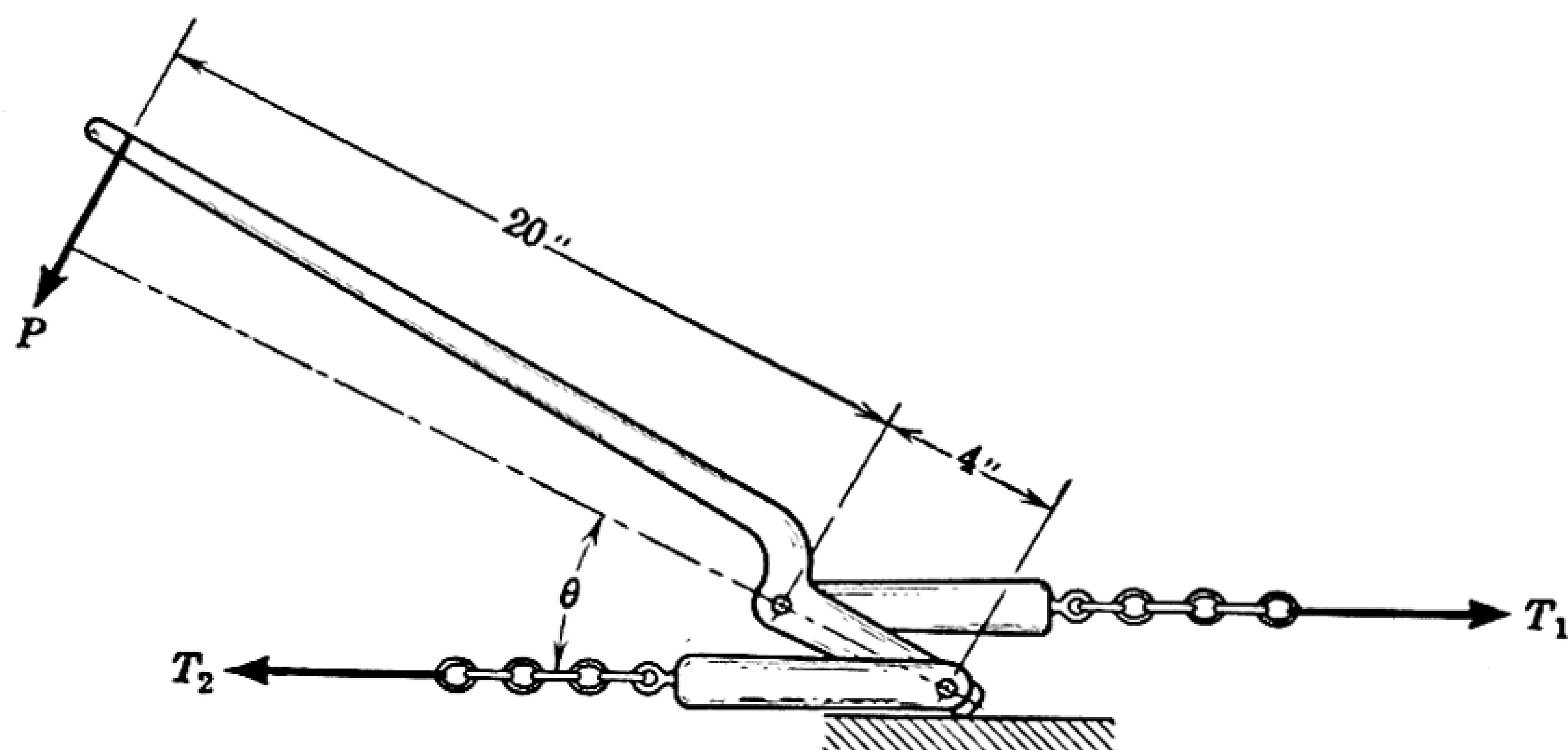
141. If the handles of the pliers are gripped with a force of 50 lb. as shown, determine the compressive force P on the tube and the reaction R on the pin at A. *Ans.* $P = 150$ lb., $R = 180$ lb.



PROB. 141

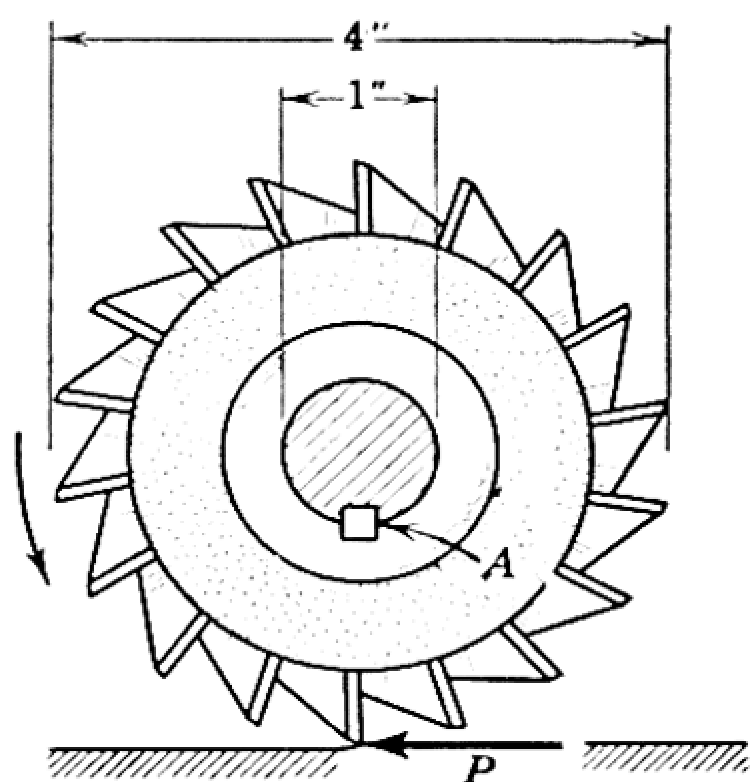
142. A chain binder is used to secure loads of logs, lumber, pipe, and the like. If the tension T_1 is 500 lb. when $\theta = 30$ deg., determine the force P required on

the lever and the tension T_2. Assume that the surface upon which the binder rests is perfectly smooth. *Ans.* $P = 41.7$ lb., $T_2 = 479$ lb.

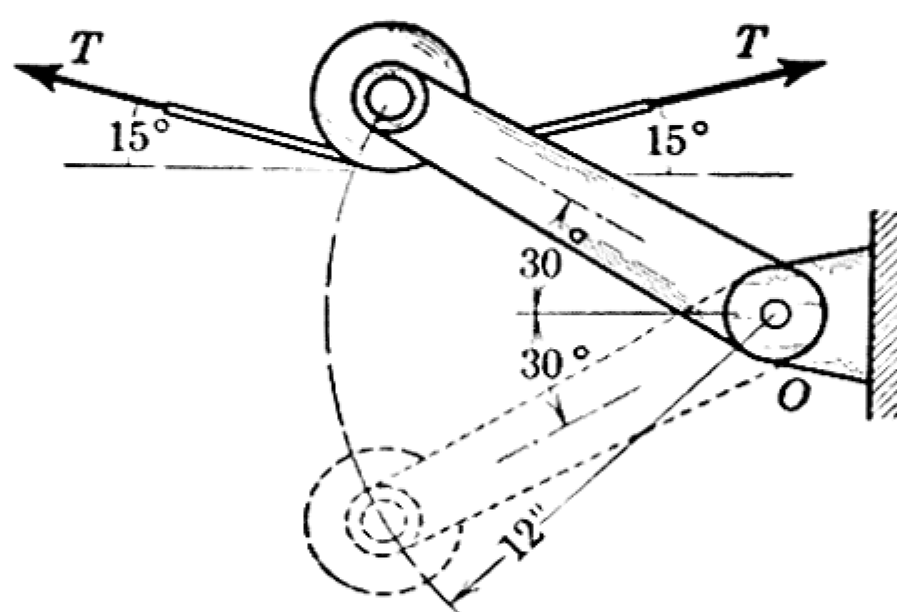


PROB. 142

143. A milling cutter is prevented from turning on its arbor by means of a key which bears horizontally against the cutter at point A in the position shown. If the motor drive supplies a torque of 100 lb. ft. to the arbor, determine the force F transmitted by the key to the cutter, the reaction R between the arbor and the smooth inner periphery of the cutter, and the tooth load P. *Ans.* $F = 2400$ lb., $R = 1800$ lb., $P = 600$ lb.



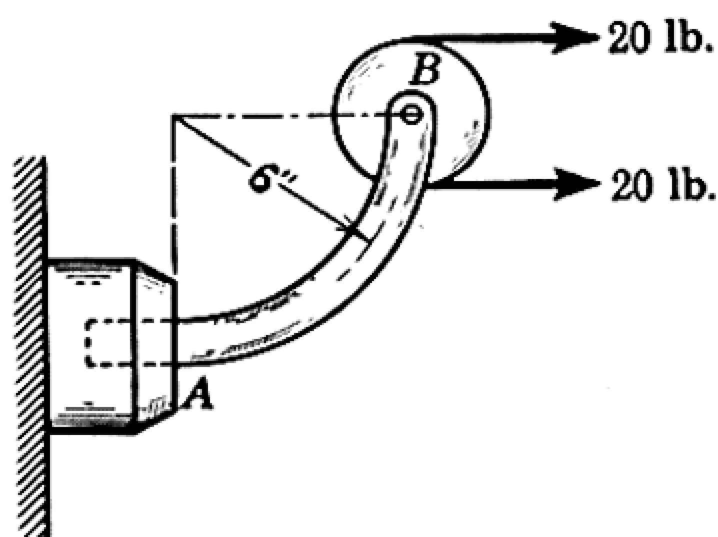
PROB. 143



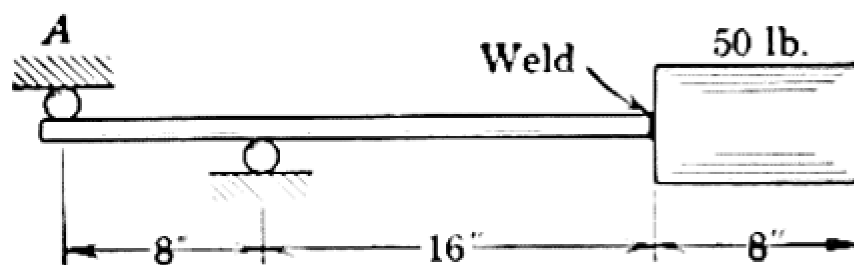
PROB. 144

144. The arm of an idler pulley is attached to one end of a coiled spring which fits over the shaft at O and is fixed at its other end. If a torque of 8 lb. in. will wind the spring through 1 deg. and the spring is untwisted with the arm in the dotted position, determine the tension T in the belt. Neglect the weight of the arm and pulley. *Ans.* $T = 89.3$ lb.

145. Draw the free-body diagram of the arm AB and determine the action of the fixed collar on the arm at A.



PROB. 145

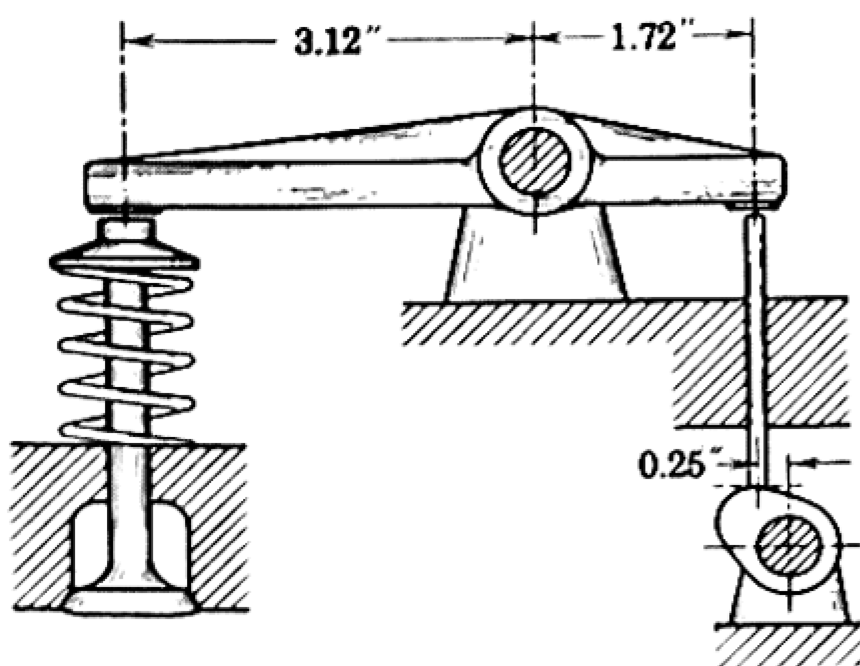


PROB. 146

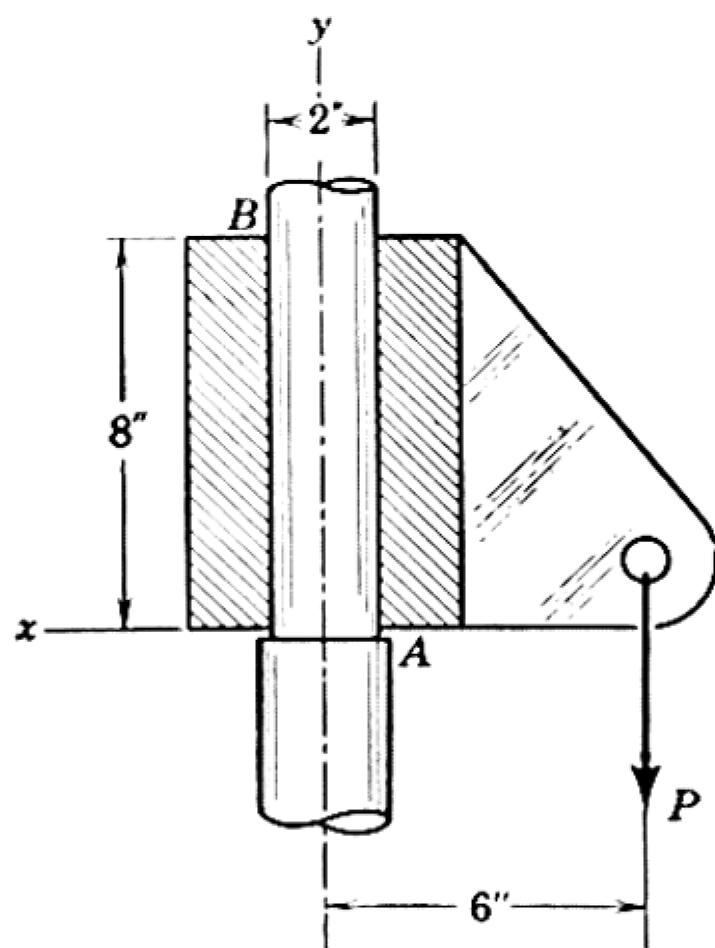
146. A 50 lb. steel cylinder 8 in. long is welded to the end of a light bar supported as indicated. Determine the reaction at A by considering, first, the free-body diagram of the bar and cylinder combined and, second, the free-body diagrams of the bar and cylinder separated.

147. The valve assembly shown contains a spring whose constant is 60 lb./in. When the valve is closed and before the push rod is in contact with the rocker arm, the spring has an initial compression of 0.82 in. Assume the contact surfaces between the cam and the push rod are perfectly smooth and determine the torque M on the cam shaft necessary to open the valve.

Ans. M = 22.3 lb. in.



PROB. 147



PROB. 149

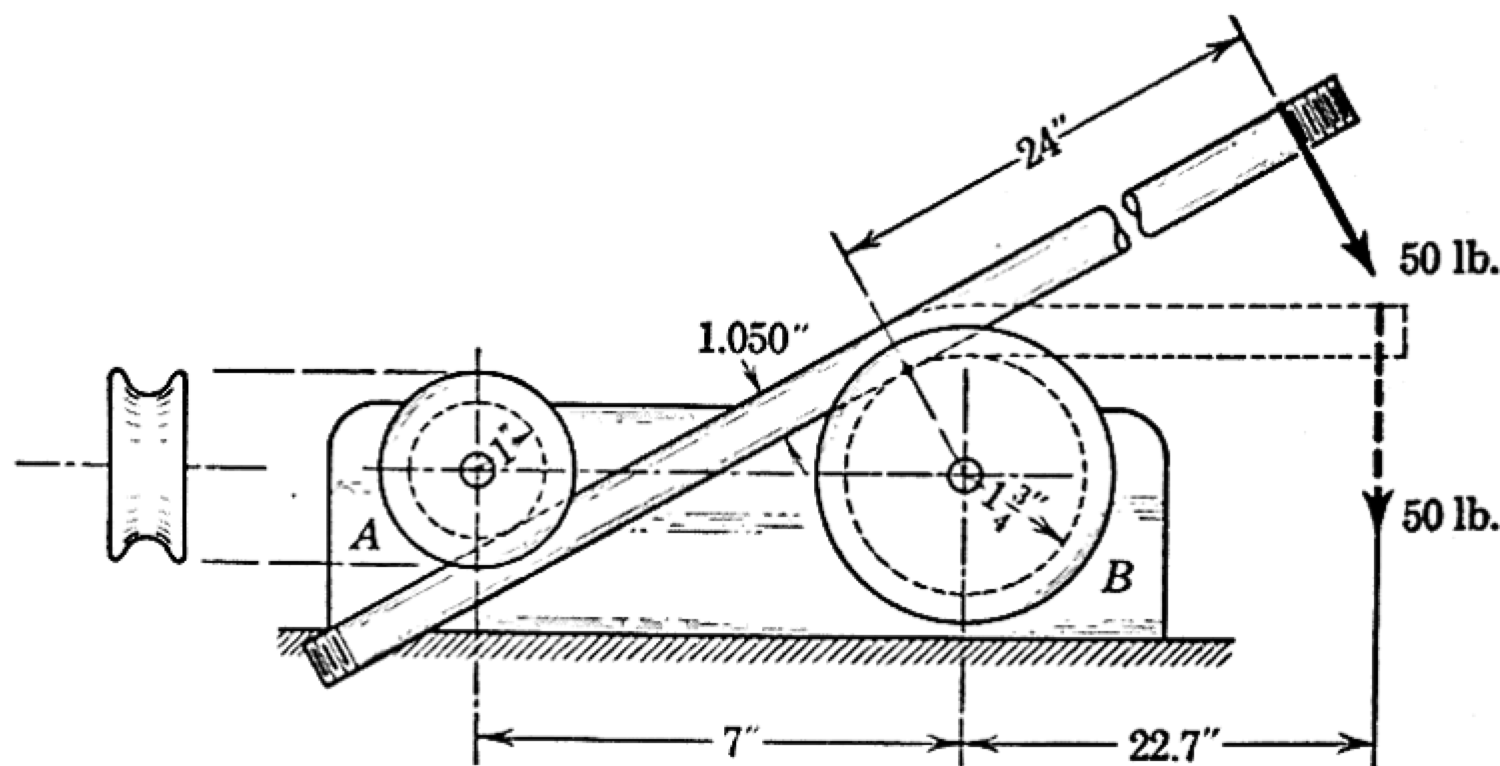
148. When the milling cutter of Prob. 143 has revolved through 270 deg. beyond the position shown, determine the reaction R between the arbor and the smooth inner periphery of the cutter.

149. The swivel bracket fits loosely on the fixed shaft and is supported vertically by the small shoulder at A. Determine the forces acting on the bracket for P = 1000 lb.

Ans. B_x = 625 lb. left, A_x = 625 lb. right, A_y = 1000 lb. up

150. A pipe bender consists of two grooved pulleys free to turn and mounted on a fixed frame as illustrated. For a 50 lb. load on the straight pipe determine the forces A and B on the bearings of the two pulleys similarly labeled.

Ans. A = 204 lb., B = 254 lb.

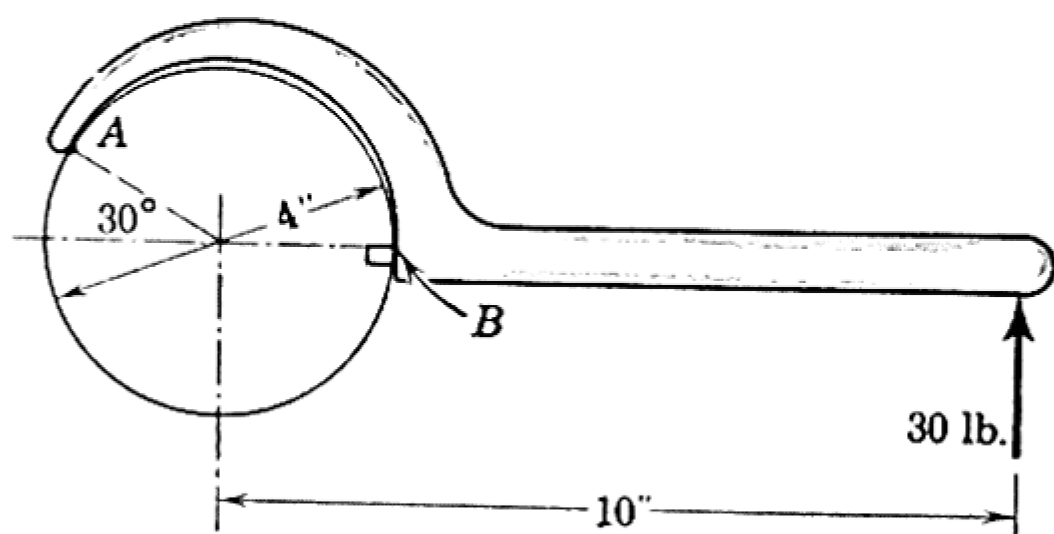


PROB. 150

151. When the pipe in Prob. 150 is bent into the dotted position shown, determine algebraically the forces A and B on the two pulley bearings.

152. Determine graphically the forces A and B fcr Prob. 151.

153. For the hook wrench shown determine the forces at the contact points A and B. Assume B to be on the horizontal center line.



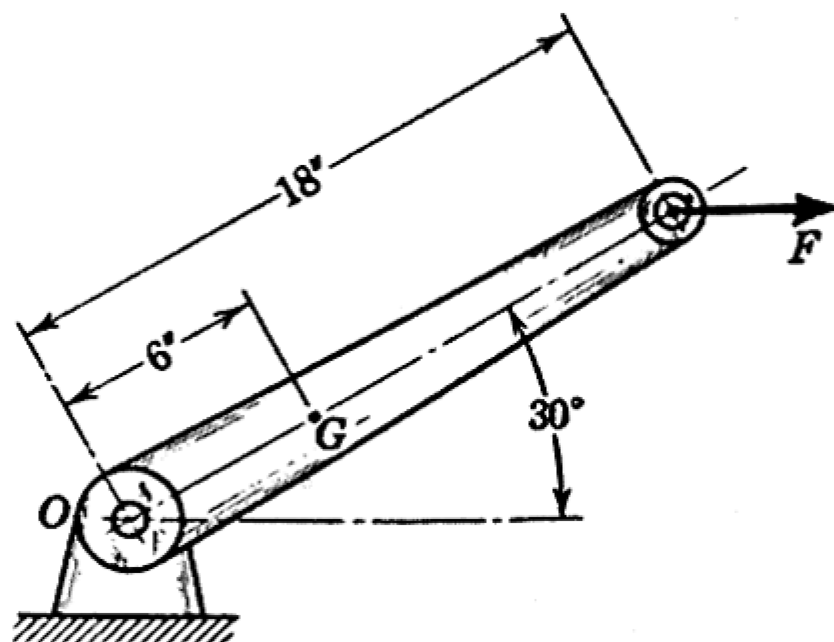
PROB. 153

154. On its way up the incline the lawn roller of Prob. 130 must roll over the edge of a 2 in. plank. If the handle is swung to the other side, determine the minimum pull P and the corresponding angle θ with the incline required to pull the roller over the plank.

Ans. P = 146 lb., θ = 36° 52′

155. The lever exerts a torque of 50 lb. ft. on its attached fixed shaft when the horizontal force F is acting. If the lever weighs 10 lb. and its center of gravity is at G, determine F and the bearing reaction at O.

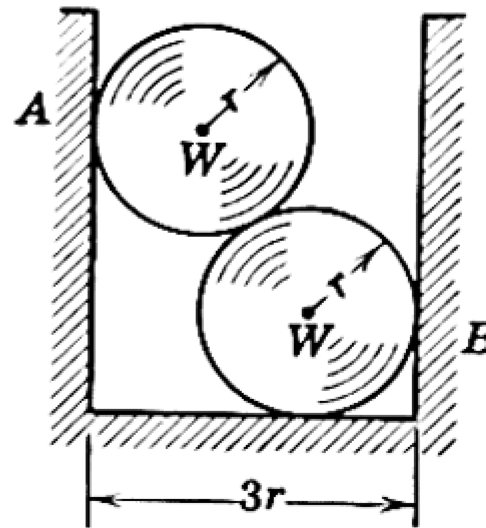
Ans. F = 60.9 lb., O = 61.7 lb.



PROB. 155

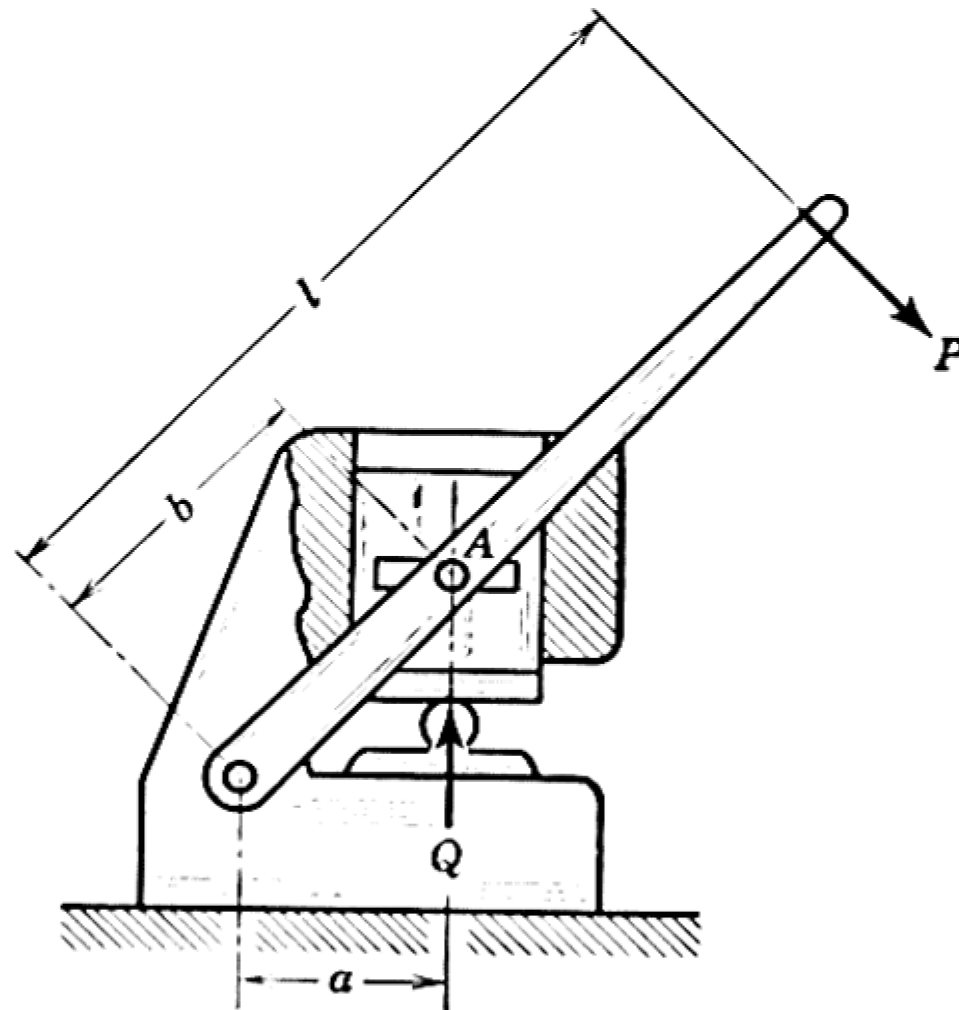
156. Find the reaction at A and B without considering the reaction between the cylinders. All surfaces are smooth.

Ans. $A = B = \dfrac{W}{\sqrt{3}}$



PROB. 156

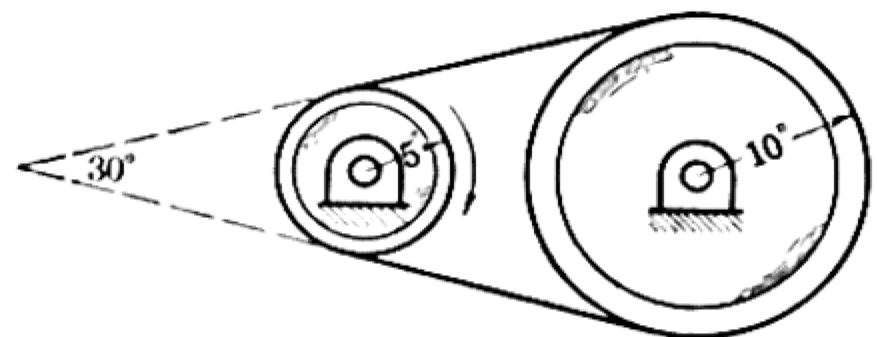
157. The blade of the bench shear illustrated is controlled by the hardened steel roller A which is attached to the handle and is confined to move in the slot of the blade. Treat all surfaces as perfectly smooth and determine the shearing force Q in terms of the applied load P.



PROB. 157

158. The motor which drives the small pulley supplies a pure torque (couple) of 250 lb. in. in the direction indicated. The tension T in the tight side of the belt is twice that in the slack side. Determine the torque M delivered by the shaft of the driven pulley, the belt tension T, and the bearing reaction R at the large pulley. Neglect pulley weight.

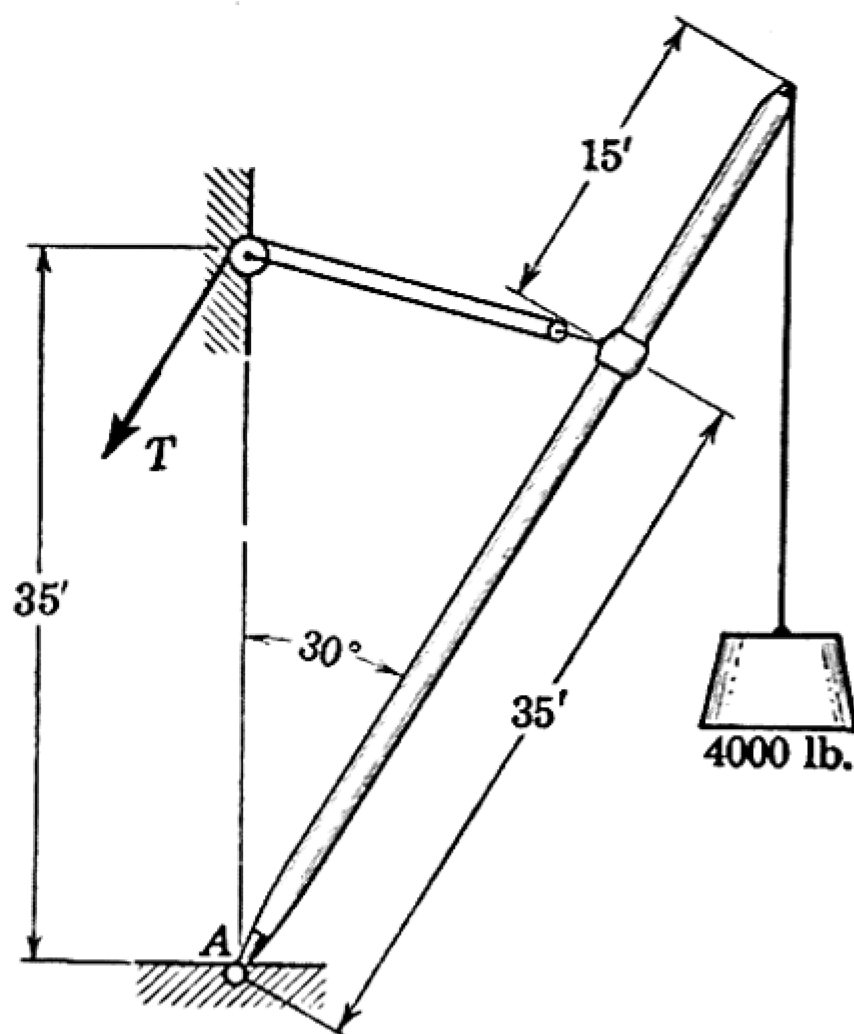
Ans. $M = 500$ lb. in., $T = 100$ lb., $R = 146$ lb.



PROB. 158

159. The boom weighs 1500 lb., and its center of gravity is at the midpoint of its length. Determine the tension T in the cable and the reaction on the ball joint at A.

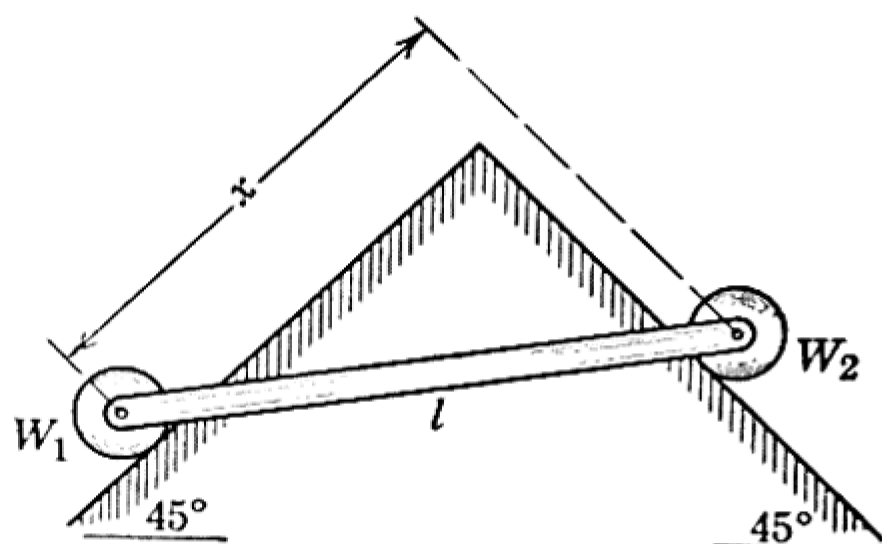
Ans. $T = 1760$ lb., $A = 5710$ lb.



Prob. 159

160. The two wheels weighing W_1 and W_2 are connected by a bar of length l whose weight is negligible. If the wheels are perfectly free to turn, determine the distance x for equilibrium when they are placed on the 45 deg. inclines.

Ans. $x = \dfrac{W_1 l}{\sqrt{W_1^2 + W_2^2}}$

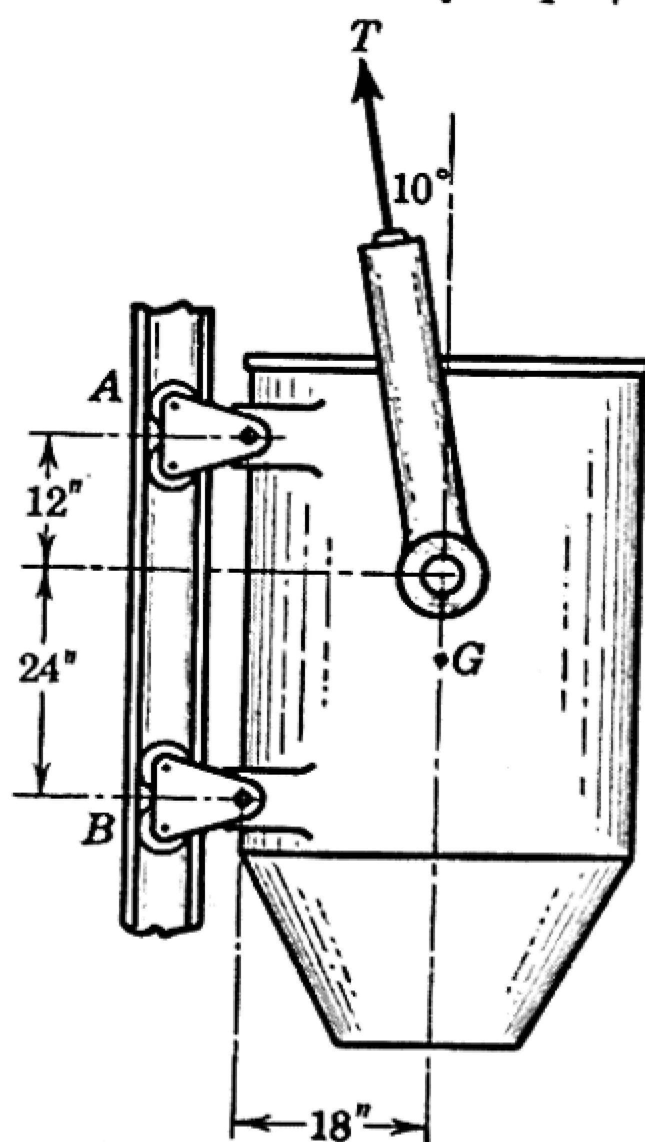


Prob. 160

161. Work Prob. 160 graphically for the case of $W_1 = 2W_2$ without resorting to a trial-and-error process.

162. A loaded concrete hopper with center of gravity at G weighs 6 tons and is elevated at constant velocity along its vertical guide by the cable tension T. Determine the side reactions at each set of guide wheels A and B.

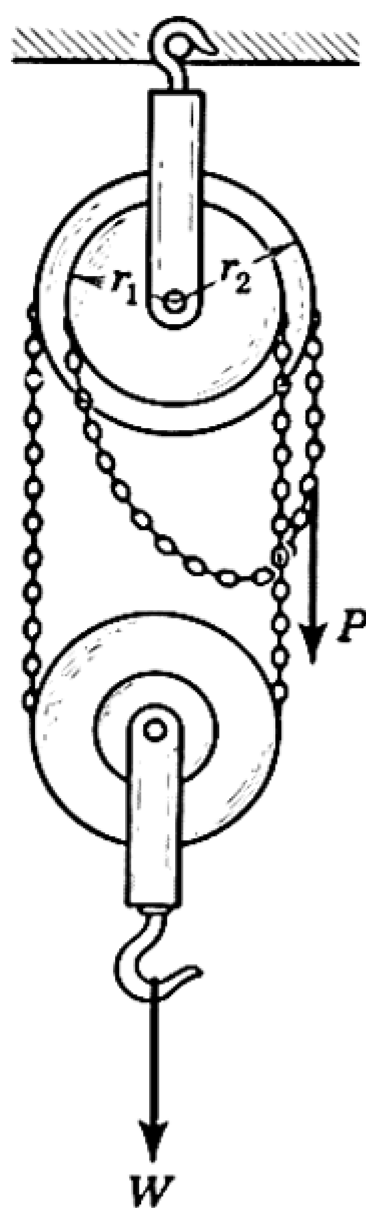
Ans. $A = 1410$ lb., $B = 706$ lb.



Prob. 162

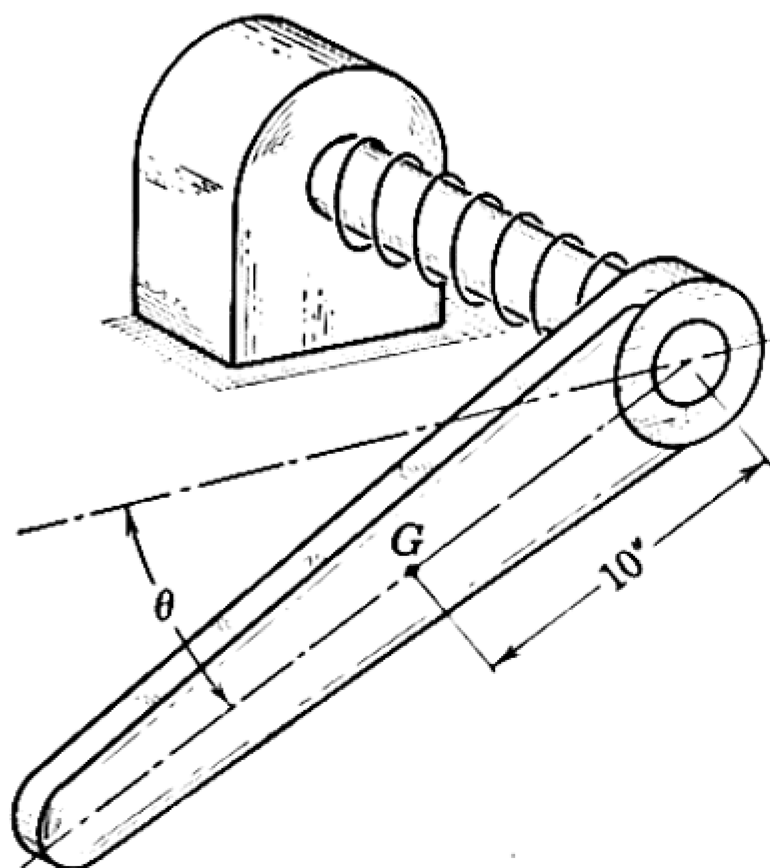
163. Determine the pull P required to lift the load W for the differential chain hoist shown. The two upper pulleys are fastened together, and the chain cannot slip on the pulleys.

Ans. $P = \frac{r_2 - r_1}{2r_2} W$



Prob. 163

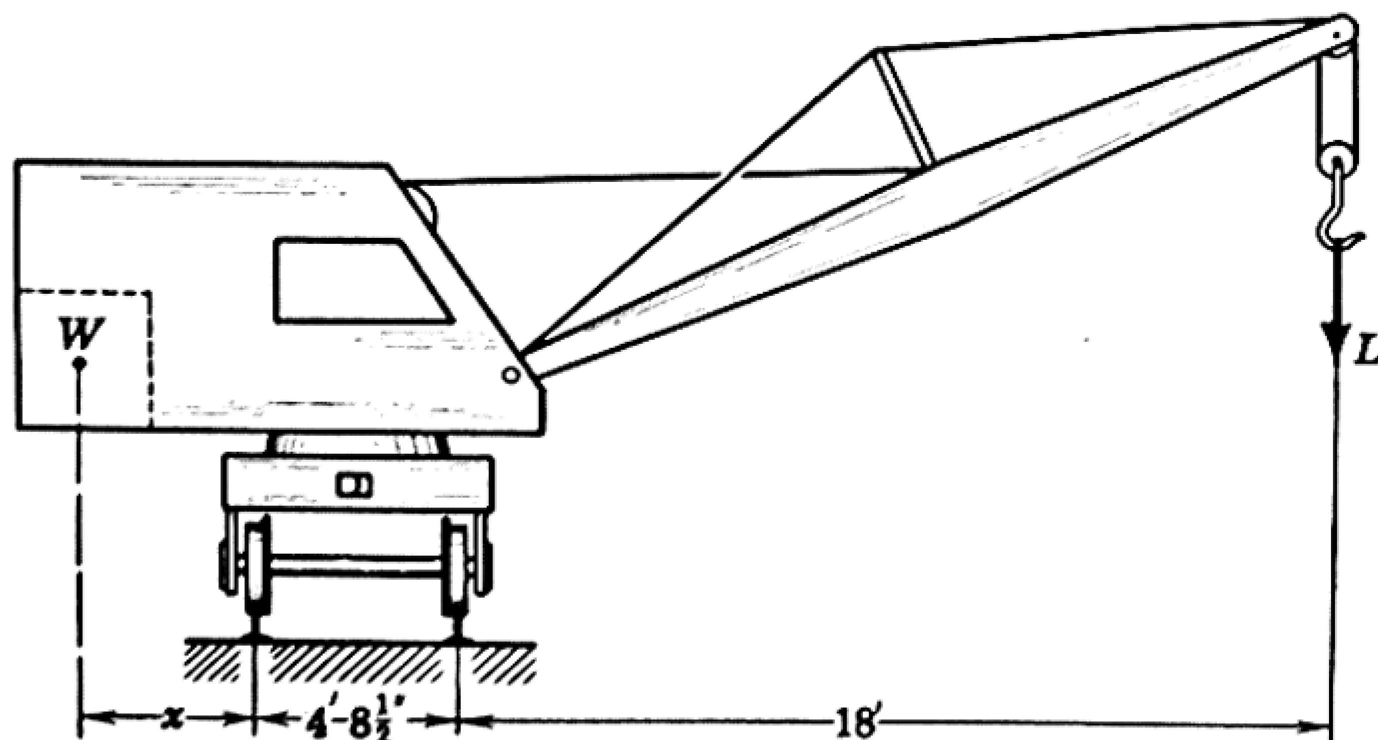
164. The center of gravity of the 50 lb. lever is at G. The lever is attached to the coil spring but otherwise is free to rotate the attached shaft in the bearing. If 4000 lb. in. of moment are required to twist the spring through 1 rev. and if the spring is untwisted when $\theta = 0$ in the horizontal position, determine θ for equilibrium.



Prob. 164

Ans. $\theta = 36.2$ deg.

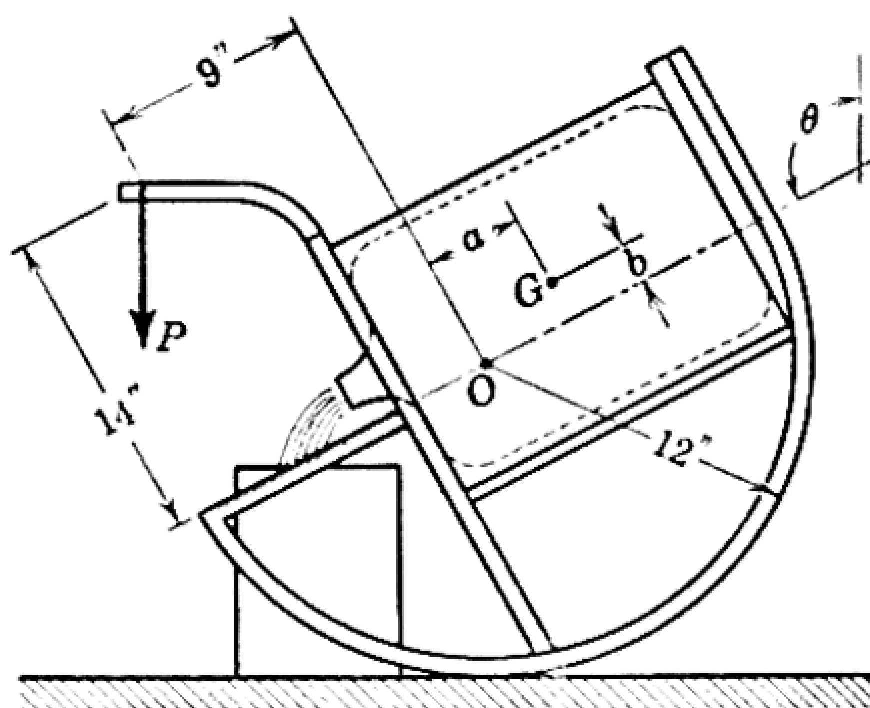
165. The center of gravity of a railway crane which weighs 50 tons including cab, boom, and car but excluding the counterweight W is on a vertical line midway between the rails. The maximum load L for which the crane is designed is 20 tons. Determine the values of W and x such that the car will not tip over for any load from zero to the maximum. *Ans.* $W = 26.4$ tons, $x = 4.45$ ft.



Prob. 165

166. In designing a carboy tilter it is desired that a full 96 lb. bottle with center of gravity at G will return to an upright position when released from an angle θ as great as 160 deg. Also it is desired that for $\theta = 135$ deg. the vertical tilting force P shall not exceed 10 lb. Locate the proper position for G by finding the distances a and b. Neglect the weight of the frame.

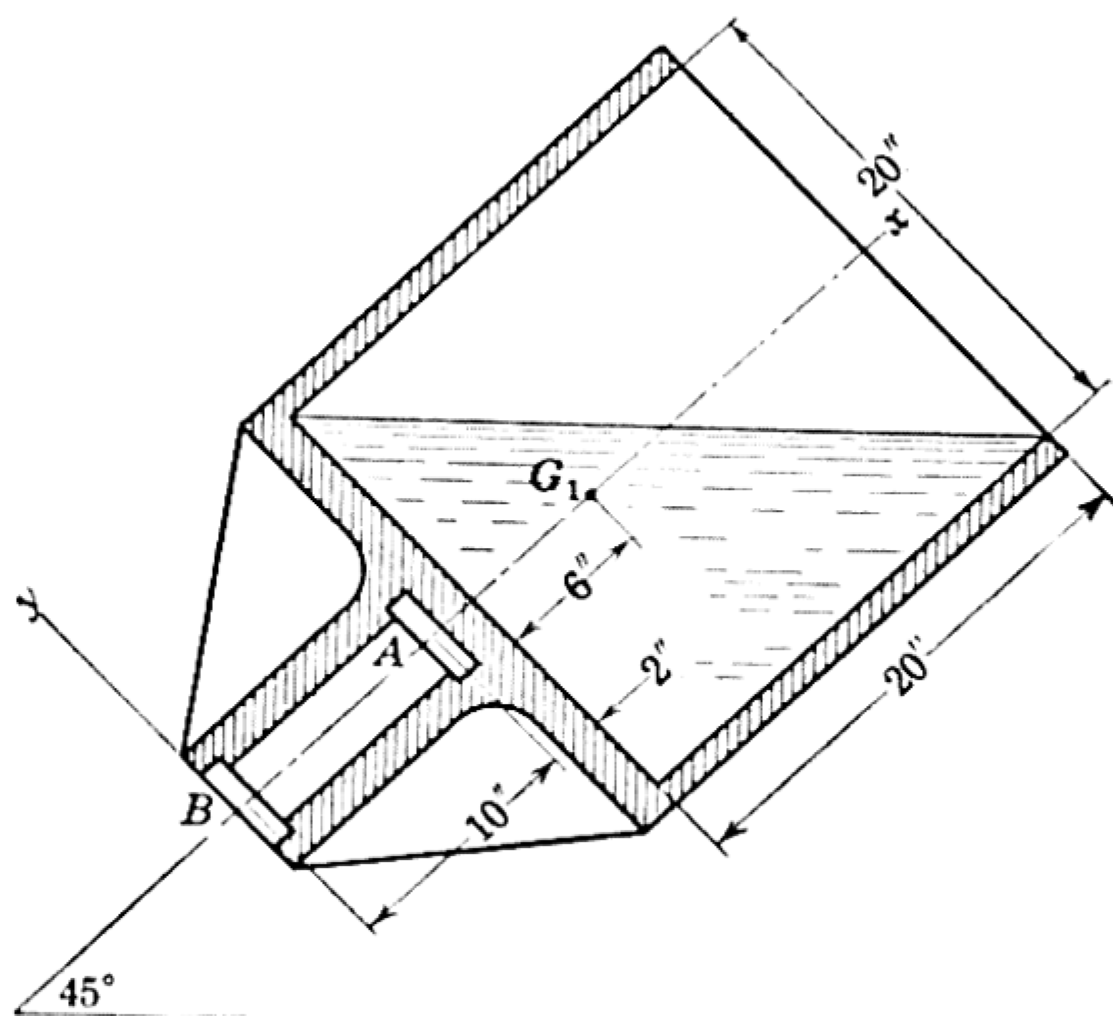
Ans. $a = 3.77$ in., $b = 1.37$ in.



Prob. 166

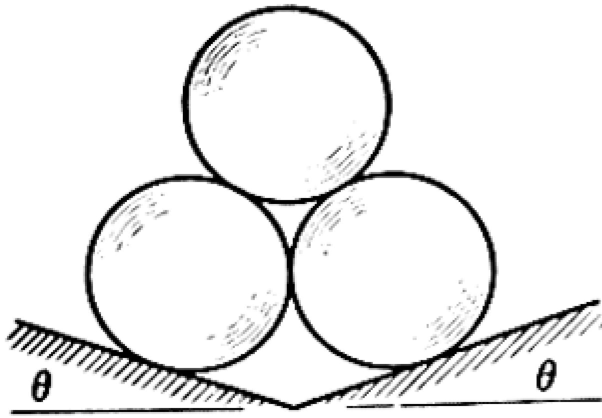
167. The revolving drum of a mixer has a diameter of 20 in. and is supported by the two ball bearings A and B. Assume that the entire thrust in the direction of the axis of the bearings is taken by A and determine the total forces supported by the bearings when the drum is tilted 45 deg. and filled to capacity with a liquid whose specific gravity is 1.20. The empty drum assembly weighs 60 lb., and its center of gravity is at G_1. (*Hint:* Use the results of Prob. 355 for the center of gravity of the liquid.)

Ans. $A_x = 139$ lb., $A_y = 276$ lb., $B_y = 137$ lb.

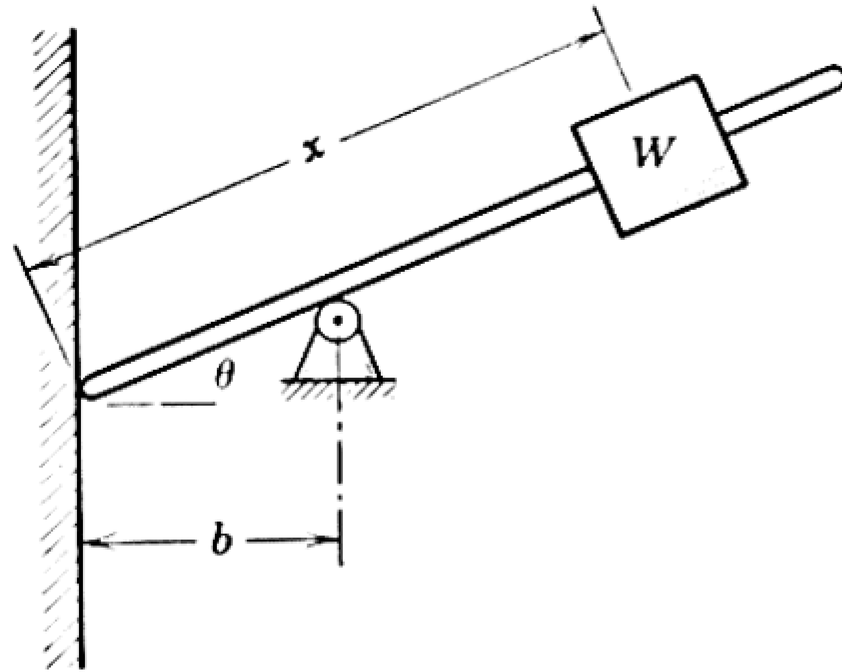


Prob. 167

168. Find the minimum angle θ such that the pile of three identical smooth rollers will not collapse. *Ans.* $\theta = 10° 55'$



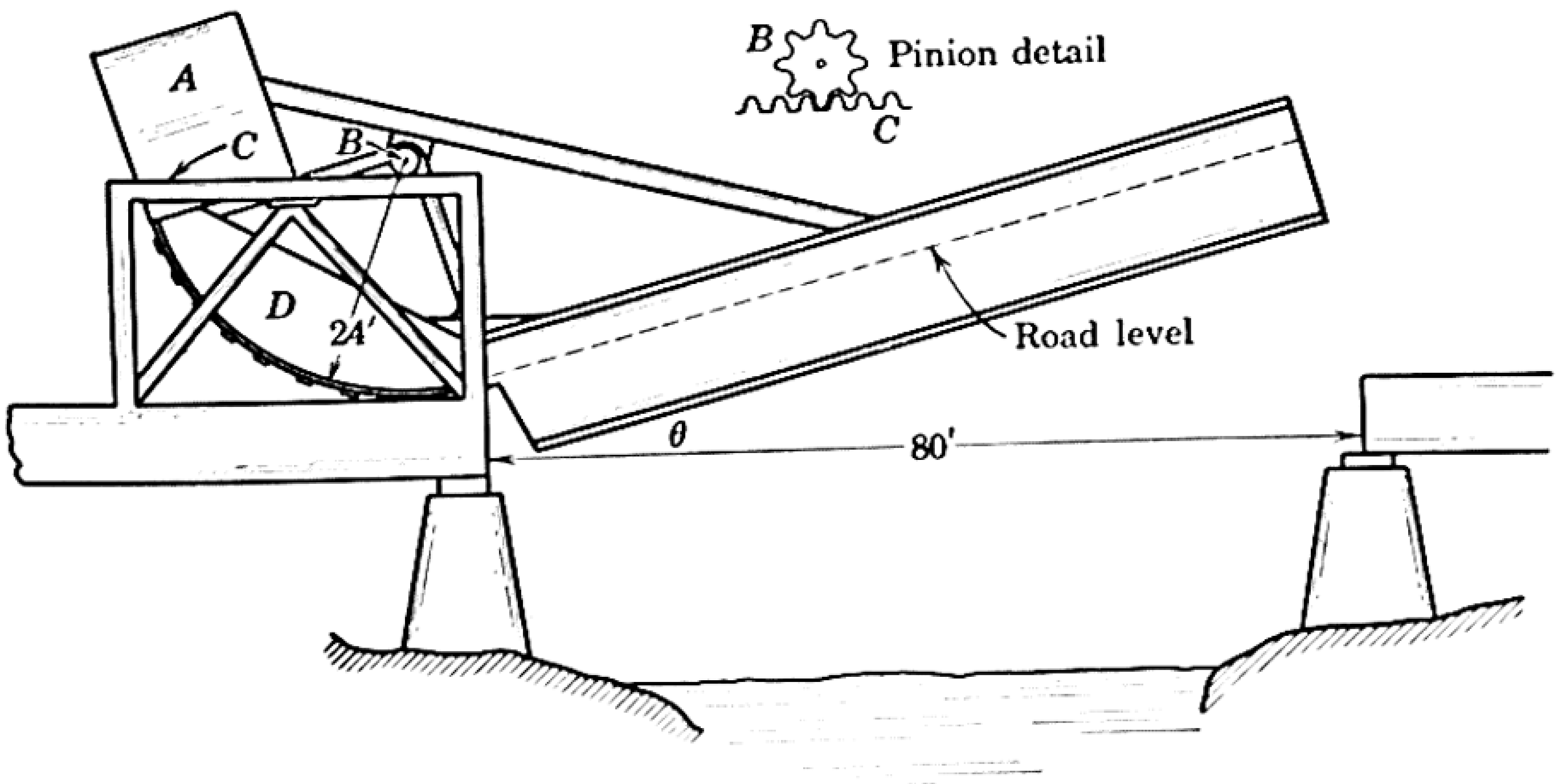
Prob. 168



Prob. 169

169. A smooth rod with negligible weight and small transverse dimensions carries a load W whose position is adjustable. The rod rests on a small roller and bears against the smooth vertical wall. Determine the distance x for any given value of θ so that the rod will be in equilibrium. *Ans.* $x = \frac{b}{\cos^3 \theta}$

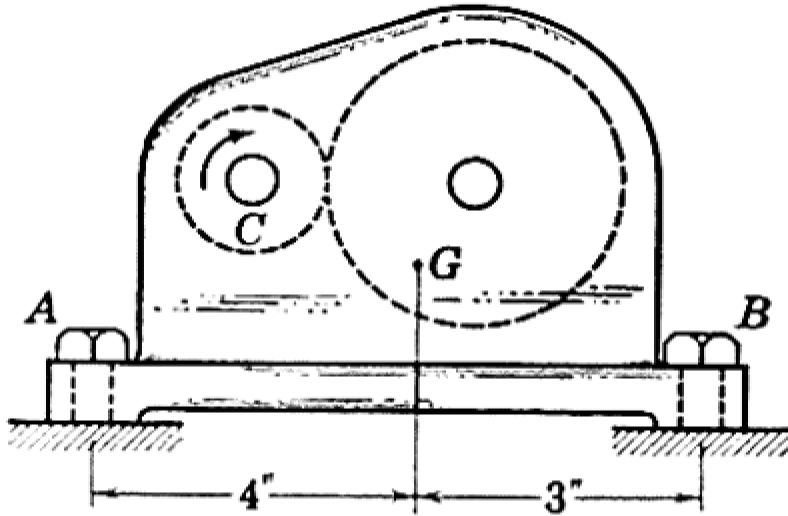
* **170.** When erected, the movable bridge was perfectly balanced for all positions. Later the bridge roadway was paved uniformly with 2.5 tons of asphalt, but the counterweight A was not increased. The two pinions at B (one on each



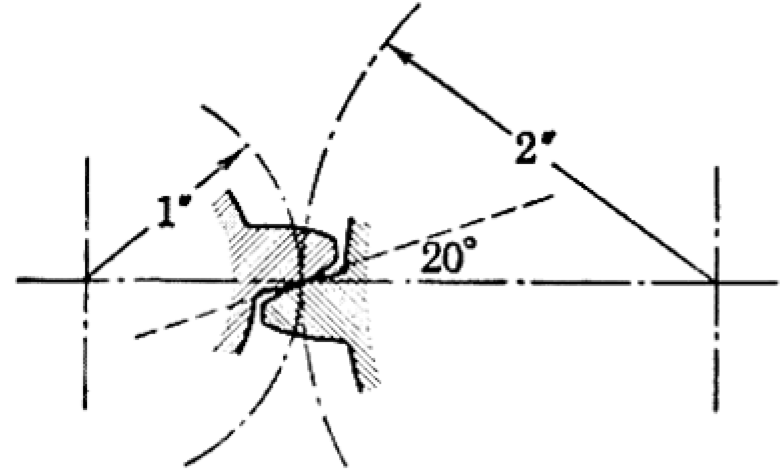
Prob. 170

side of the bridge) are driven by a single connecting shaft and mesh with the fixed rack C. This action rolls the bridge on the rocker segment D. Determine the torque M which must be applied to the pinion drive shaft in order to overcome the added roadway weight for $\theta = 20$ deg. The mean radius of each pinion is 5 in. *Ans.* $M = 4050$ lb. ft.

* **171.** The 2:1 speed reducer runs at constant speed and receives an input torque of 200 lb. in. at shaft C which turns clockwise. The unit weighs 12 lb., and its center of gravity is at G. Determine the reactions at the bolts A and B on the reducer. *Ans.* $A = 80.6$ lb. down, $B = 92.6$ lb. up



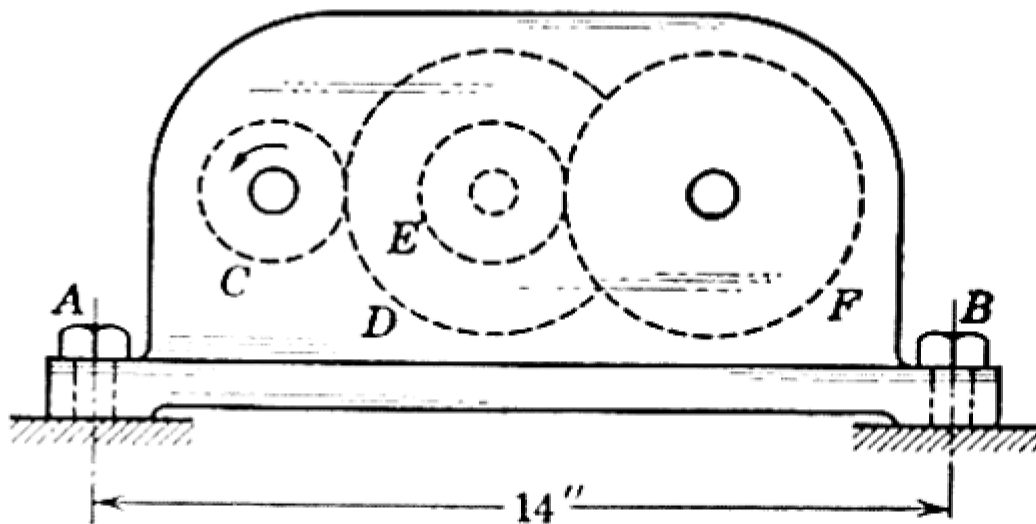
PROB. 171



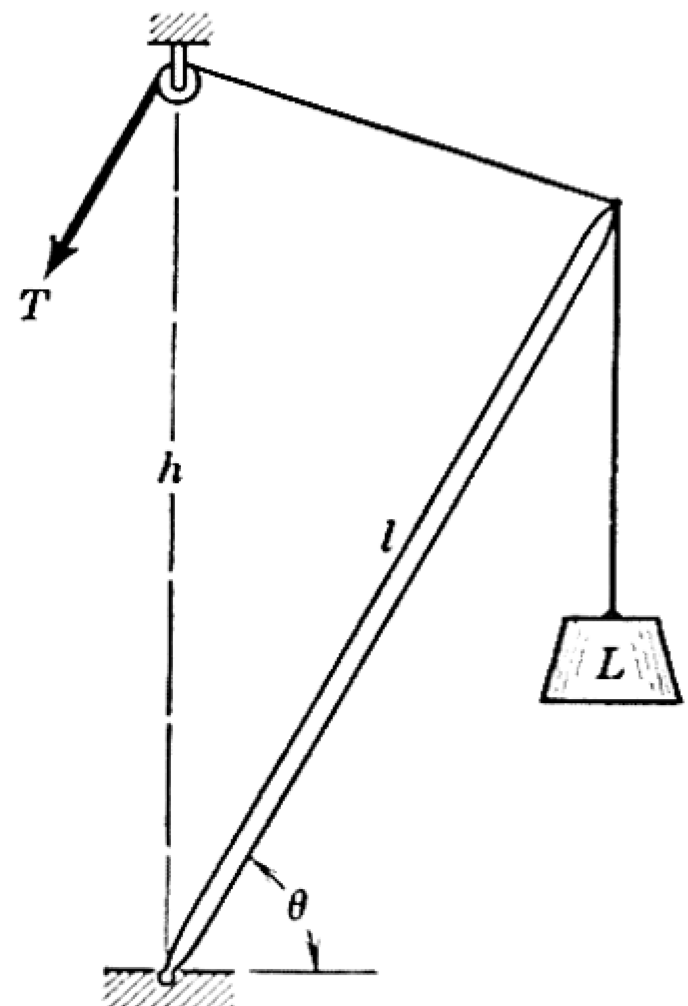
PROB. 172

172. The gear-tooth action for the reducer of Prob. 171 is shown in the accompanying sketch. Determine the resultant force exerted by each gear on its bearing.

* **173.** The speed reducer shown contains the pinion C which drives gear D with attached pinion E. Pinion E, in turn, drives gear F which runs the output shaft. Both speed reductions are 2:1. Determine the resultant forces exerted on the reducer at A and B if the reducer is operating with a counterclockwise input torque of 200 lb. in. on the shaft of pinion C. Neglect the weight of the reducer.



PROB. 173



PROB. 175

174. If the radii of the pinions C and E of the reducer of Prob. 173 are both 1.75 in. and the gear-tooth action is as shown in the figure for Prob. 172, determine the total bearing forces for the three gears C, D, and F.

* **175.** If the weight of the boom is negligible compared with the load L, find the compression C in the boom and prove algebraically that C is constant for all values of θ. Find the limiting value of the tension T as θ approaches 90 deg.

Ans. $C = \frac{l}{h} L$; $T_{\theta=90°} = \pm\left(1 - \frac{l}{h}\right) L,\ h \gtrless l$

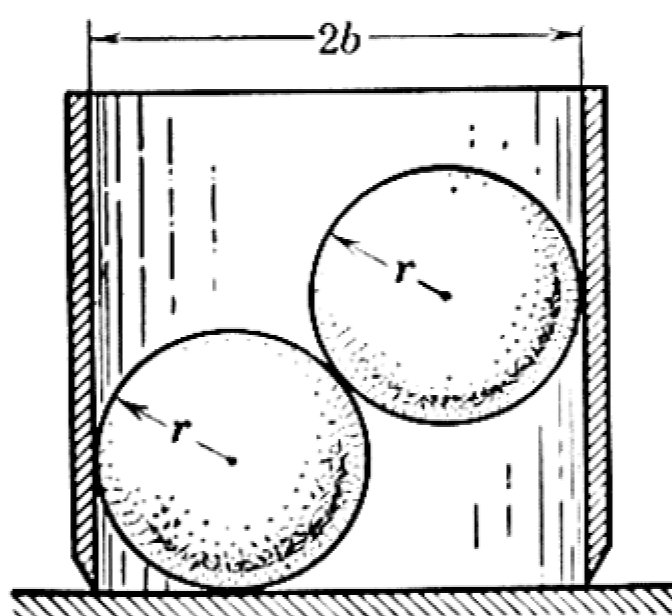
176. Solve Prob. 175 by using the similarity in the force polygon for the boom and the triangle defined by h, l, and the cable.

* **177.** As the angle θ for the chain binder of Prob. 142 becomes smaller, the chain tensions increase. If T_1 and T_2 are zero when $\theta = 90$ deg. and both tensions increase by 200 lb. for every inch which each chain is stretched, determine the maximum force P required to bind the chain and the corresponding angle θ at which the maximum pull occurs. Both chains remain parallel.

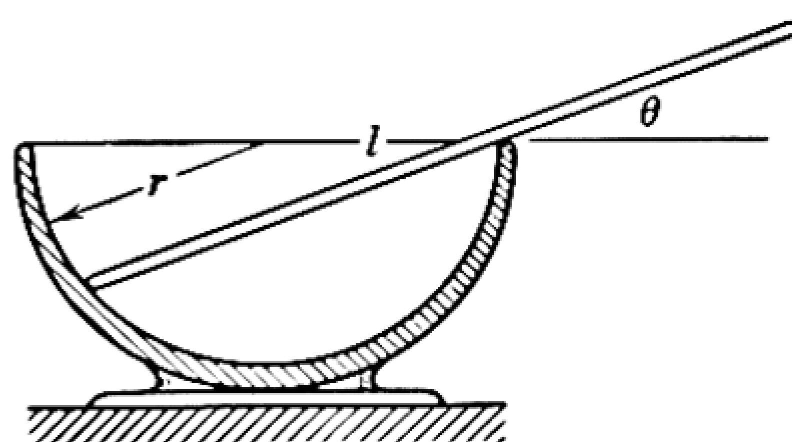
Ans. $P = 34.8$ lb., $\theta = 46° 15'$

* **178.** The "magic tube," if too short, will not stand upright on its base because of the overturning moment due to the hidden balls inside. Determine the minimum weight W of the tube such that it will not overturn. The weight of each ball is w, and $b < 2r$.

Ans. $W = 2\left(1 - \frac{r}{b}\right) w$



Prob. 178



Prob. 179

* **179.** A smooth uniform rod of length l rests in a smooth hemispherical dish of radius r as shown. Find the angle θ for equilibrium, assuming $2r < l < 4r$.

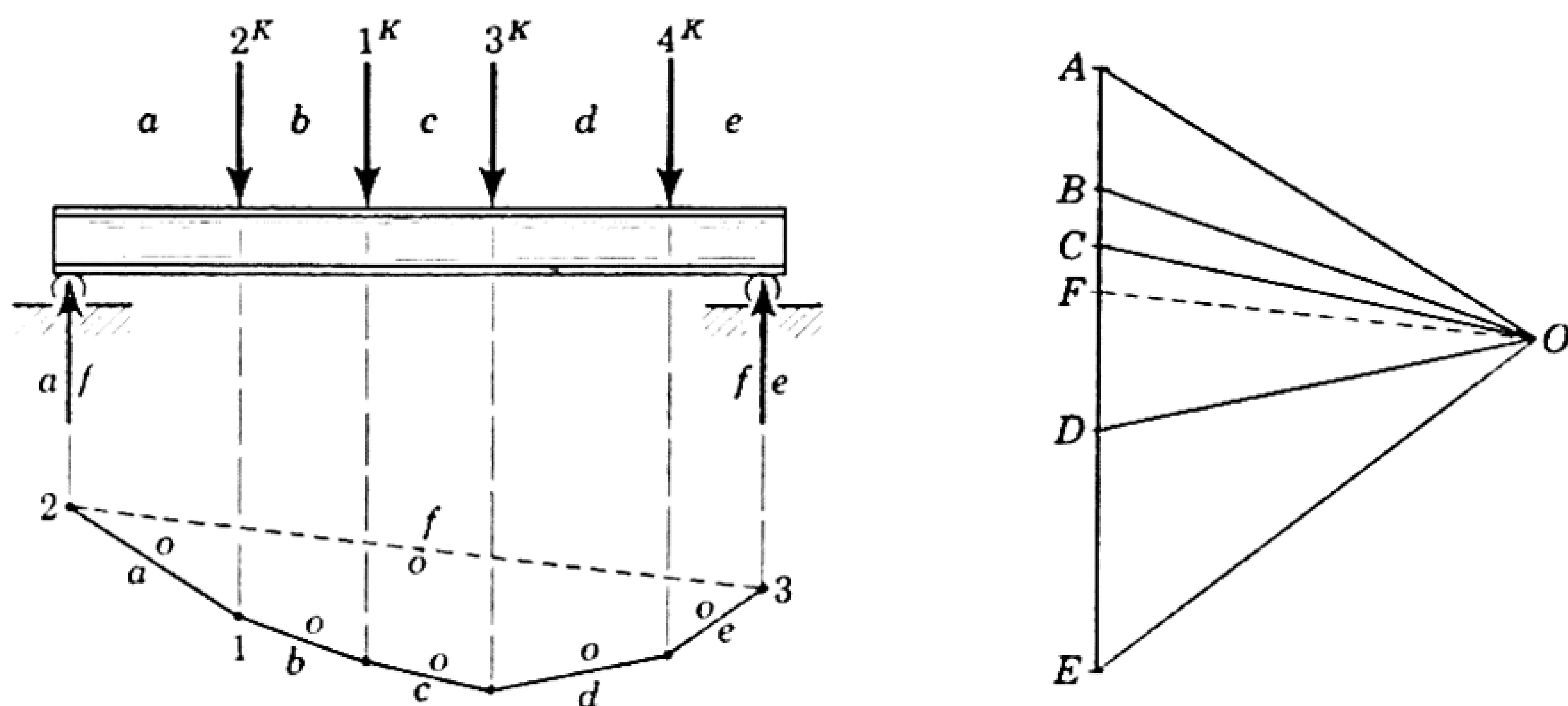
Ans. $\cos\theta = \frac{l}{16r} + \sqrt{\left(\frac{l}{16r}\right)^2 + \frac{1}{2}}$

23. Equilibrium by Funicular Polygon. In Art. 18 the resultant of a coplanar force system was found by the funicular polygon. If the resultant of the force system is zero, the body upon which it acts is in equilibrium. It follows that the force polygon must close (zero resultant force) and that the string polygon must also close (zero resultant moment or couple). In Fig. 22a the body would be in equilibrium under the action of F_1, F_2, F_3, and a force equal and opposite to the resultant AD. Such a force is called the *equilibrant.* Thus the equilibrant and the resultant force are always equal, opposite, and collinear. The equilibrant for a system of forces is that force which when added to the system produces equilibrium. In the event a system of forces has no resultant force but is not in equilibrium, then the equilibrant would be a couple equal and opposite to the residual couple of the given system.

In dealing with the equilibrium of a coplanar system of forces there may be as many as three unknown quantities since Eqs. (13) represent three independent conditions to be satisfied. Thus there are several possible combinations of magnitude, direction, and line of action which may be the unknowns. No general set of rules can be stated which covers the procedure for constructing the force and string polygons for all possible combinations of unknowns. The procedures for two frequently encountered combinations are described in the sample problems which follow.

SAMPLE PROBLEMS

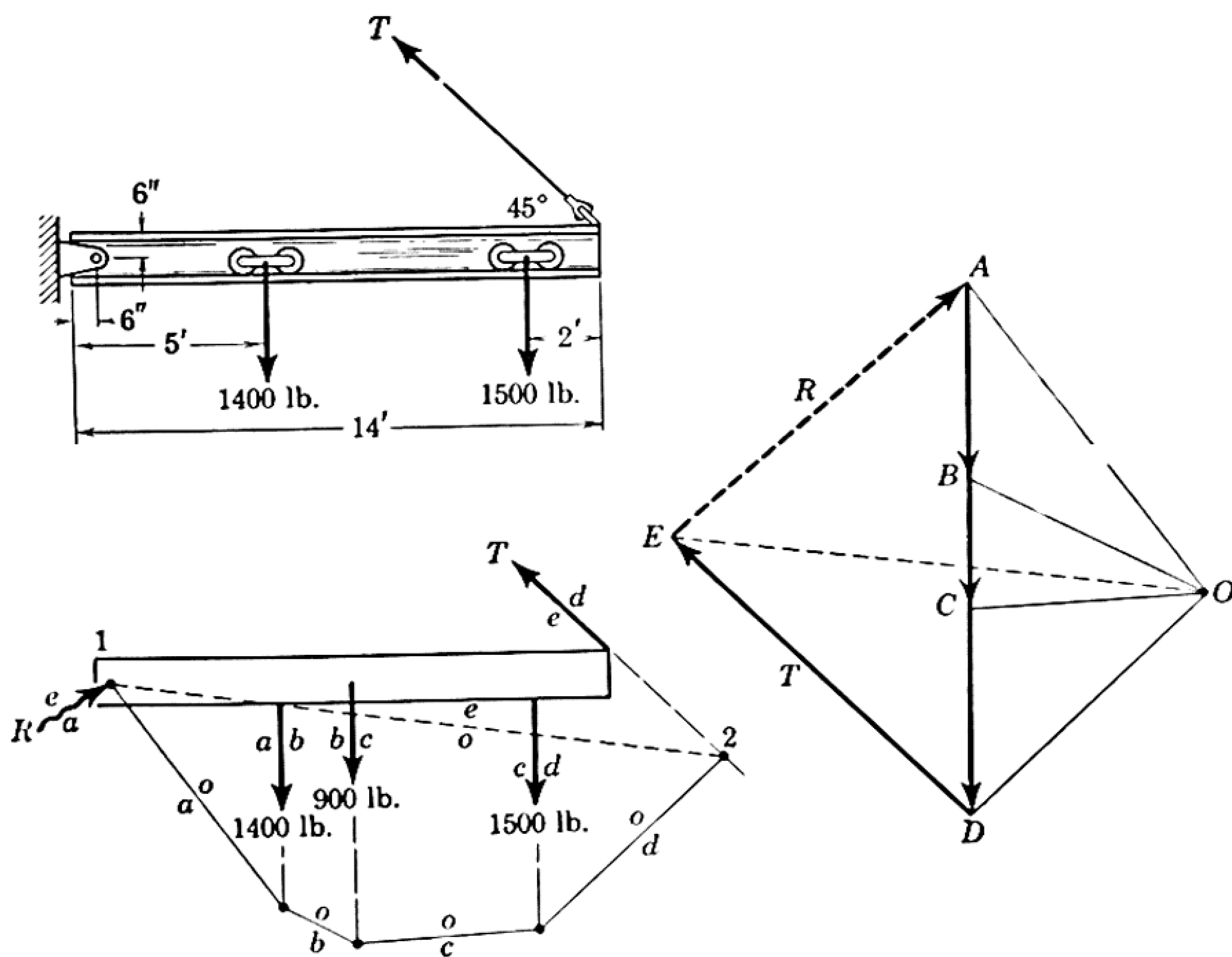
180. Determine by means of a funicular polygon the reactions at each end of the loaded beam of negligible weight.



Prob. 180

Solution: The beam is drawn to scale and the lines of action of the forces are labeled as shown. Adding the known loads designated by capital letters gives the line $ABCDE$. The pole O is arbitrarily selected and the rays AO, BO, CO, DO, and EO are drawn. The string polygon is begun at any position, such as point 1, on the line of action of one of the known loads. The strings are constructed parallel to their corresponding rays as explained in Art. 18. By this process points 2 and 3 are located. In order that the resultant shall not be a couple, the string polygon must close. Thus the closing string *fo* may be drawn connecting points 2 and 3. It follows that the unknown ray FO in the force polygon may now be drawn parallel to *fo*, and point F is thereby determined. The reaction on the right end is EF and scales 6.3 kips. The reaction FA on the left end scales 3.7 kips.

181. The uniform I-beam of the jib crane weighs 900 lb. and is loaded as shown. Determine by means of a funicular polygon the reaction R at the left end of the beam and the tension T.



PROB. 181

Solution: The free-body diagram of the beam is drawn to scale, and the reaction R, unknown in magnitude and direction, is shown by a wavy arrow. The lines of action of the forces are labeled with small letters in the order in which they will be added, and the force polygon is begun by adding the three known forces AB, BC, and CD. A line through D with the known direction of the tension DE is also constructed. Point E is still unknown. The pole O is arbitrarily selected and the rays drawn. Point 1 is the only known point on the line of action of R so the string polygon should begin here. The strings *ao*, *bo*, *co*, and *do* are constructed parallel to their respective rays in the force polygon, and point 2 on the line of action of T is established. Since the string polygon must close for equilibrium, the closing string *eo* is drawn connecting points 1 and 2. Point E on the force polygon is now located by constructing the ray OE through O parallel to the string *oe*. The magnitude of the tension T is now established, and R, the closing leg of the polygon, is obtained in magnitude and direction. Scaling the values from the polygon gives

$$T = 2970 \text{ lb.} \qquad \text{and} \qquad R = 2700 \text{ lb.} \qquad \textit{Ans.}$$

PROBLEMS

Solve the following problems by means of a funicular polygon.

182. Prob. 122.

183. Determine the vertical reactions at A and B for the beam of Prob. 67.

184. Determine the reactions at each end of the truss of Prob. 68.

185. In Prob. 127 determine the direction of the pin reaction at A from the funicular polygon and not from the concurrency principle.

186. Prob. 119.

187. Determine the reaction on the rocker C of the truss in Prob. 74.

Ans. $C = 11.0$ tons

188. Prob. 159.

189. Determine the reaction R at the pin support for the truss of Prob. 72.

Ans. $R = 707$ lb.

190. Prob. 162.

191. Cut the beam in Prob. 66 at its support and determine the resultant vertical force (shear) Q and the bending moment M acting on the cut section.

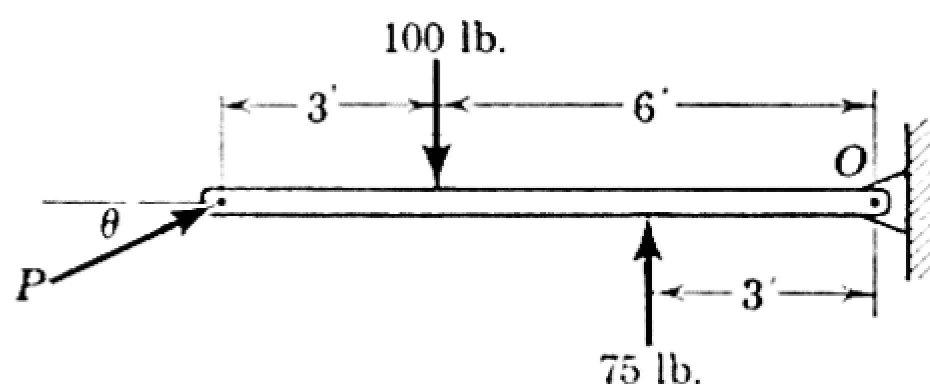
Ans. $Q = 260$ lb., $M = 520$ lb. ft.

192. If the pulley in Prob. 76 is in equilibrium, determine the tension T and the bearing reaction R.

193. Cut the beam in Prob. 82 at its support and determine the total force R and the bending moment M acting on the center of the cut section.

Ans. $R = 599$ lb., $M = 870$ lb. ft.

194. Determine F, P, and θ for Prob. 83 by introducing the magnitude of the applied couple into the known part of the string polygon.



PROB. 195

195. Determine the magnitude of P and the corresponding acute angle θ which will cause the reaction at O to be 100 lb. in the equilibrium position shown. (*Hint:* The vertical components of P and the reaction at O can be determined independently of the horizontal components.)

24. Equilibrium in Three Dimensions. When applied to the general three-dimensional problem, the vector equations of equilibrium, Eqs. (12), may be written in scalar form as

$$\Sigma F_x = 0, \qquad \Sigma M_x = 0,$$
$$\Sigma F_y = 0, \qquad \Sigma M_y = 0, \qquad (16)$$
$$\Sigma F_z = 0, \qquad \Sigma M_z = 0.$$

Physically the first three of these six equations state that for a body in equilibrium there is no resultant force acting on it in any of the

three directions. Also the second three equations express the requirement that there is no resultant twist or moment about any of the coordinate axes or about axes parallel to the coordinate axes. These six equations are the necessary conditions for complete equilibrium. Equations (16) are entirely independent, and any of them may hold without the others, in which case the body would be in partial equilibrium only. If, for instance, $\Sigma M_x \neq 0$ but the remaining five equations are satisfied, then this combination would describe a body, such as a wheel, fixed at its center of gravity but accelerating in its rotation about the x-axis.

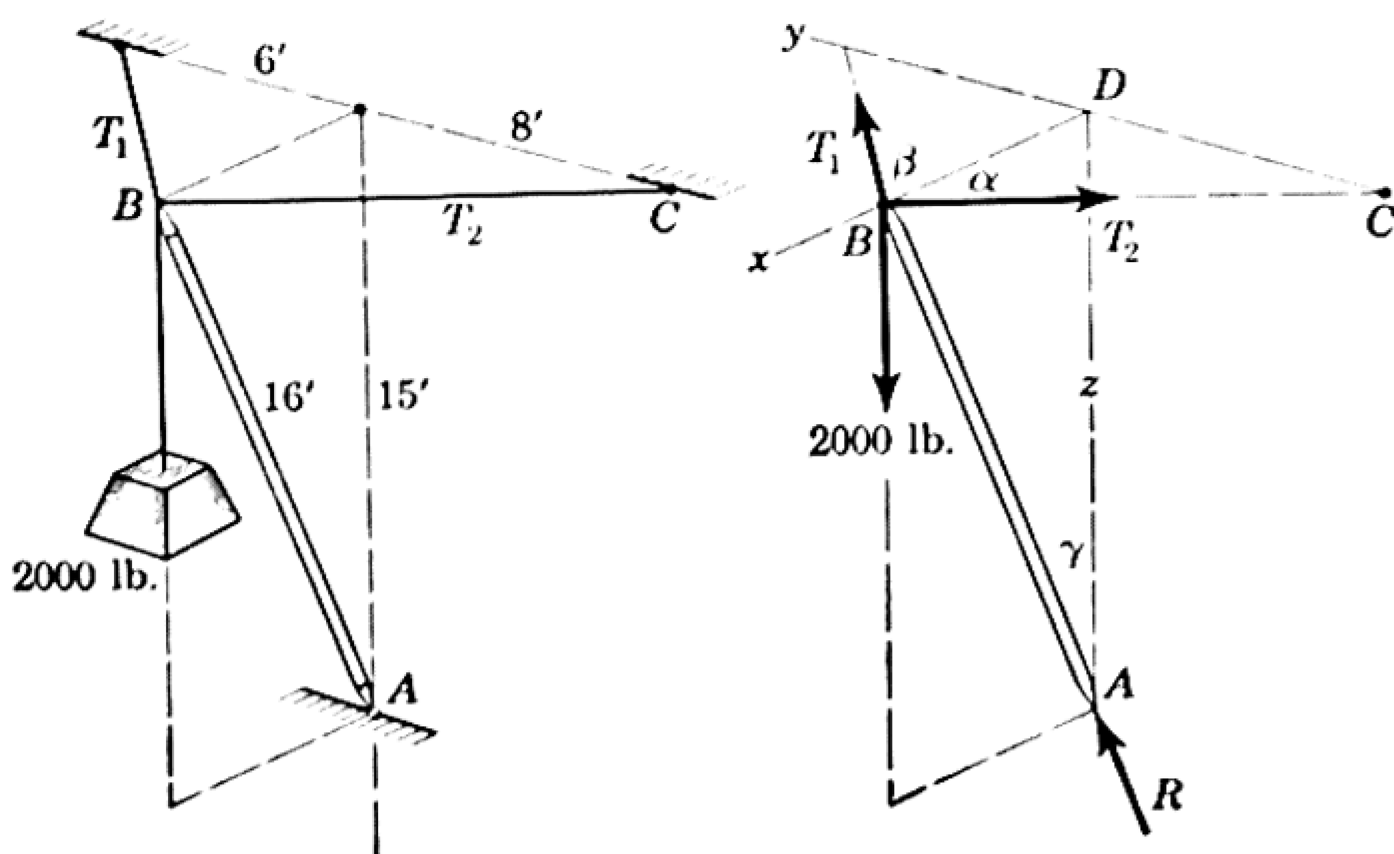
Most three-dimensional problems are simplified if they are broken down into three related two-dimensional problems. Projection of the forces on the three coordinate planes results in the three two-dimensional systems involving the force components in the respective planes.

When equilibrium is produced by concurrent forces, the origin of coordinates is conveniently chosen at the point of concurrency. Since the moments of all forces are zero about any axis through this point, it follows that the equilibrium of concurrent forces is specified completely by the first three of Eqs. (16).

In the case of equilibrium produced by parallel forces, again only three conditions must be satisfied. If the forces are all perpendicular to the x–y plane, for example, then $\Sigma F_z = 0$. The moments of the forces about the other two axes must also be zero, so that $\Sigma M_x = 0$ and $\Sigma M_y = 0$.

SAMPLE PROBLEMS

196. Determine the compression R in the boom and the tensions T_1 and T_2 in the supporting cables if the weight of the boom is neglected compared with the applied load.



Prob. 196

Solution: A space view of the free-body diagram of the boom is shown. For equilibrium the compression R must pass through B, the point of concurrency of the other three forces, and is therefore in the direction of the boom axis. From the geometry of the figure the following values, needed in the computation, are obtained,

$$BD = 5.57 \text{ ft.}$$

$$\cos\gamma = 0.938, \qquad \sin\gamma = 0.348,$$

$$\cos\beta = 0.680, \qquad \sin\beta = 0.733,$$

$$\cos\alpha = 0.571, \qquad \sin\alpha = 0.821.$$

It may be observed that T_1 and T_2 lie in the x–y plane, and, hence, R may be computed by a force summation in the z-direction. Thus

$$[\Sigma F_z = 0] \qquad R\cos\gamma = 2000, \qquad R = 2130 \text{ lb.} \qquad \textit{Ans.}$$

Equating to zero the moments about an axis through A parallel to the y-axis eliminates R from the equation and gives

$$[\Sigma M_{A_y} = 0] \qquad (T_1\cos\beta)15 + (T_2\cos\alpha)15 - 2000 \times 16\sin\gamma = 0,$$

$$0.680T_1 + 0.571T_2 = 742.$$

Equating to zero the forces in the y-direction gives

$$[\Sigma F_y = 0] \qquad T_1\sin\beta - T_2\sin\alpha = 0, \qquad 0.733T_1 = 0.821T_2.$$

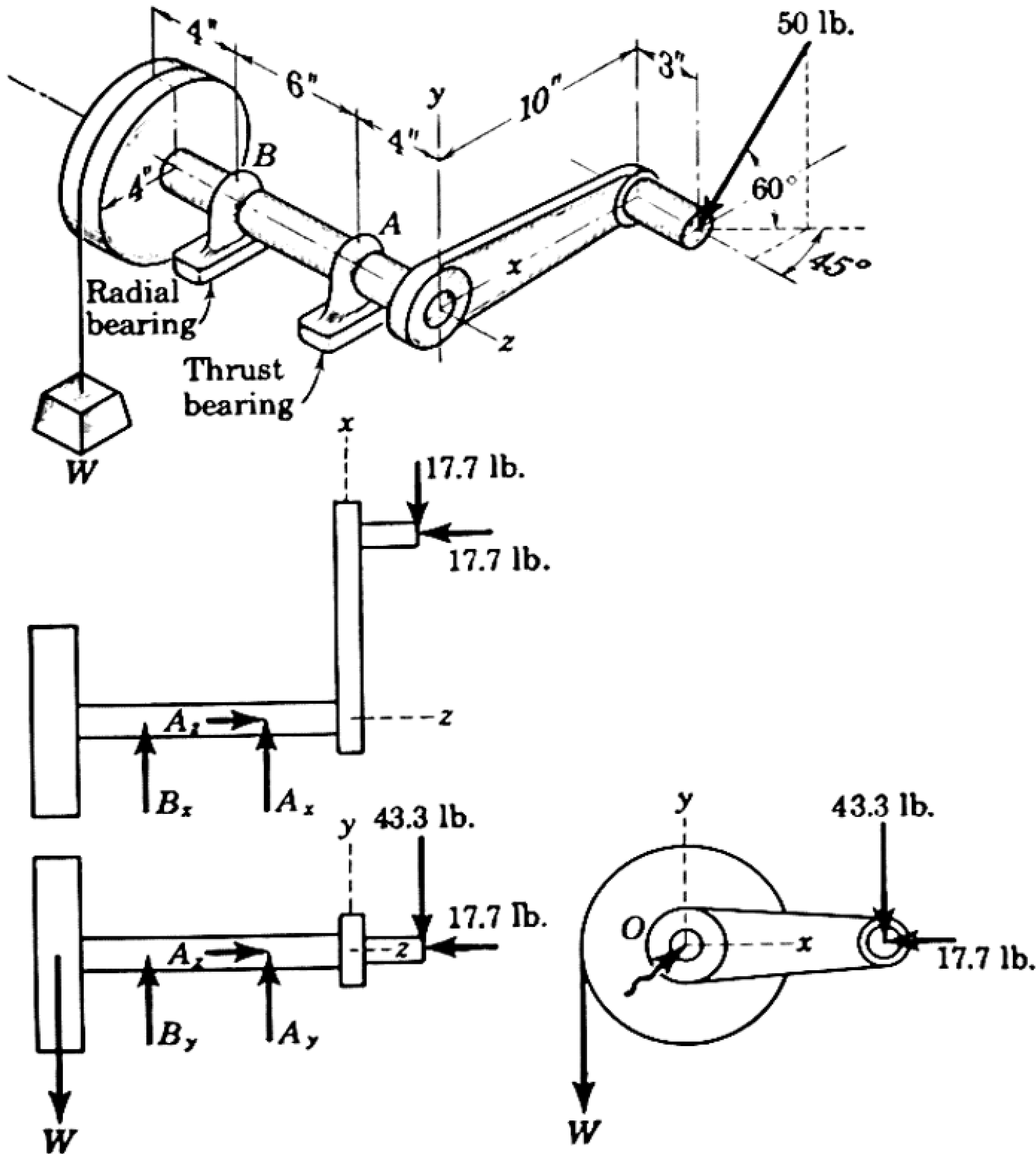
Solution of the two equations yields

$$T_1 = 624 \text{ lb.}, \qquad T_2 = 557 \text{ lb.} \qquad \textit{Ans.}$$

The procedure used here is only one of several possible ones for obtaining the answers. In place of the moment equation about A the principle $\Sigma F_x = 0$ could have been used to obtain one of the two necessary relations between T_1 and T_2. It would be possible to avoid a simultaneous solution by equating the moments about a line passing through A and C to zero. In this event T_1 could be found from one equation. However, the added difficulty of computing moment arms to such an inclined axis would offset the small labor of a simultaneous solution in this case.

It should be noted that if the weight of the boom is not negligible the reaction at A will no longer be in the direction of AB since the forces are no longer concurrent.

197. A 50 lb. force is applied to the handle of the hoist in the direction shown. The bearing A supports the thrust (force in the direction of the shaft axis) while bearing B supports only radial load (load normal to the shaft axis). Determine the weight W which can be supported and the total radial force exerted on the shaft by each bearing.



PROB. 197

Solution: The free-body diagram of the shaft, lever, and drum considered as a single body could be shown by a space view if desired but is represented here by its three orthogonal projections. The 50 lb. applied force is resolved into its three components, and each of the three views shows two of these components. The correct directions of A_x and B_x may be seen by inspection if it is observed that the resultant of the two 17.7 lb. forces exerts a counterclockwise moment about A. The correct directions of A_y and B_y cannot be determined until the magnitudes of the moments are obtained, so these forces may be arbitrarily assigned. The x–y projection of the bearing forces is shown by a wavy arrow since its direction is unknown. The addition of A_z and W completes the free-body diagrams. It should be noted that the three views represent three two-dimensional problems related by the corresponding components of the forces.

From the x–y projection

$[\Sigma M_O = 0]$ $4W - 10 \times 43.3 = 0,$ $W = 108.3$ lb. *Ans.*

From the x-z projection

$[\Sigma M_A = 0]$ $6B_x + 7 \times 17.7 - 10 \times 17.7 = 0,$ $B_x = 8.85$ lb.,

$[\Sigma F_x = 0]$ $A_x + 8.85 - 17.7 = 0,$ $A_x = 8.85$ lb.

The y–z view gives

$[\Sigma M_A = 0]$ $6B_y + 7 \times 43.3 - 10 \times 108.3 = 0,$ $B_y = 130$ lb.,

$[\Sigma F_y = 0]$ $A_y + 130 - 43.3 - 108.3 = 0,$ $A_y = 21.6$ lb.,

$[\Sigma F_z = 0]$ $A_z = 17.7$ lb.

The total radial forces on the bearings become

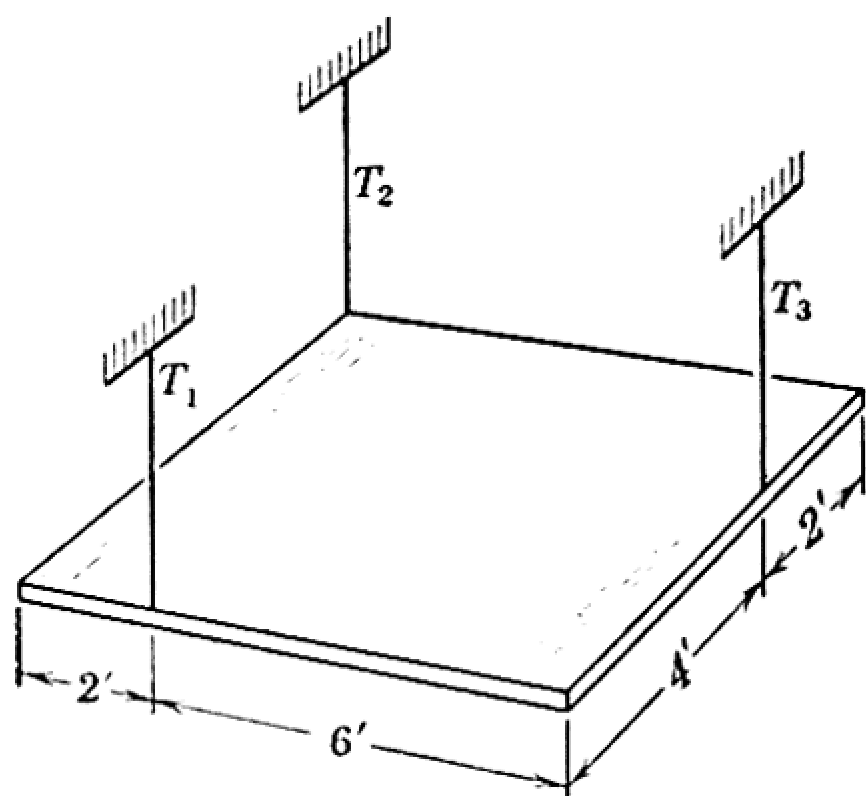
$[A_r = \sqrt{A_x^2 + A_y^2}\,]$ $A_r = \sqrt{(8.85)^2 + (21.6)^2} = 23.3$ lb. *Ans.*

$[B = \sqrt{B_x^2 + B_y^2}\,]$ $B = \sqrt{(8.85)^2 + (130)^2} = 130.2$ lb. *Ans.*

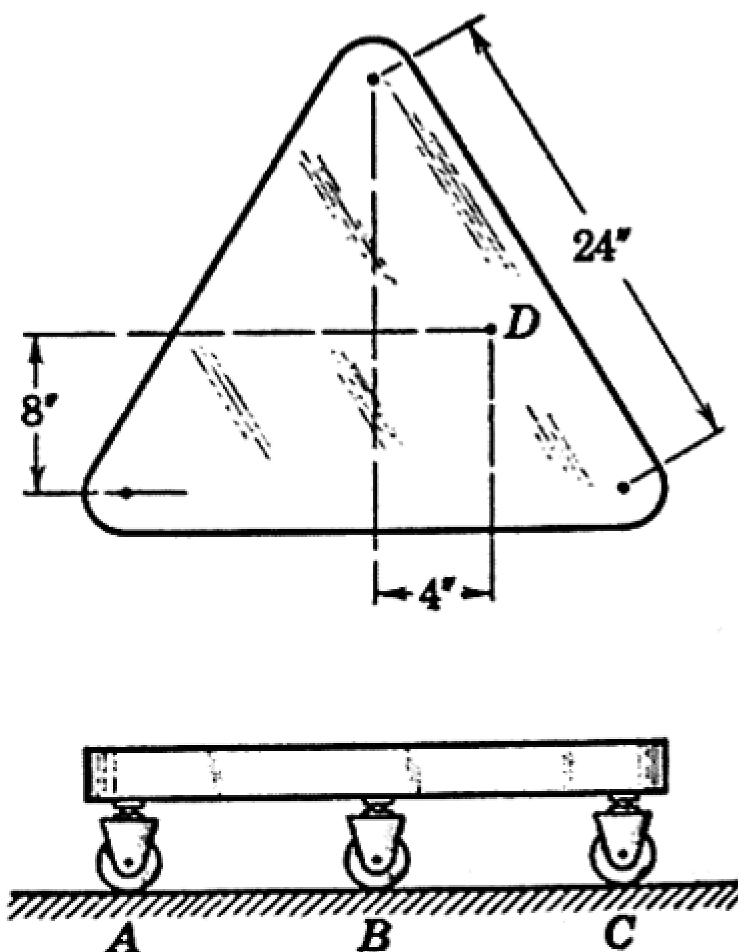
PROBLEMS

198. The uniform rectangular steel plate weighs 1500 lb. Determine the tensions in the three supporting cables.

Ans. $T_1 = 546$ lb., $T_2 = 340$ lb., $T_3 = 614$ lb.



Prob. 198

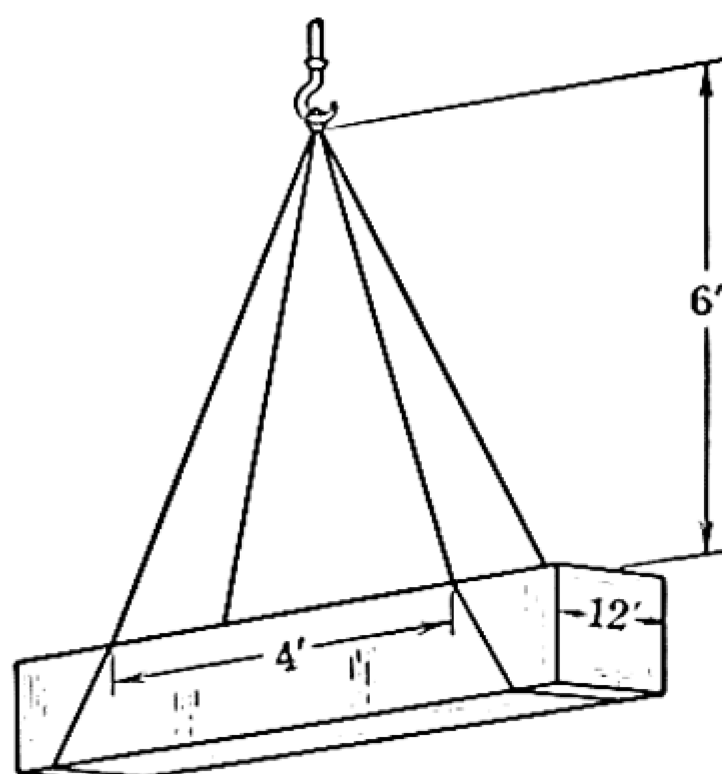


Prob. 199

199. A three-wheel service truck is provided with swivel casters spaced 24 in. apart to form an equilateral triangle. If a 500 lb. load with center of gravity above point D is carried, determine the force supported by each wheel.

Ans. $A = 70.5$ lb., $B = 192.5$ lb., $C = 237$ lb.

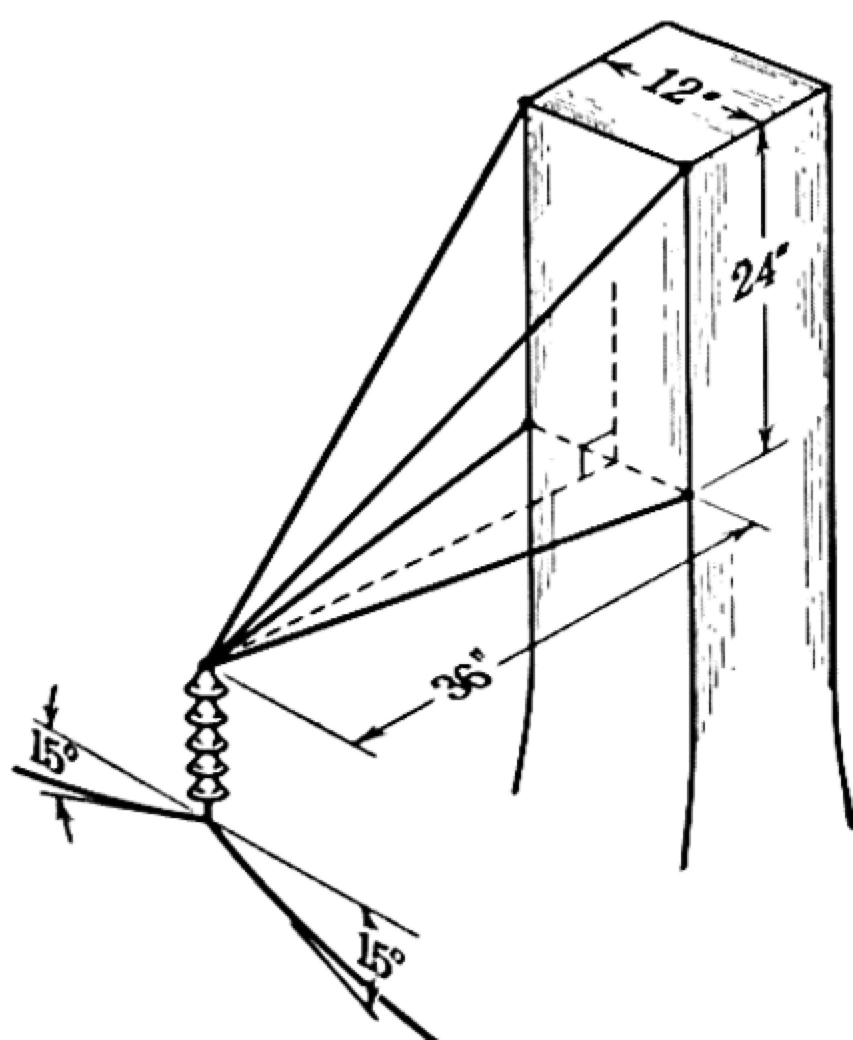
200. Determine the tension T in the cables all of which are symmetrically spaced and equally loaded. The marble slab weighs 1000 lb.



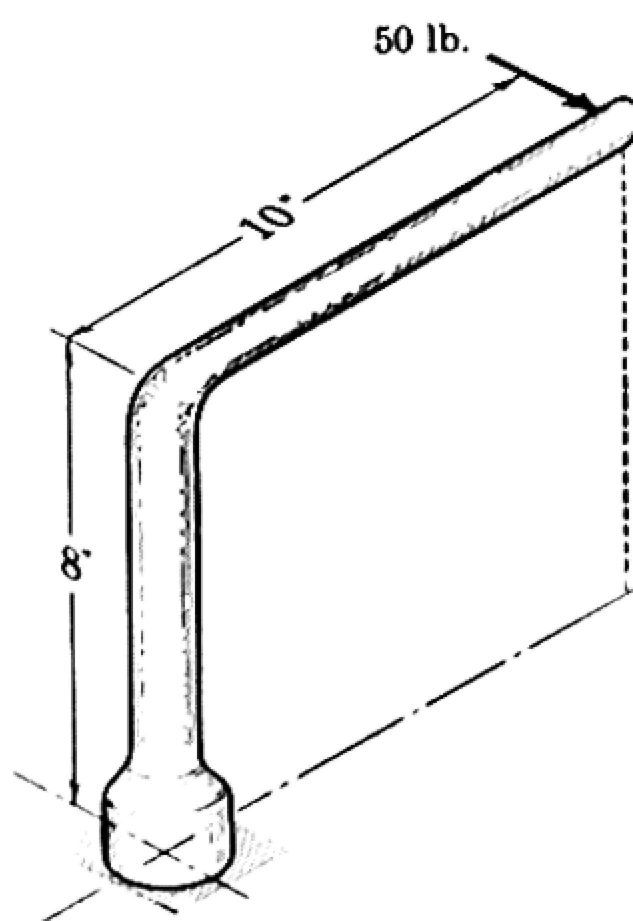
PROB. 200

201. Determine the tension T in the upper two members and the compression C in the lower two members which support the power line. The tension in the line is 400 lb. at the insulator. (*Hint:* Take advantage of symmetry.)

Ans. $T = 188$ lb., $C = 156$ lb.



PROB. 201

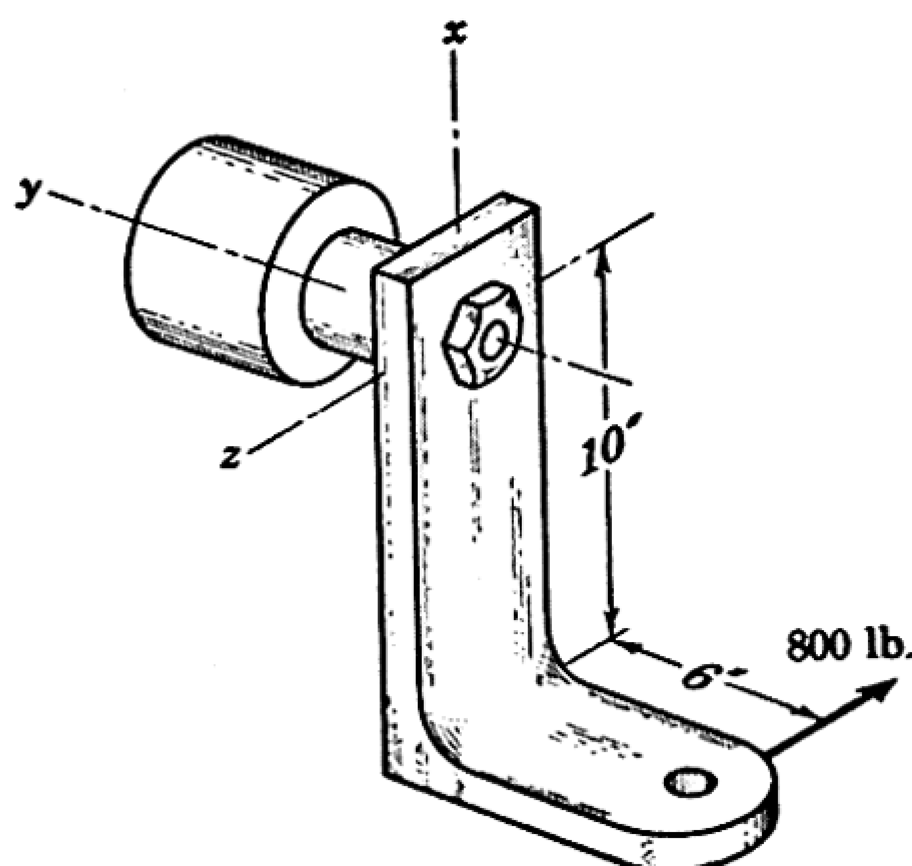


PROB. 202

202. A socket wrench with a 50 lb. applied force is used to tighten a bolt (hidden from view). What reaction is exerted on the wrench by the bolt? What additional force should be applied to the wrench in order to improve its operation?

Ans. Reaction is 50 lb. force and 640 lb. in. couple

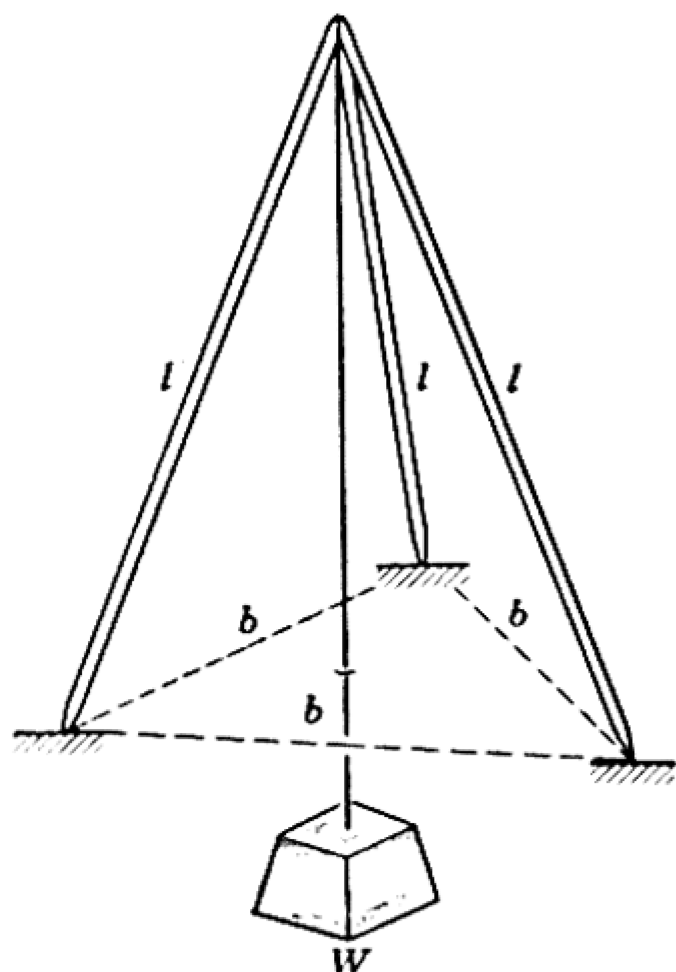
203. The bracket is secured to the end of the fixed shaft by a single bolt which produces sufficient force to hold it in place. Determine the resultant force and moment exerted by the bolt and shaft on the bracket.



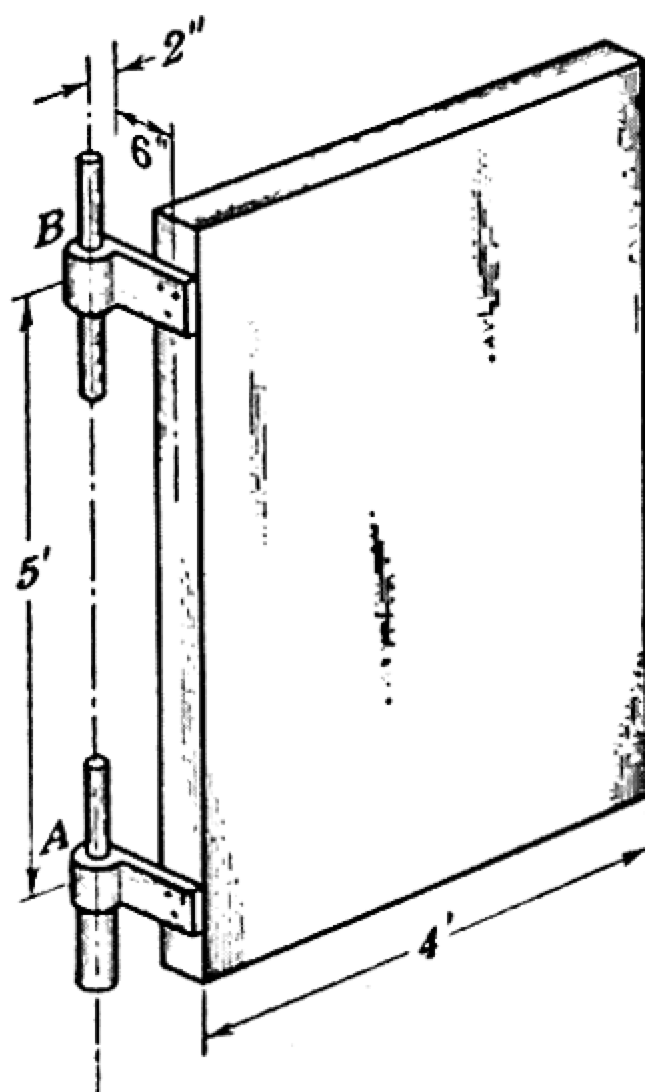
PROB. 203

204. The derrick is constructed from three identical legs of length l whose weight is negligible compared with the load W. If the base of the legs forms an equilateral triangle of side b, determine the compression C in the members.

Ans. $C = W \Big/ 3\sqrt{1 - \frac{b^2}{3l^2}}$



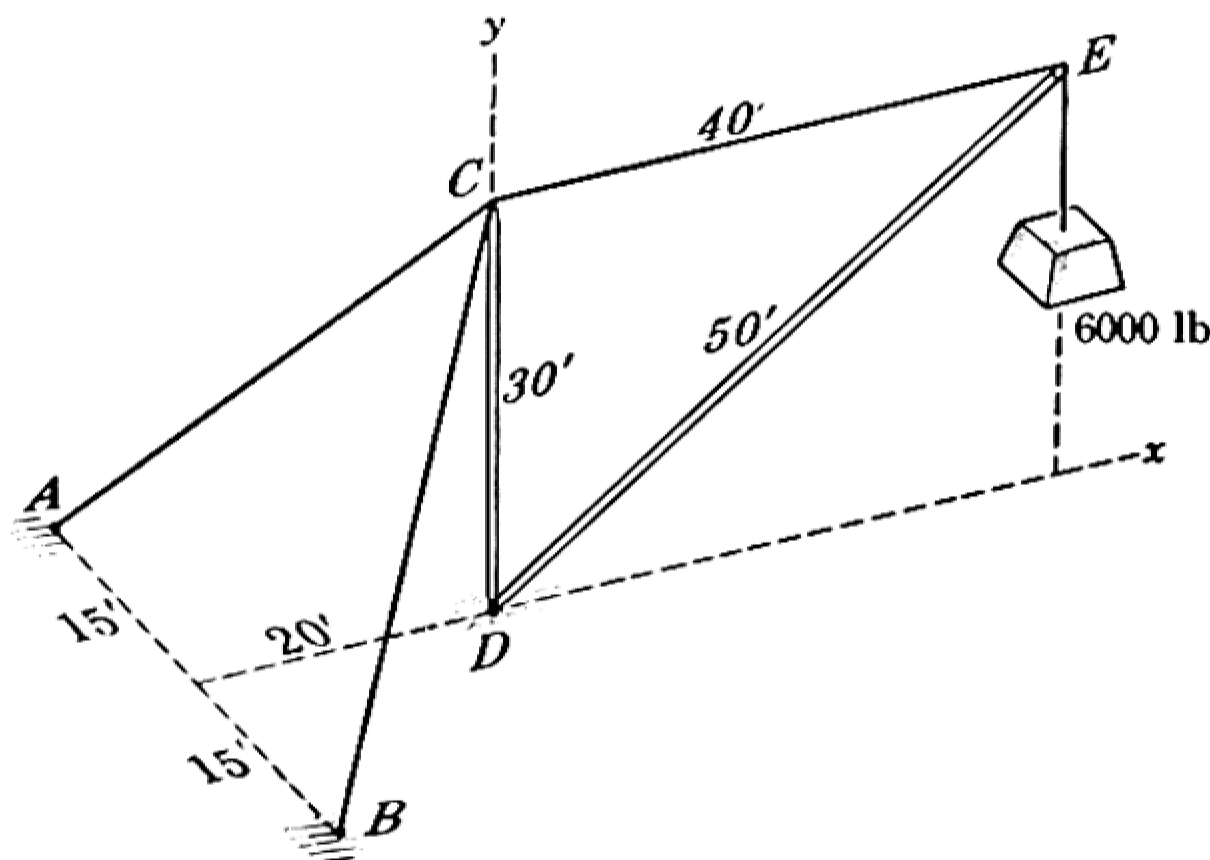
PROB. 204



PROB. 205

205. The weight of the homogeneous 100 lb. hinged door is supported entirely at A. Determine the force exerted on the upper hinge B and the total horizontal force exerted on the lower hinge A. *Ans.* $|A|_{\text{Horiz.}} = |B| = 44.4$ lb.

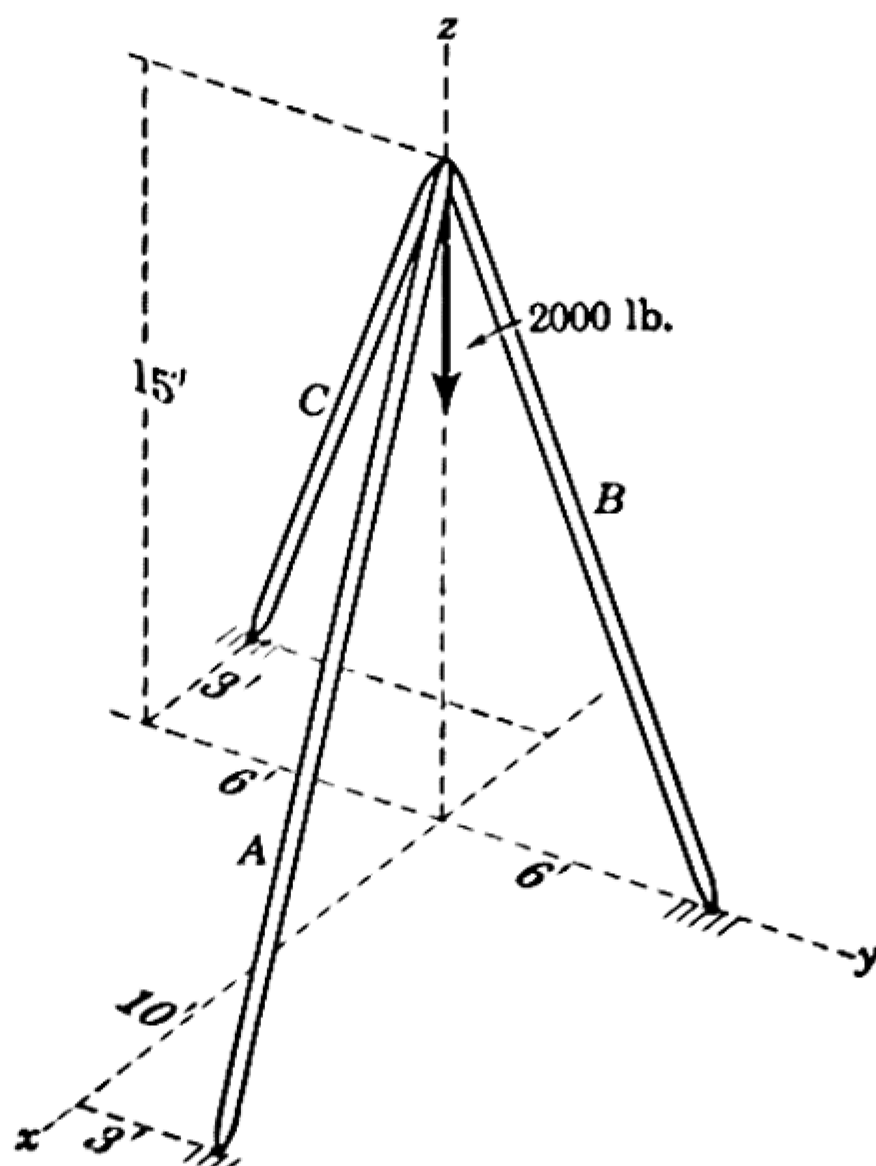
206. Find the tension T in the cables AC and BC and the compression C in the mast CD. The 6000 lb. load is large compared with the weights of the mast and boom. *Ans.* $T = 7810$ lb., $C = 12{,}000$ lb.



PROB. 206

207. If the plane of the boom and cable CE of the derrick in Prob. 206 is rotated through an angle of 20 deg. from the x-y plane with the 6000 lb. load in place, determine the larger of the two tensions in the cables BC and AC.

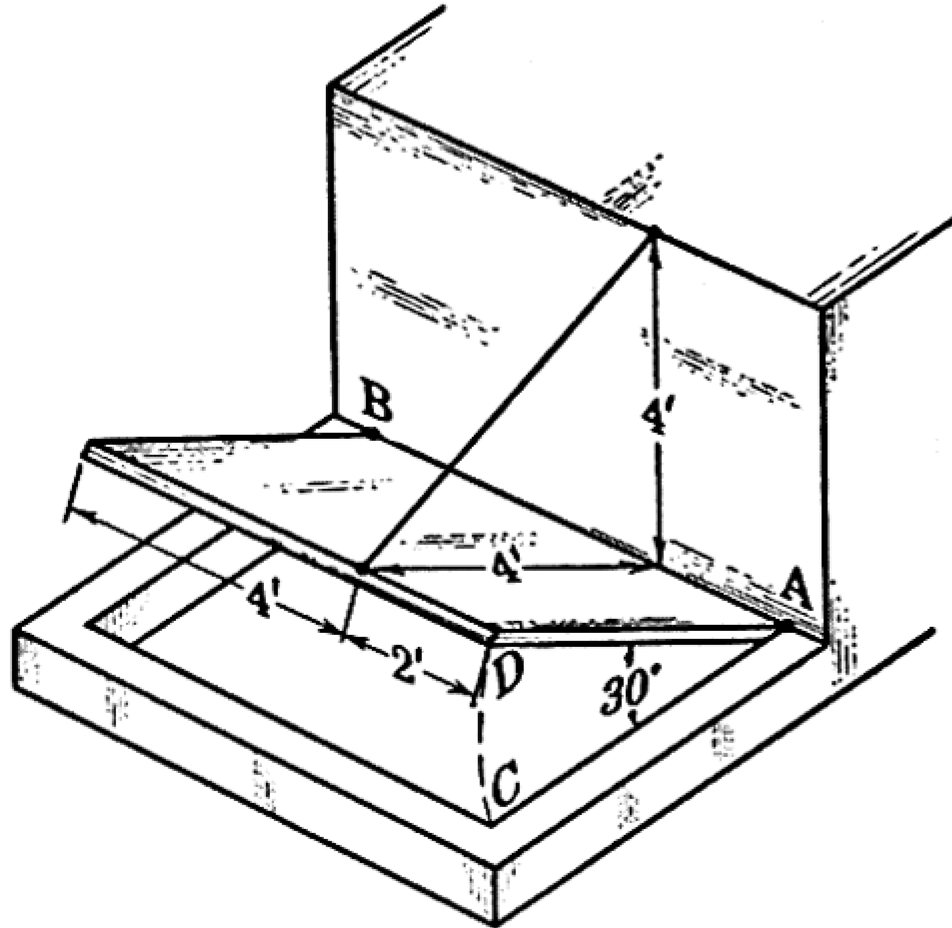
208. Determine the compression in each of the three unequal legs of the derrick. The applied load is large compared with the weights of the legs. *Ans.* $A = 340$ lb., $B = 852$ lb., $C = 1019$ lb.



PROB. 208

209. A homogeneous trap door weighing 60 lb. is hinged at A and B and secured by a cable in the position shown. Determine the forces on the two hinges.

Ans. $|A| = |B| = 26.5$ lb.

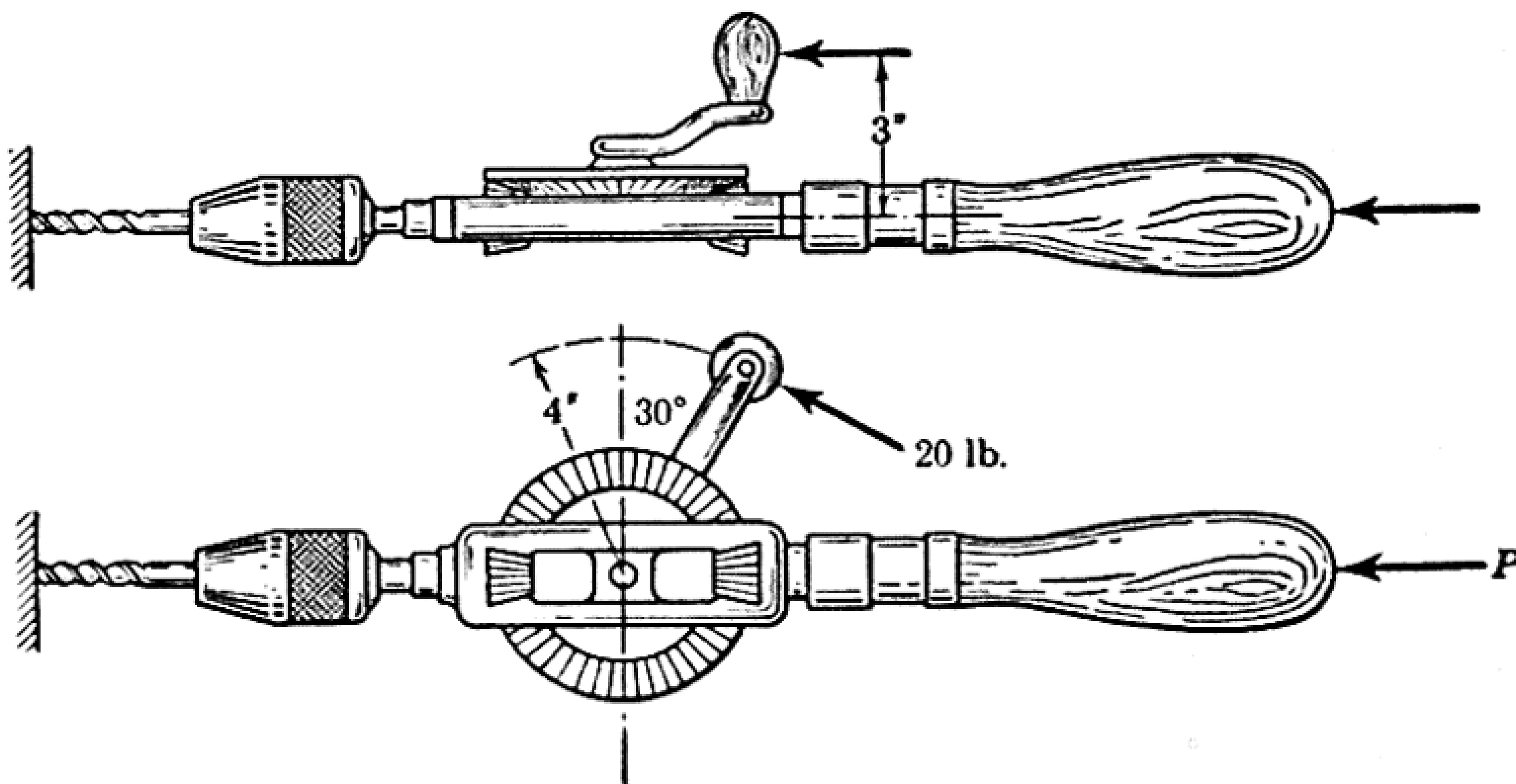


Prob. 209

210. If the trap door of Prob. 209 is supported in the 30 deg. position by a strut between C and D in place of the cable, determine the forces on the hinges A and B.

211. A force of 20 lb. is required on the crank of the hand drill to start the drill in the position shown with a thrust P applied to the handle. The gear ratio is 3 to 1. Determine the moment M about the axis of the drill which must be supplied by the hand grip to prevent the handle from turning.

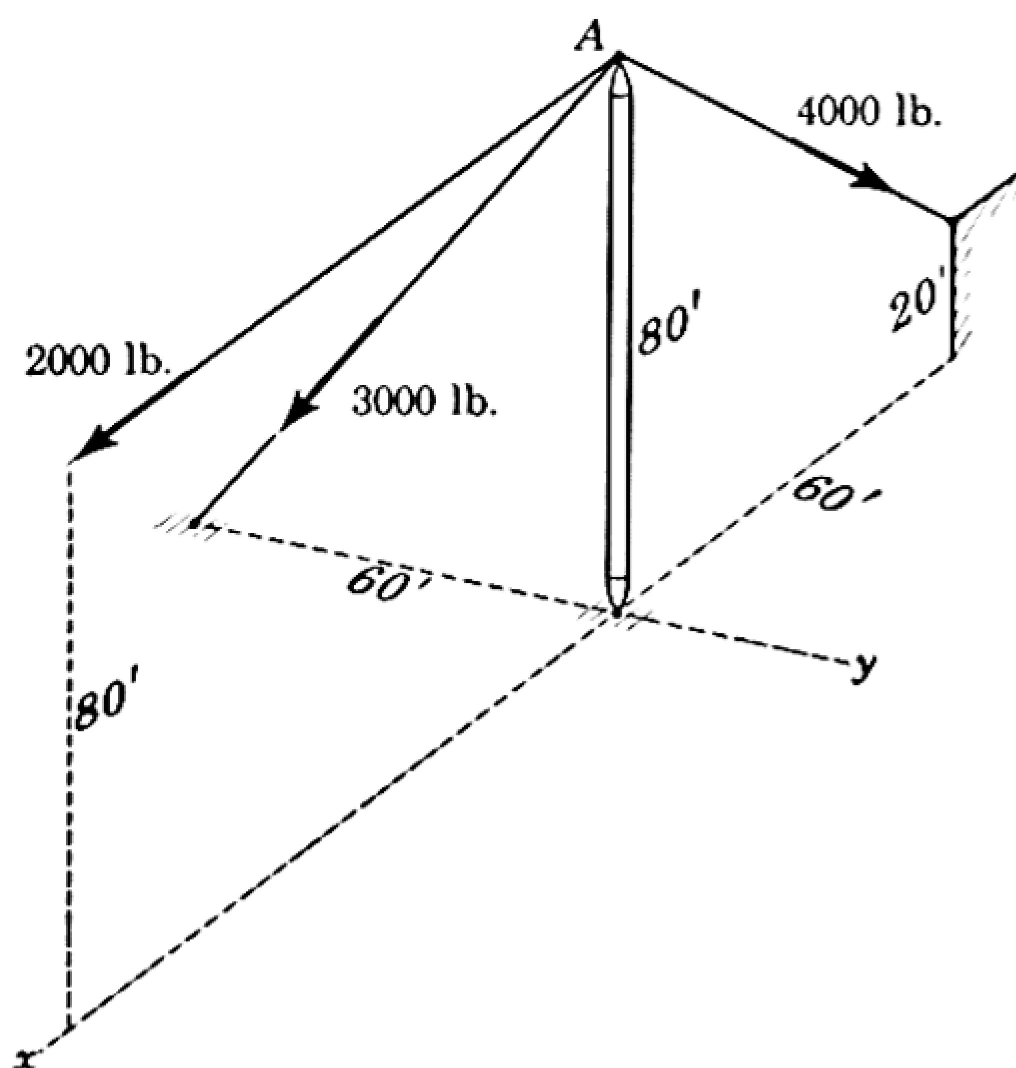
Ans. $M = 56.7$ lb. in.



Prob. 211

212. The 80 ft. mast is to support the three cable tensions shown. It is desired to secure the mast with an additional single 100 ft. cable from A to the

ground. Determine the x- and y-coordinates of the cable anchorage and the tension T in the cable. *Ans.* $x = 25.1$ ft., $y = 54.5$ ft., $T = 3300$ lb.



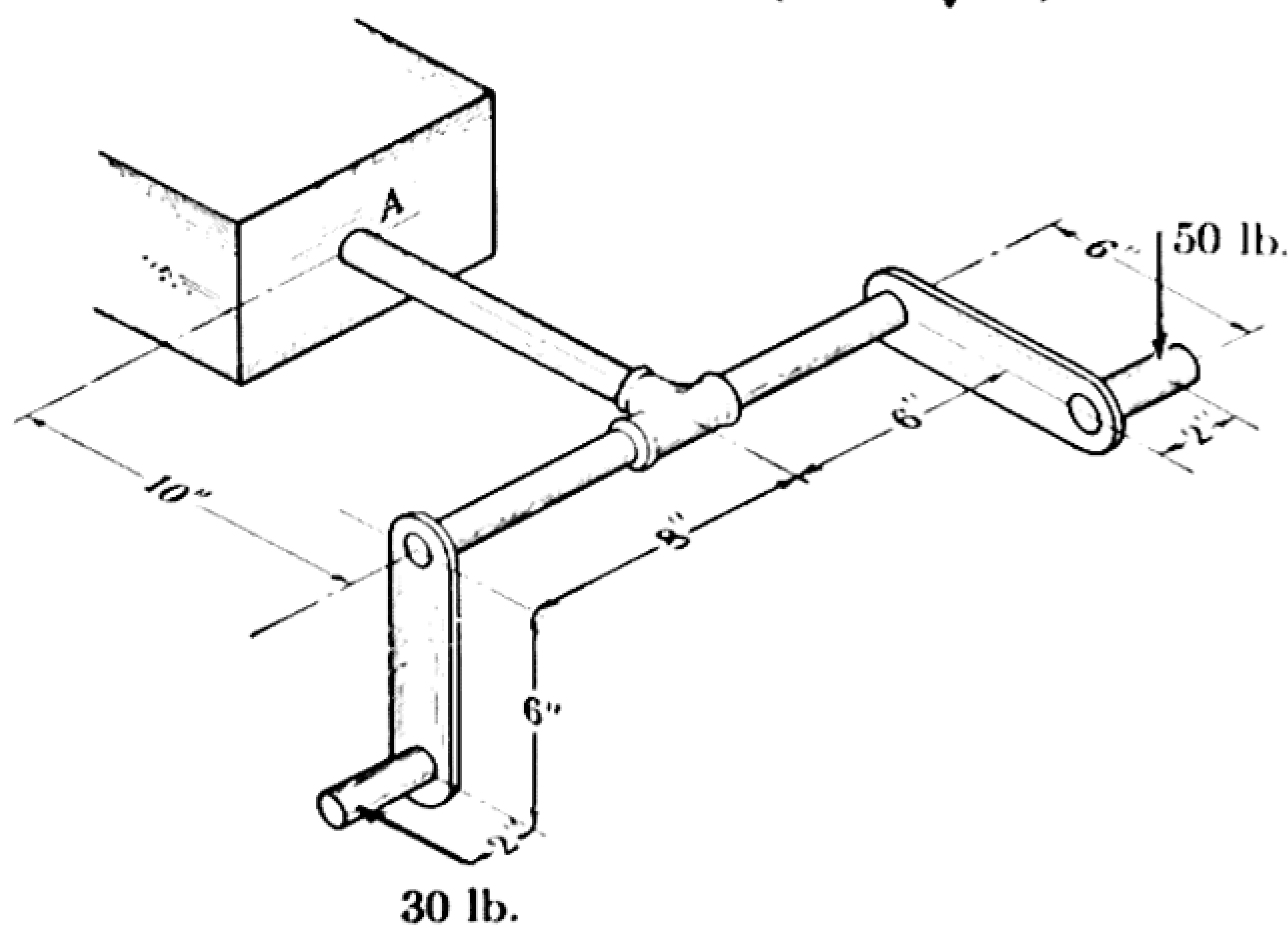
Prob. 212

213. Three identical smooth steel balls of radius r and weight W are placed in a smooth hemispherical bowl of radius R. Determine the force of contact P between the balls for their equilibrium position.

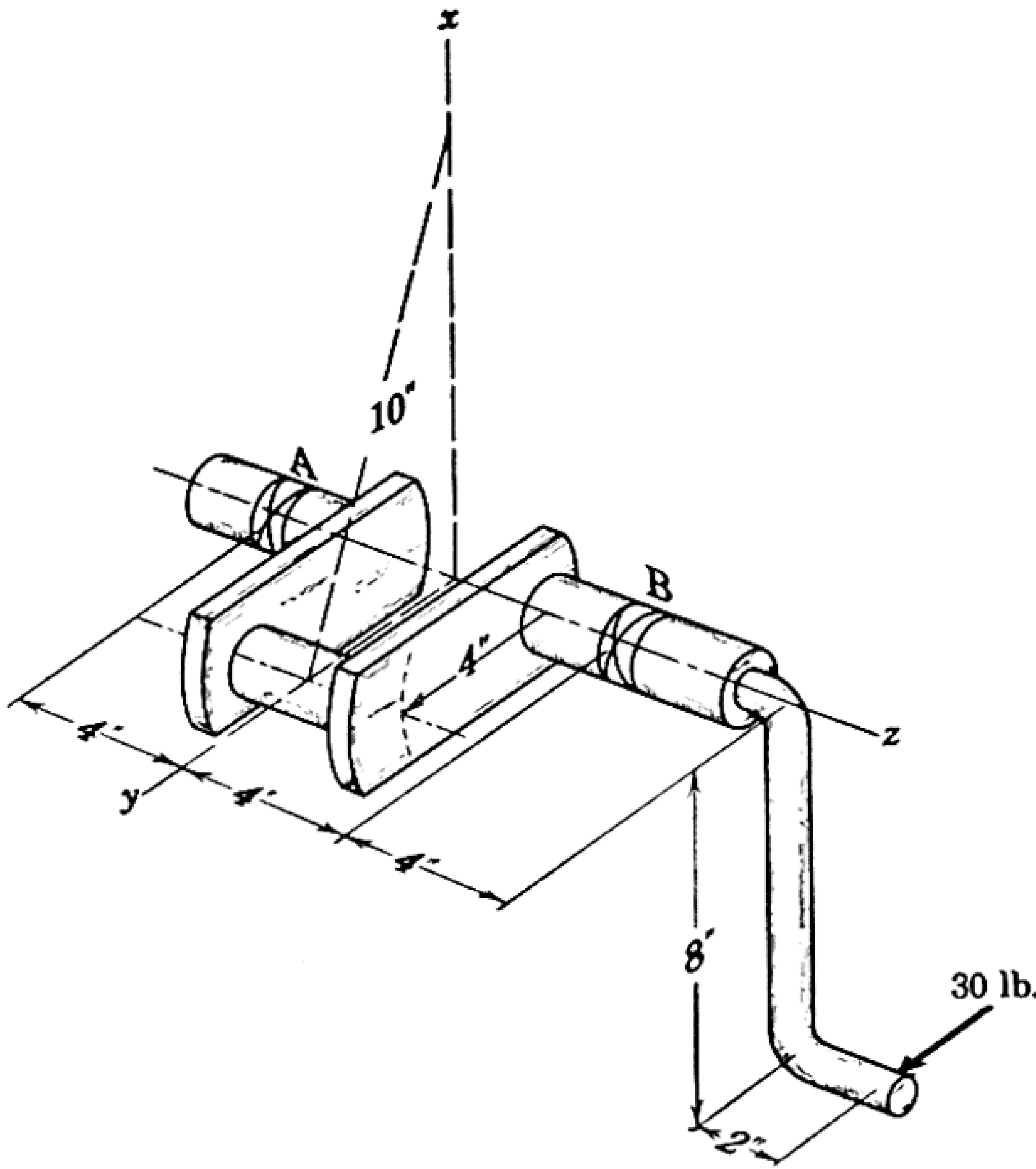
$$\textit{Ans.}\quad P = \frac{2Wr}{3\sqrt{R^2 - 2Rr - \frac{r^2}{3}}}$$

214. If a fourth identical ball is placed centrally on the three balls of Prob. 213, determine the maximum value of R so that the lower three balls will not separate. *Ans.* $R = (1 + 2\sqrt{11})r = 7.63r$

215. Determine the force F and moment M exerted on the rigidly attached shaft at A by the fixed block.



Prob. 215



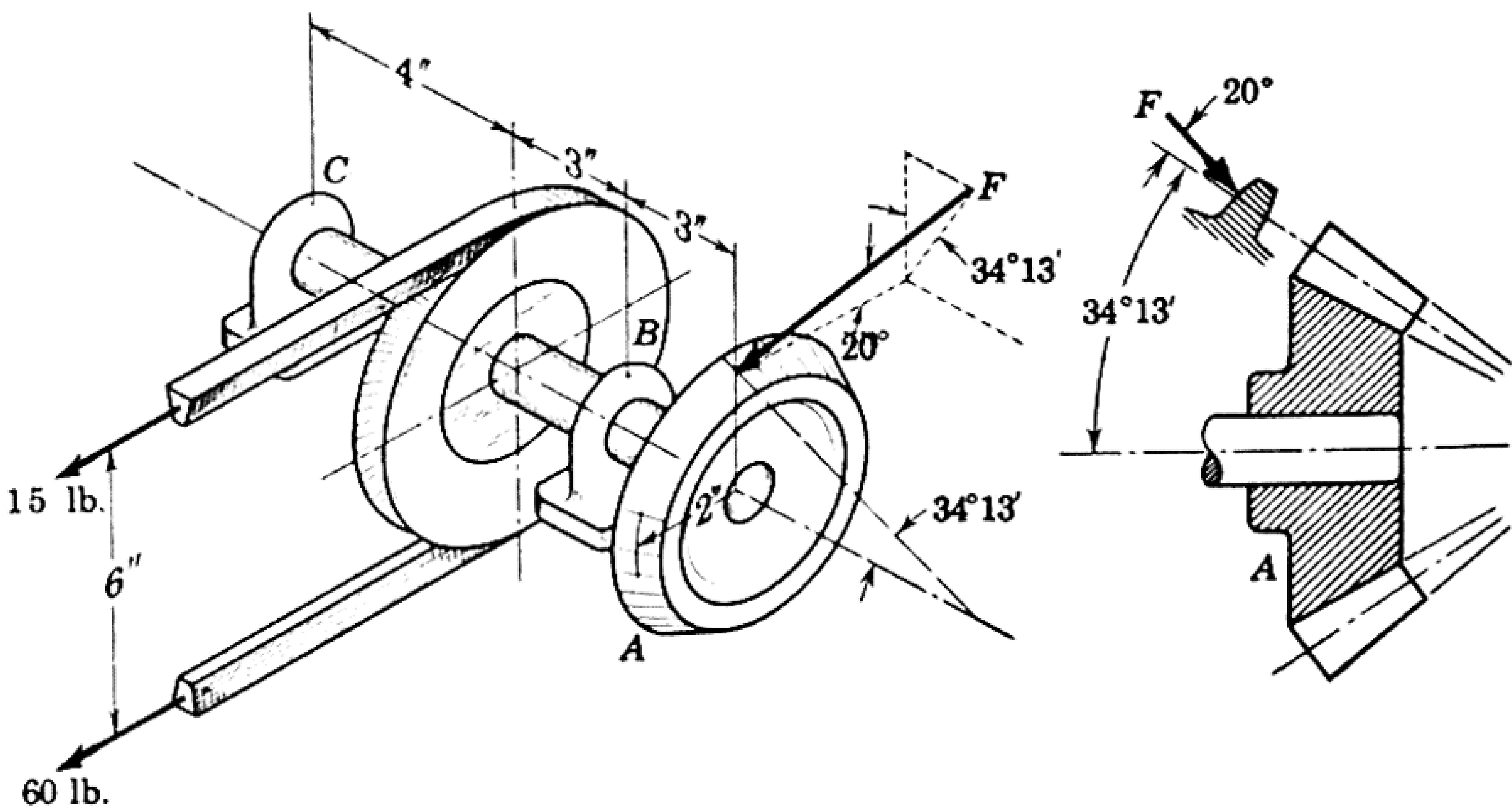
PROB. 216

216. Determine the forces on the bearings A and B if the 30 lb. force is required to turn the crankshaft of the single-cylinder engine against the compression in the cylinder for the position shown. The axis of the 10 in. connecting rod is shown by the dotted line.

Ans. $A = 31.4$ lb., $B = 72.2$ lb.

* **217.** The bevel gear A drives the shaft, pulley, and V-belt at constant speed. The gear has a pitch angle of 34° 13′ and a pressure angle of 20 deg. as shown in the separate view. Bearing B is capable of supporting thrust, while bearing C can support radial load only. For the given belt tensions determine the gear-tooth load F and the total radial force exerted on each bearing.

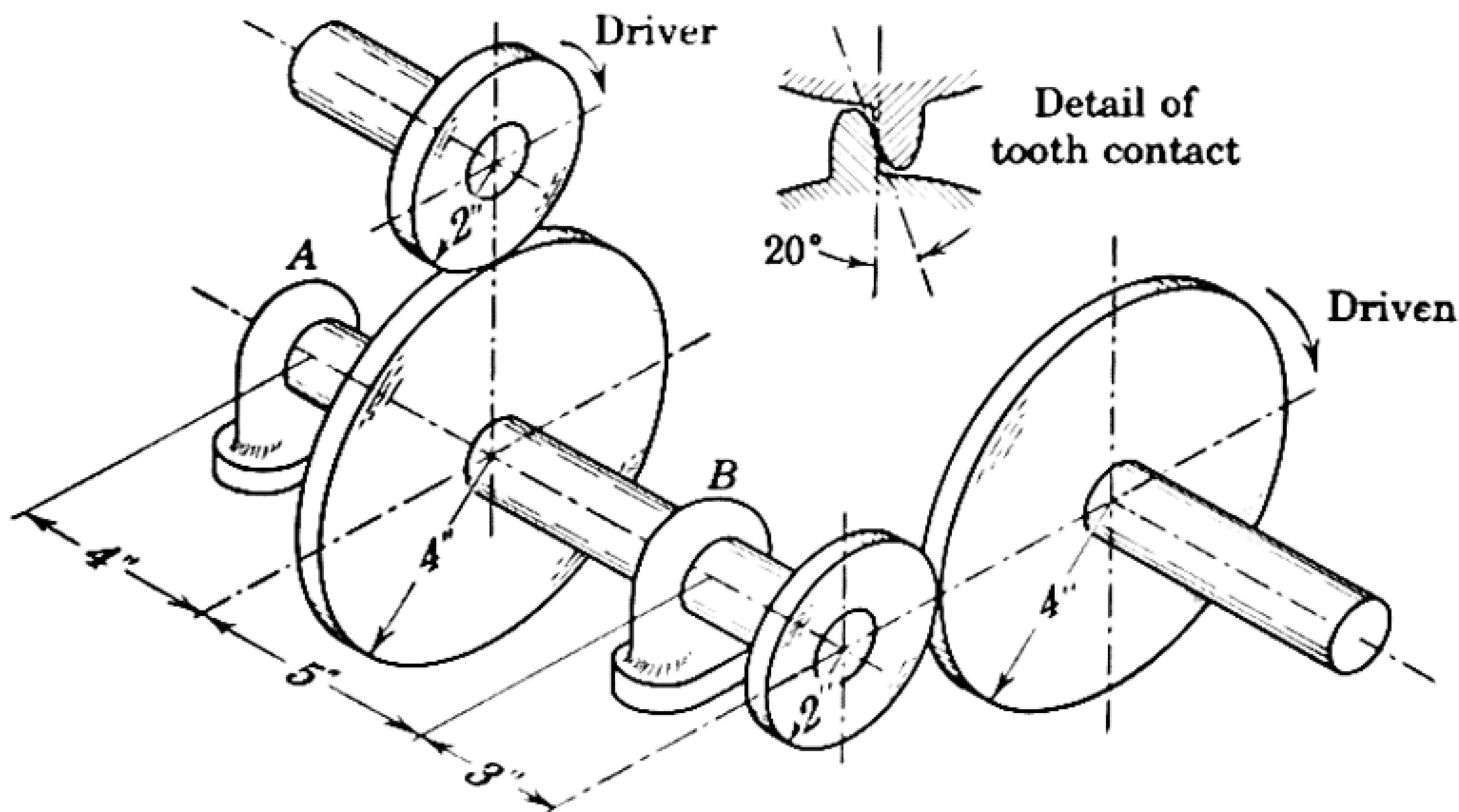
Ans. $F = 71.9$ lb., $B_r = 142$ lb., $C = 5.8$ lb.



PROB. 217

* **218.** The 4:1 reduction drive has a torque of 1000 lb. in. supplied to the shaft of the driving pinion. If the tooth action for both pairs of gears is as shown,

determine the total forces exerted by the bearings A and B on the shaft. The bearings for the shafts of the driving pinion and driven gear are not shown. The weights of the parts may be neglected. *Ans.* $A = 279$ lb., $B = 1580$ lb.

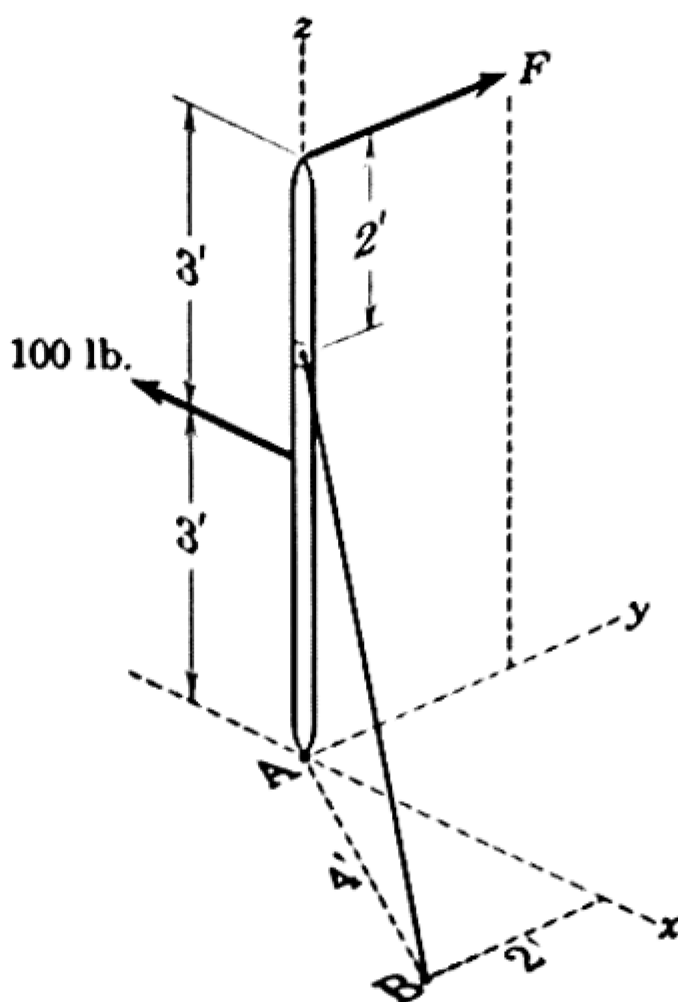


PROB. 218

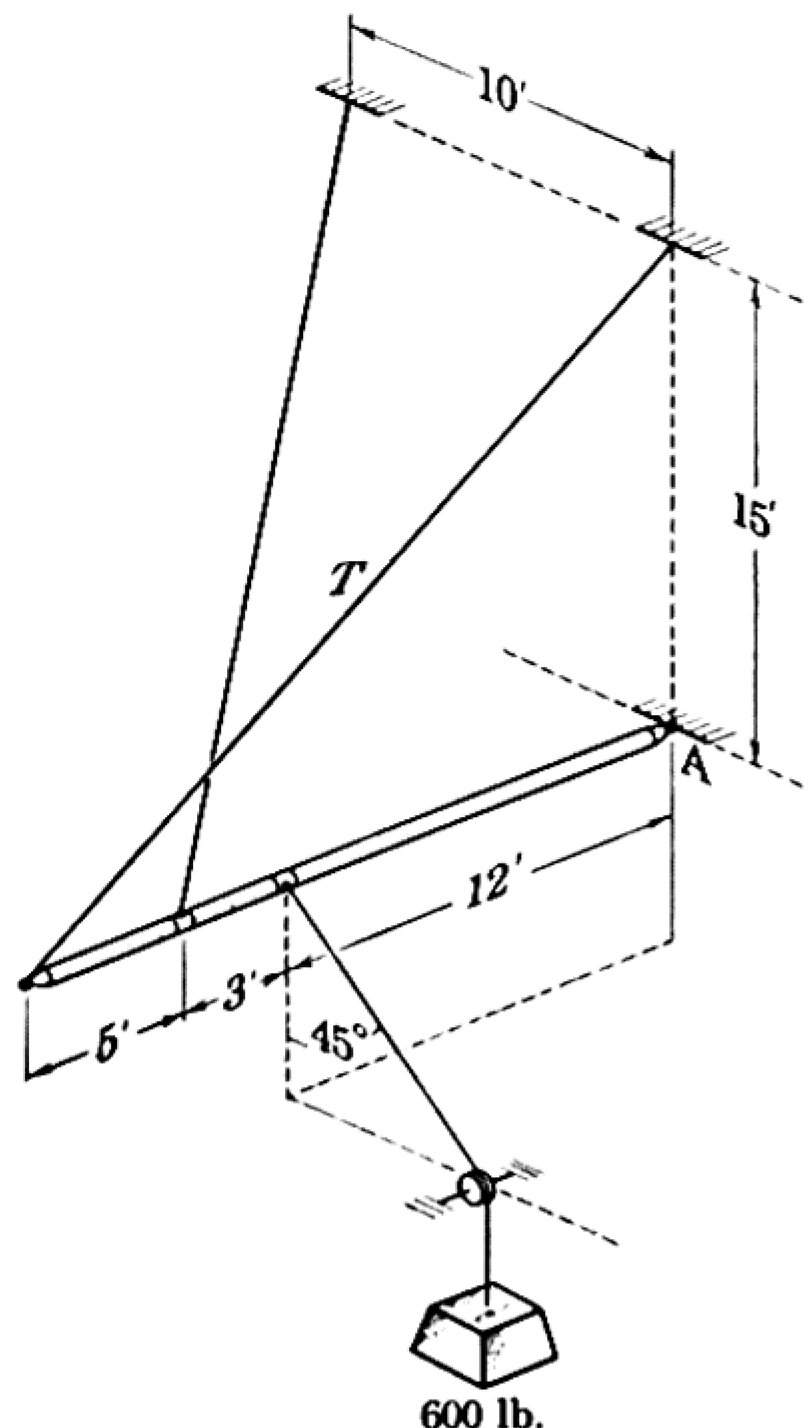
* **219.** The 20 lb. post is supported at its lower end by a ball and socket joint. Determine the value of F to maintain equilibrium in the position shown. Also find the total force exerted on the joint at A. *Ans.* $F = 28.9$ lb., $A = 110$ lb.

* **220.** Determine the tension T and the magnitude of the total reaction at A for the loaded hoist. The boom is uniform and weighs 500 lb.

Ans. $T = 203$ lb., $A = 738$ lb.

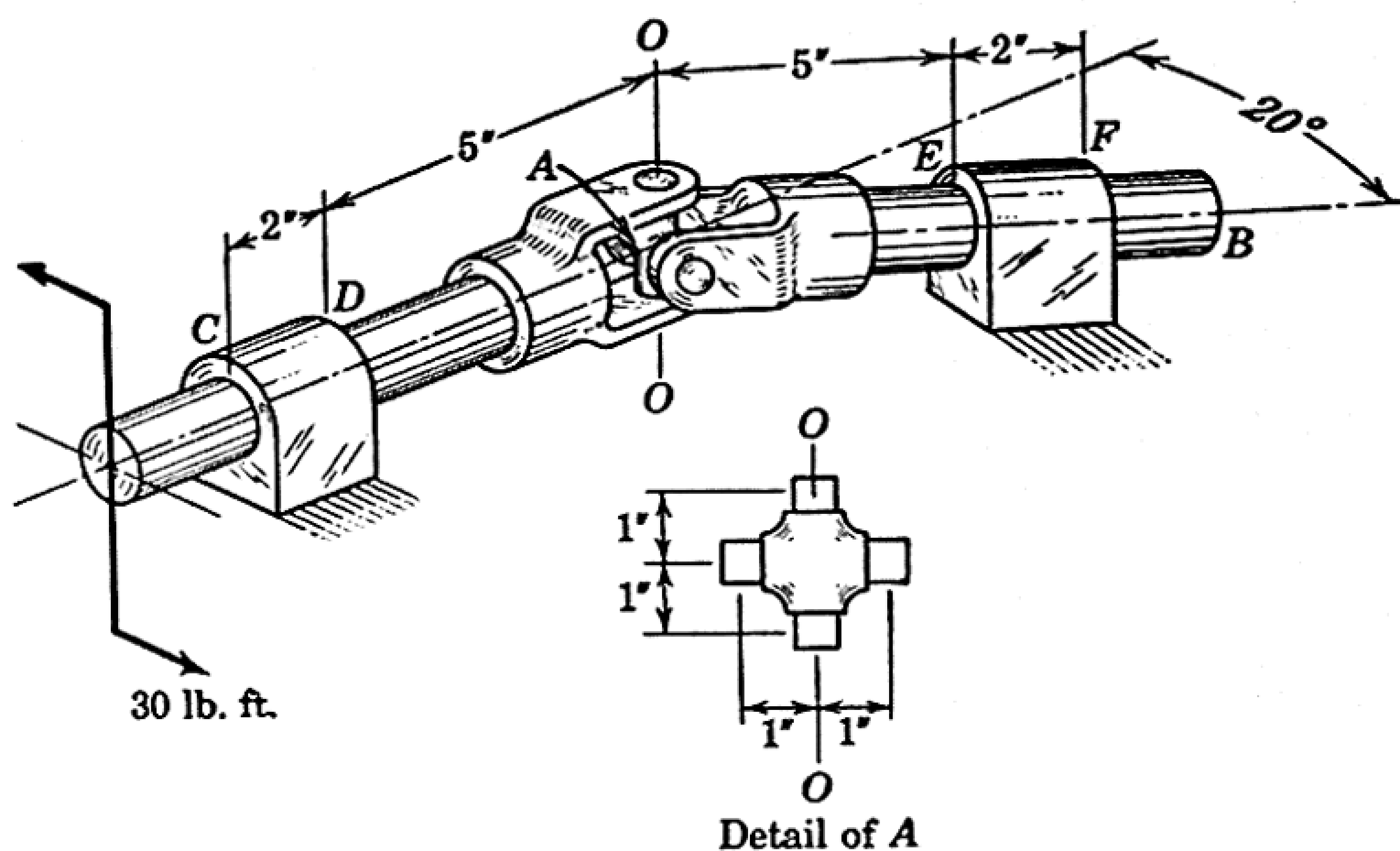


PROB. 219



PROB. 220

* **221.** A torque of 30 lb. ft. is applied to one end of a universal joint. When the joint is in the position shown, determine the resisting torque M applied to the shaft at B for equilibrium and the reactions exerted by each sleeve bearing



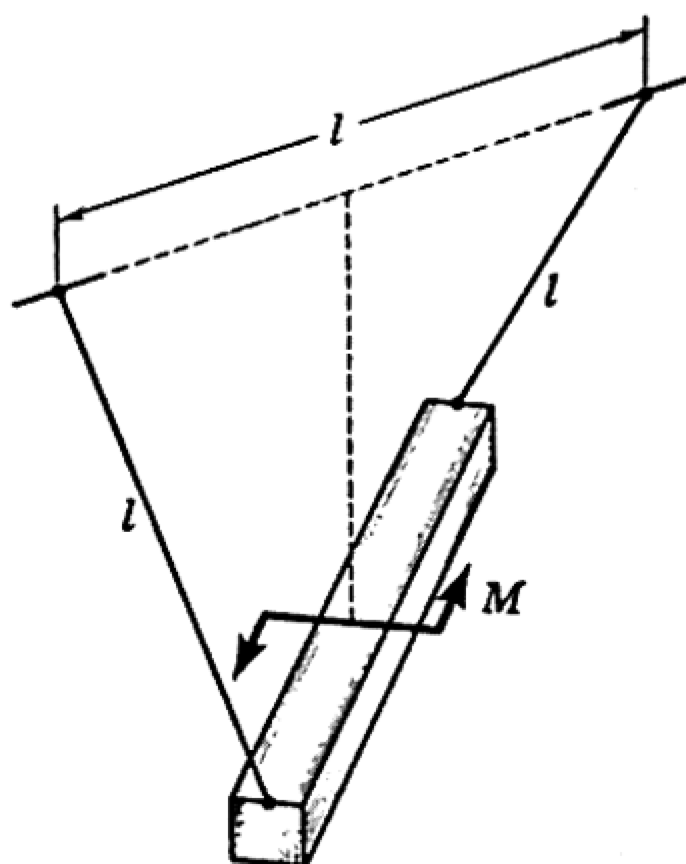
Detail of A

Prob. 221

on its shaft. The detail of the connecting cross with its four bearing surfaces is shown in the separate view. Both shafts are in the horizontal plane, and axis O–O is vertical.

Ans. $M = 31.9$ lb. ft., $C = 65.4$ lb. up on shaft, $D = 65.4$ lb. down on shaft, $E = F = 0$

* **222.** A uniform bar of weight W and length l is suspended at its ends by two cords, also of length l, from two points in the horizontal plane a distance l apart.

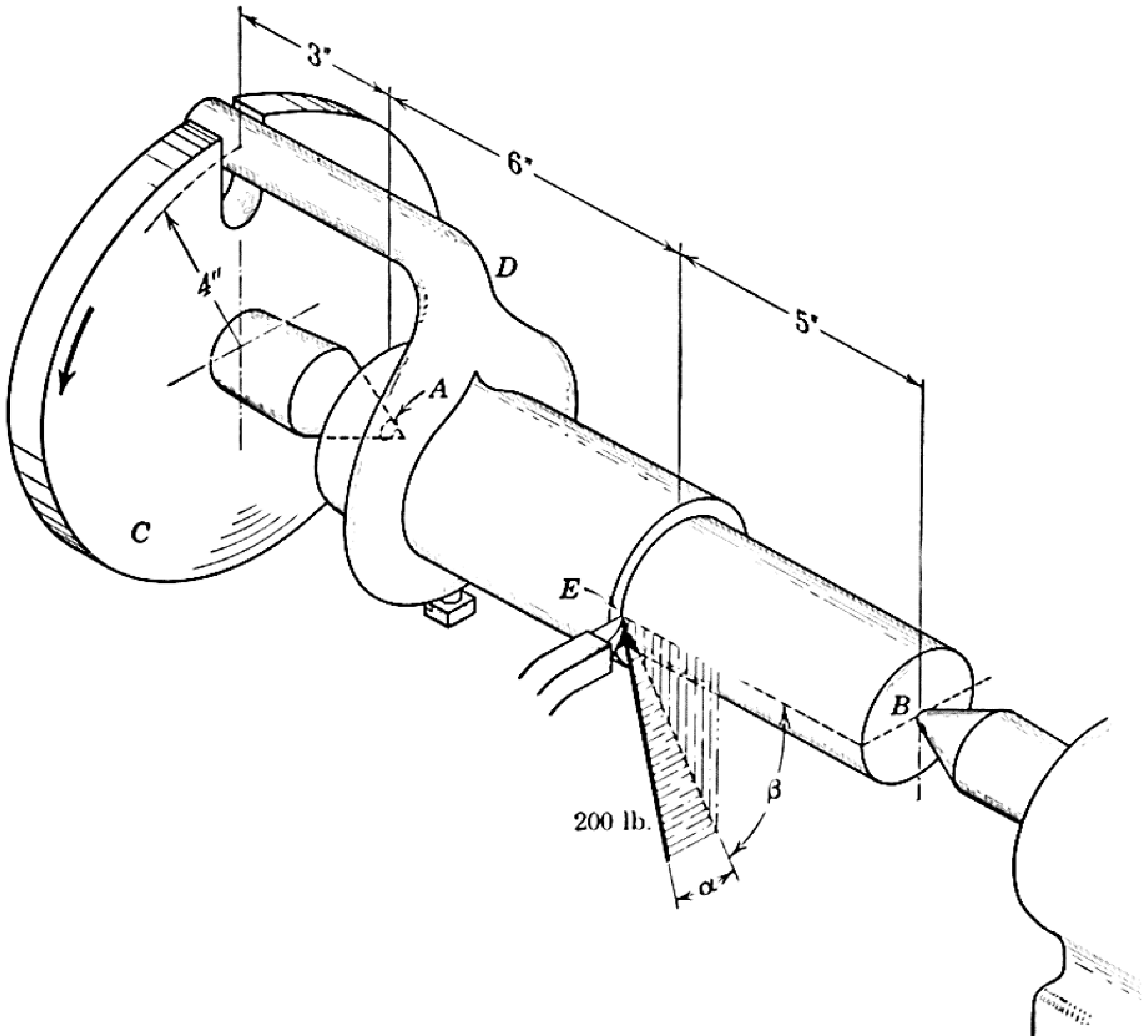


Prob. 222

A couple M in the horizontal plane is applied to the bar, causing it to rotate to an equilibrium position as shown. Determine the height h which it rises from its original untwisted position. What value of M is required to raise the bar the maximum amount l?

Ans. $h = l\left(1 - \sqrt{1 - \frac{4M^2}{W^2l^2}}\right)$, $M_{h \to l} = \frac{Wl}{2}$

* **223.** A steel shaft is mounted in a lathe between centers A and B and is driven from the face plate C in the direction shown by the tang of the clamped



PROB. 223

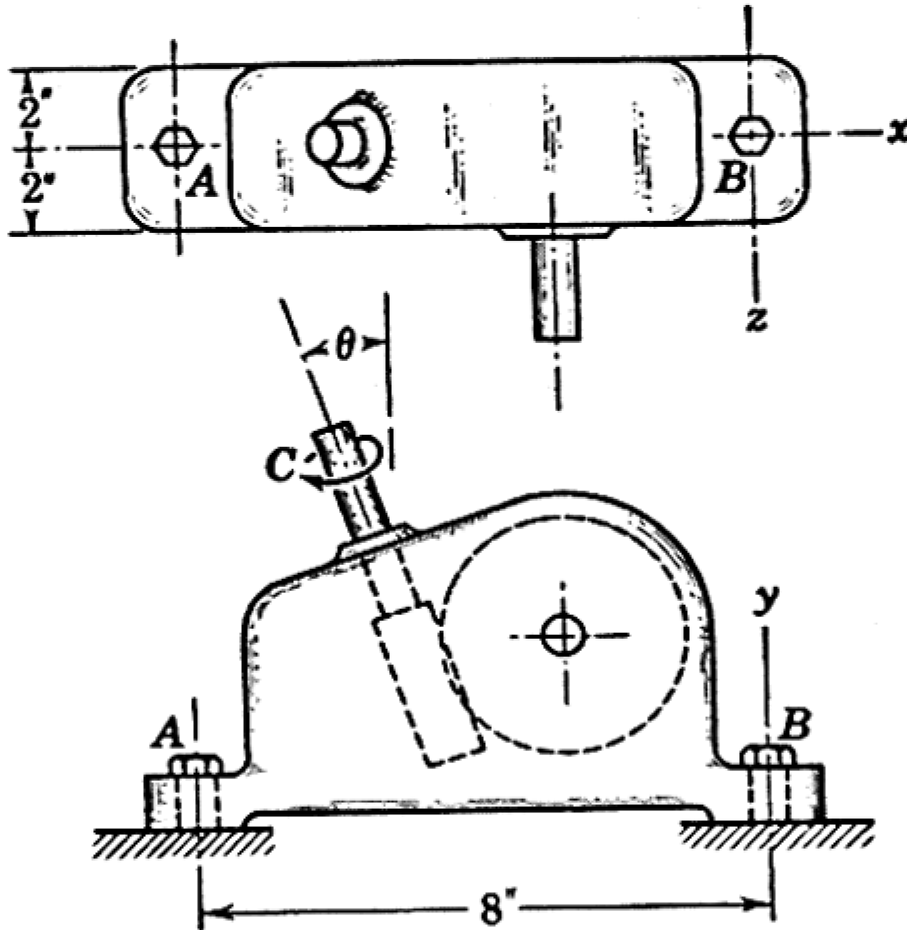
dog D. The lathe tool E exerts a force of 200 lb. on the shaft at a mean radius of 2 in. and in a direction given by $\beta = 70$ deg. and $\alpha = 10$ deg. Determine the radial components A_r and B_r (normal to the shaft axis) of the forces exerted by the centers at A and B on the shaft when the tang is in the position illustrated.

Ans. $A_r = 123$ lb., $B_r = 106$ lb.

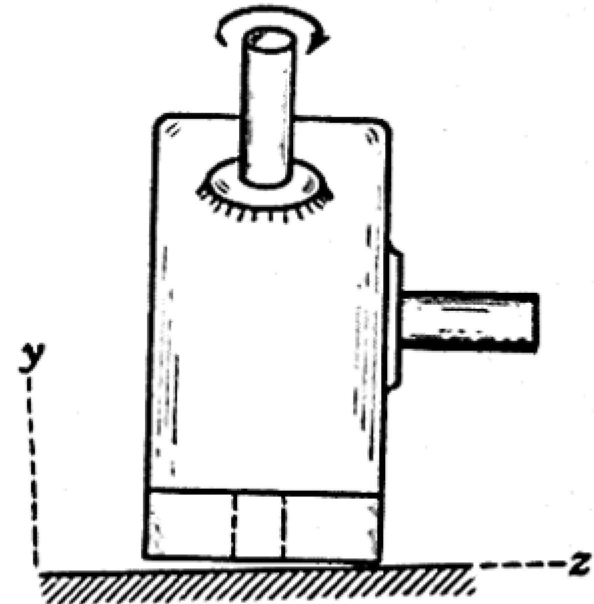
* **224.** The 40:1 right-handed worm reducer shown has an input torque of 10 lb. ft. applied to shaft C. Find the reaction components on the reducer at

the mounting holes if the input shaft is vertical ($\theta = 0$). Assume the output torque to be 400 lb. ft. and neglect the weight of the unit.

Ans. $A_y = 600$ lb., $A_z = 15$ lb., $B_y = -600$ lb., $B_z = -15$ lb.



PROB. 224

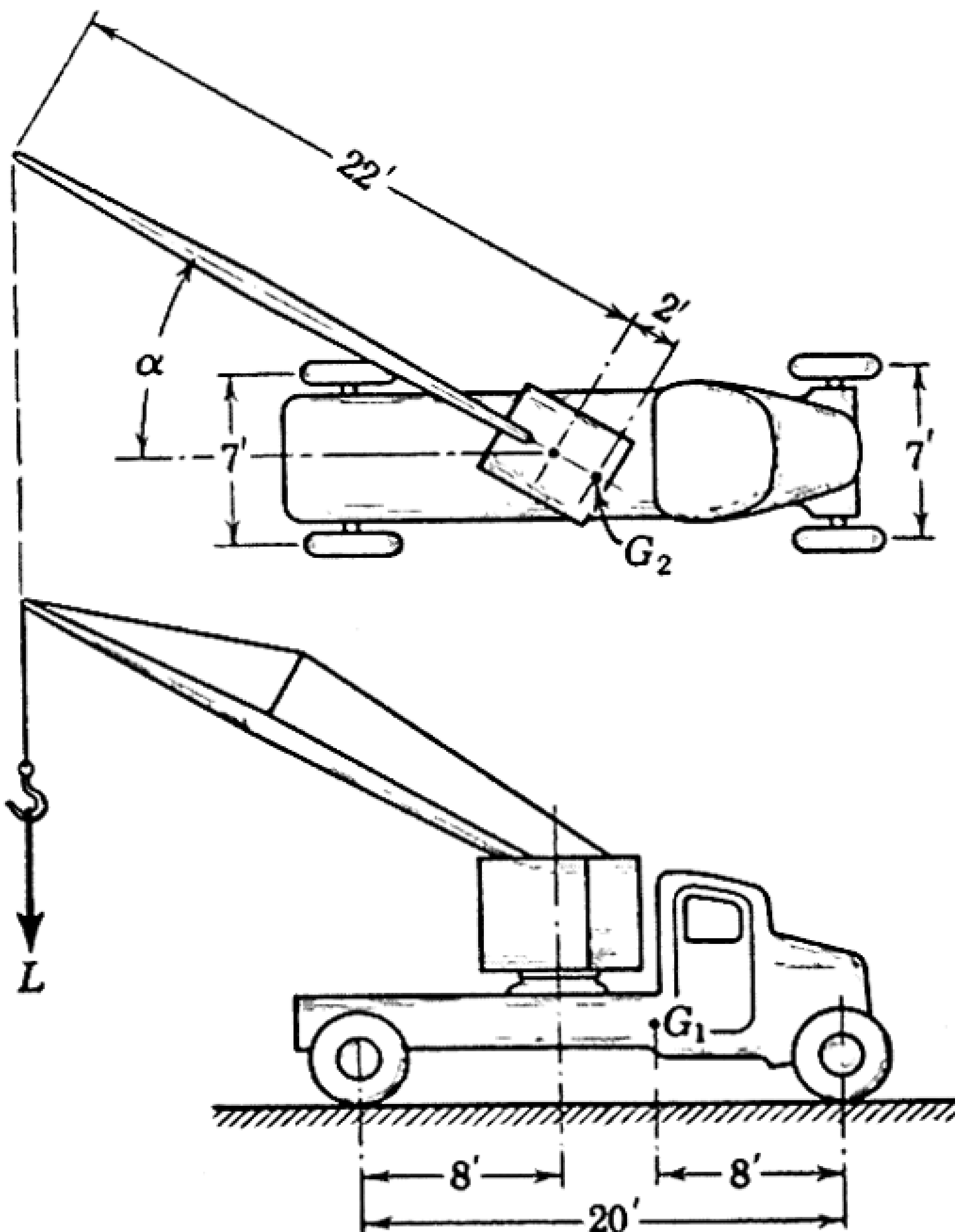


PROB. 225

* **225.** Work Prob. 224 for $\theta = 20$ deg. (*Hint:* The unit will have a tendency to tip as shown. It may be assumed that the bolts at A and B contribute equally to overcoming this tendency.)

* **226.** The weight of the truck, exclusive of boom and revolving cab, is 5 tons and its center of gravity is at G_1. The boom and revolving cab with hoisting machinery together weigh 3 tons and have a center of gravity at G_2. Plot the maximum load L which the crane can support without tipping as a function of α up to 90 deg. For what angle α and corresponding load L will the crane tip on the left rear wheel only?

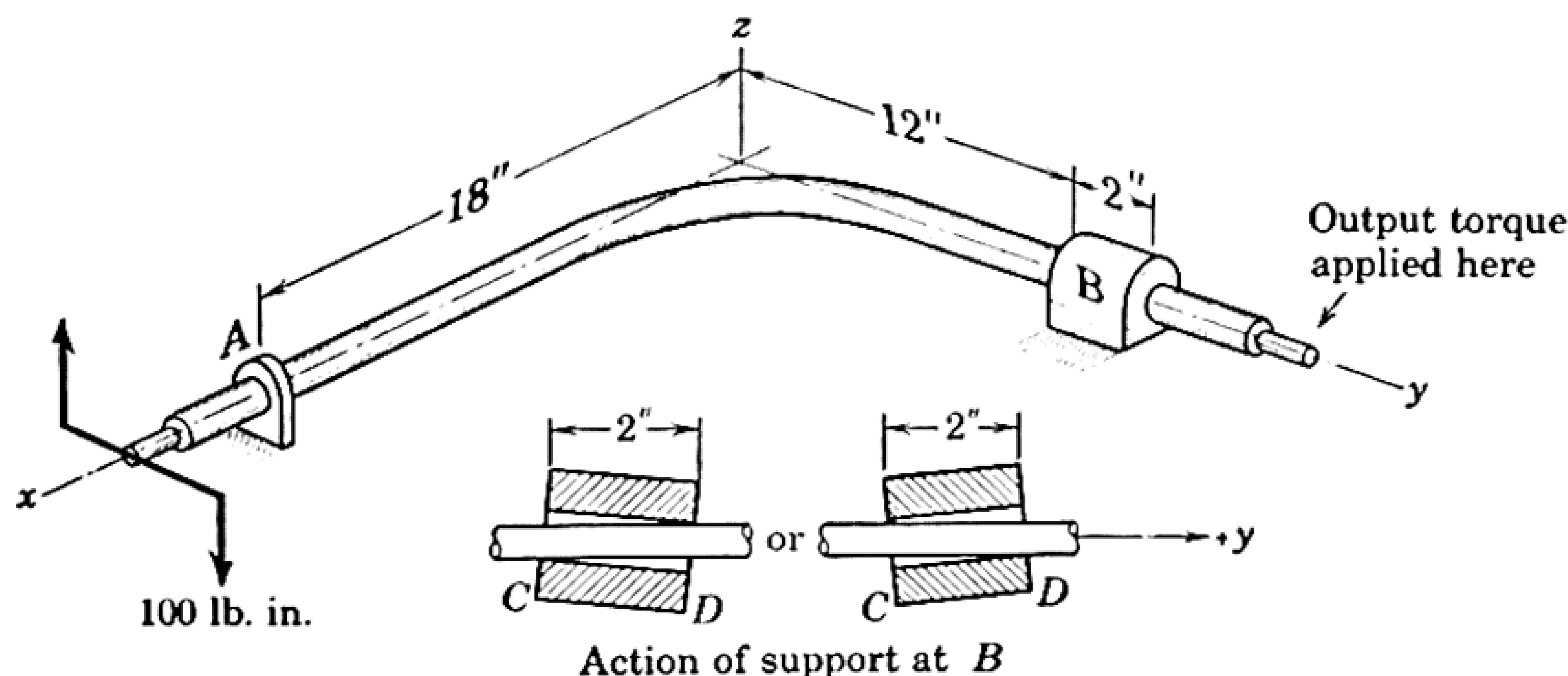
Ans. $\alpha = 21°$, $L = 7.1$ tons



PROB. 226

* **227.** A flexible shaft is bent through 90 deg. in the horizontal plane, and an input torque of 100 lb. in. is applied at one end as shown. After the initial angular twist of the shaft has taken place,

one revolution of the input end will be accompanied by one revolution of the output end, and the output torque will equal the input torque. The shaft housing is mounted in a self-aligning support at A which is capable of exerting only a single radial force normal to the shaft axis. Support B, on the other hand, is a loose-fitting sleeve and can support the shaft and housing as shown



PROB. 227

in the separate sectional views. It may be assumed that the shaft housing acts like a rigid tube for any position in which it is bent. Determine the forces exerted by the supports on the shaft housing. Neglect the weight of the shaft.

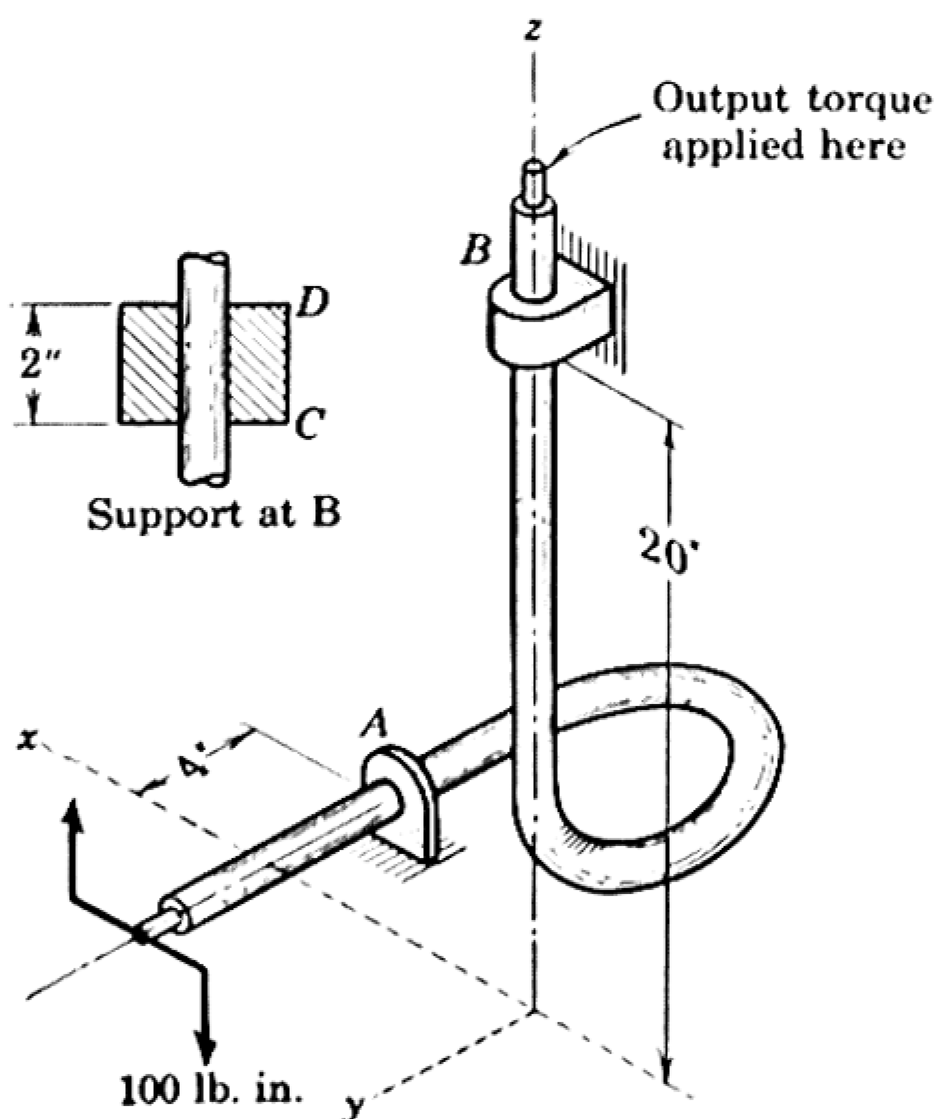
Ans. A = 5.6 lb. down, C = 11.1 lb. down, D = 16.7 lb. up

*** 228.** If the support at B in Prob. 227 is to be replaced by a support of the type at A, find the distance y from the z-axis at which it must be placed and the force exerted by the two supports on the shaft.

Ans. A = 5.6 lb. down, B = 5.6 lb. up, y = 18 in.

*** 229.** If the flexible shaft described in Prob. 227 is bent into the position illustrated with the present problem and the supports are of the type used with Prob. 227, determine the forces exerted by the supports on the shaft.

Ans. A = 25 lb. plus x-direction, C = 325 lb. minus x-direction, D = 300 lb. plus x-direction



PROB. 229

25. Graphical Solution for Three-Dimensional Equilibrium. The necessary conditions for equilibrium expressed vectorially by Eqs. (12) require that the space polygon of external forces, considered as free vectors, must close and the space polygon of moment or couple vectors must also close. For two-dimensional problems closure of the force and string polygons satisfies these two requirements as was shown in Art. 23. Extension of this procedure to cover the three-dimensional case involves a closed space polygon of forces to insure zero resultant force and a closed space couple polygon to insure zero resultant couple. The construction of these polygons on paper could be accomplished by drawing the three related orthogonal projections of the two space polygons on the three coordinate planes. Such a process involving both polygons is awkward and does not offer an attractive method of solution.

Fortunately many three-dimensional problems involve the equilibrium of concurrent forces or forces which can be easily reduced to a concurrent

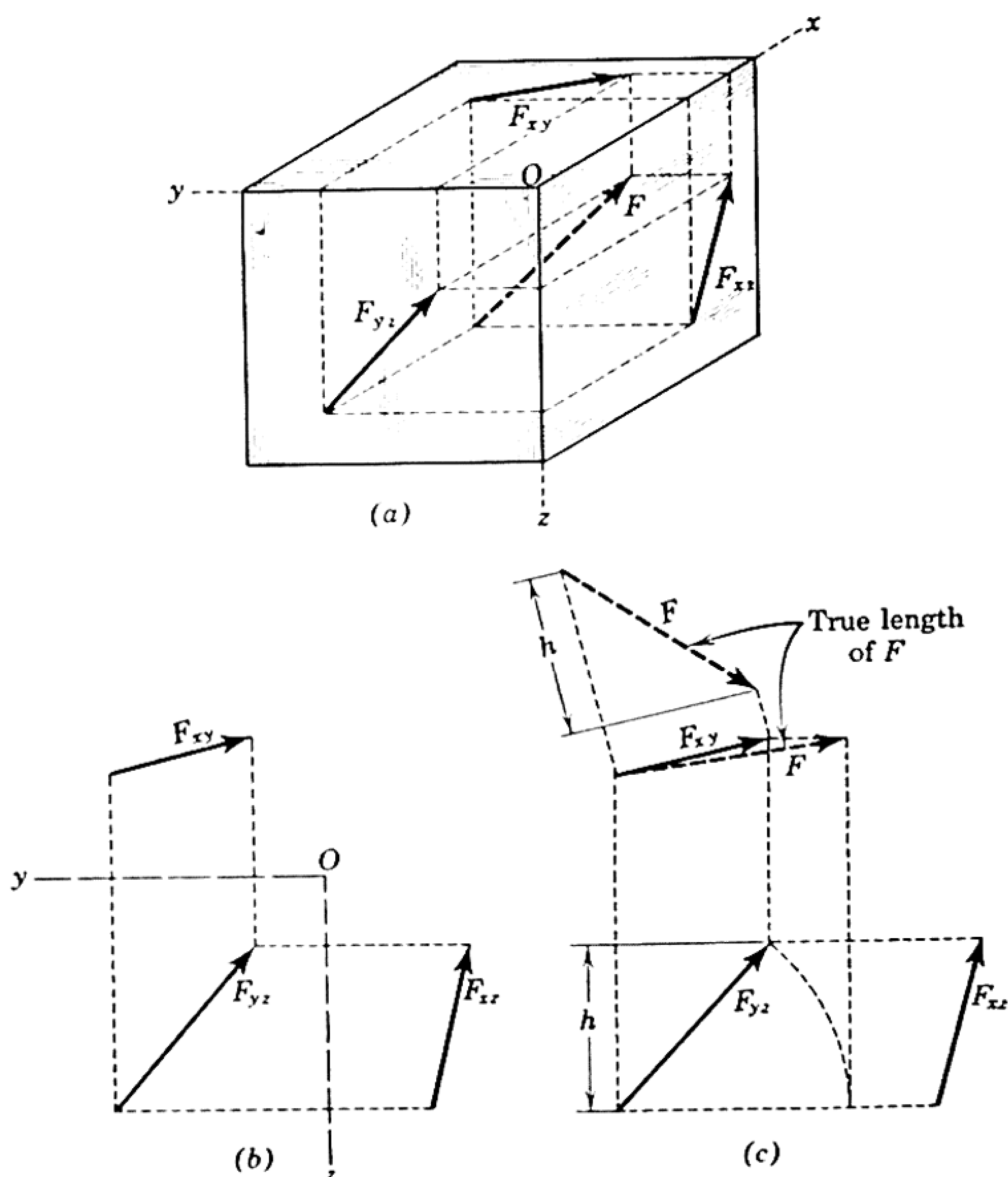


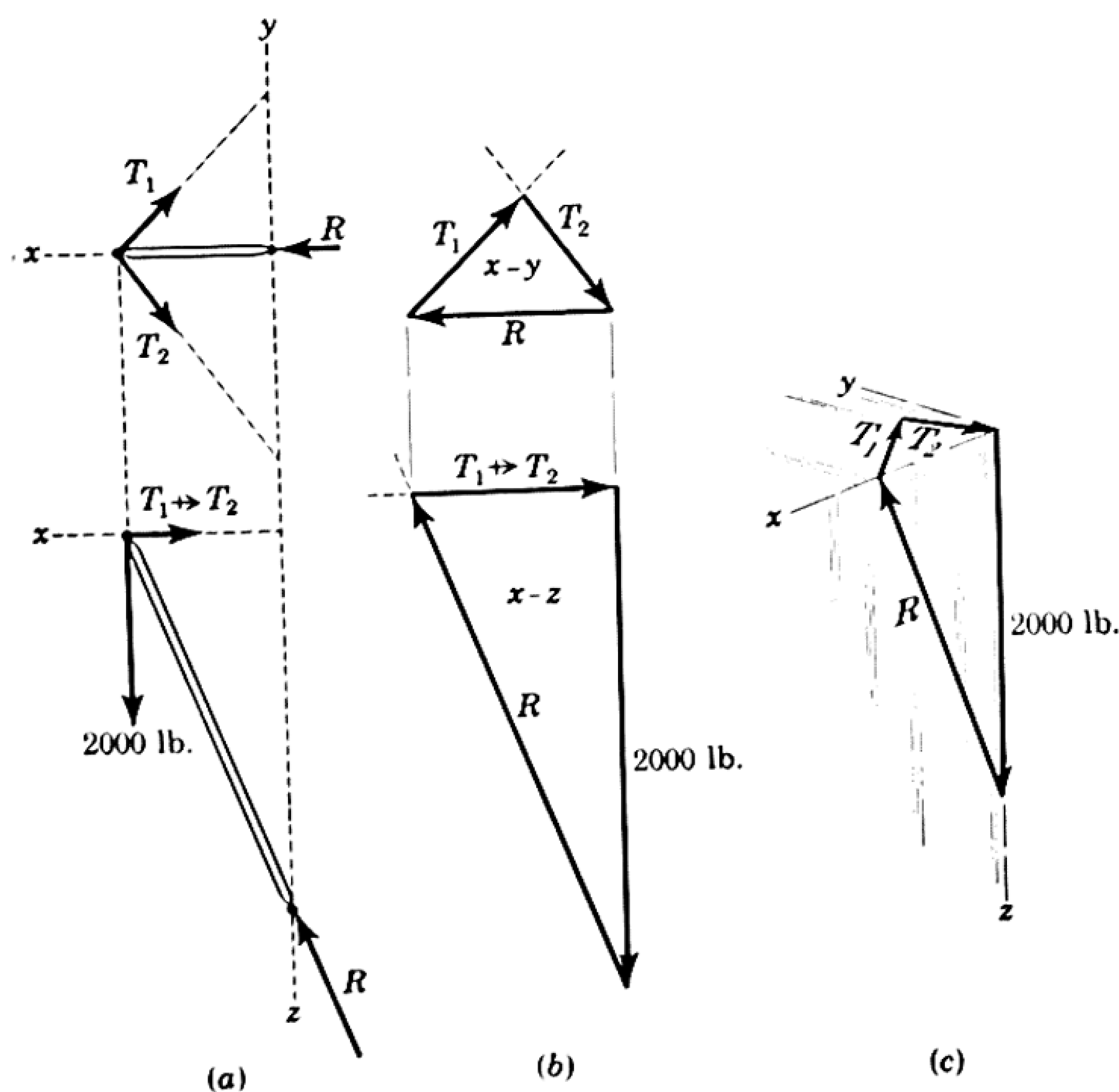
Fig. 33

system. Under these conditions the requirement of zero resultant moment is automatically satisfied, and closure of the space polygon of forces is the only condition which need be considered. Closure of the force polygon is best accomplished by constructing its orthogonal projections and treating each as a two-dimensional problem. In many instances only two projections are needed although for some problems all three projections are required.

It is essential before proceeding further that several basic laws of projective geometry be reviewed. In Fig. 33*a* the directed line segment representing the force F may be located in space by its three coordinate projections. By severing the edge Ox and unfolding along Oy and Oz there result the normal three views of third-angle projection as shown in Fig. 33*b*. The magnitude of the force is obtained in Fig. 33*c* from an auxiliary projection which shows the true length of the vector. This true length may also be found by rotation as shown.

SAMPLE PROBLEMS

230. Solve Prob. 196 graphically.



PROB. 230

Solution: Two views of the boom are laid off to scale in the *a*-part of the figure, and the free-body diagram of each view is completed by drawing the corresponding x–z and x–y projections of all forces on the two figures. In the x–z plane the x-components of T_1 and T_2 will appear as a single force and are labeled $T_1 \nrightarrow T_2$. Equilibrium of the boom requires the compression R to act along its axis.

The triangle of forces in the x–z plane is constructed in the *b*-part of the figure by starting with the known 2000 lb. load. From this figure the magnitude of R is obtained since the vector R is seen in its true length. The top or x–y view of the space polygon of forces is drawn next, and the forces are added in the same sequence as in the x–z projection. The 2000 lb. load does not appear in this view so that only the horizontal projection of R plus the two tensions T_1 and T_2 are involved. Again the triangle is drawn by starting with the known projection of R. The values of T_1, T_2, and R are scaled from the views which show them in their true length, and they are $T_1 = 624$ lb., $T_2 = 557$ lb., $R = 2130$ lb.

In the *c*-part of the figure is shown the space polygon of forces which was represented first by the two projections. Any order of addition of the forces may be used except that it is usually necessary to start with the known loads first.

231. If the weight of the mast is neglected compared with the applied load, determine graphically the two cable tensions T_1 and T_2 and the force R acting at the ball joint at A.

Solution: The three orthogonal projections of the mast and cables are constructed to scale and used for the three views of the free-body diagram of the mast. In the y–z projection only three forces appear, so that their projections in this view must satisfy the principle of concurrency of three forces in equilibrium. Thus the direction of the y–z projection of R is drawn through point O_1. Actually the three forces all pass through a line through O_1 perpendicular to the y–z plane. The y–z projection of the space polygon of forces may now be drawn as shown in the *c*-part of the figure. The two resulting vectors R and T_2 are not seen in their true length so their proper magnitudes are not yet obtained.

From the x–z view of the free-body diagram the proper direction of the projection of R is obtained by constructing it through the point of concurrency O_2 of the three vectors. The x–z view of the force polygon is now drawn with the aid of the previous construction and the x–z free-body diagram. By starting with R its x–z projection is constructed parallel to that determined in the free-body diagram, and its terminal points must lie in the same horizontal planes as its established terminal points in the y–z view. Next T_2 and T_1 are drawn with their proper directions to complete the triangle.

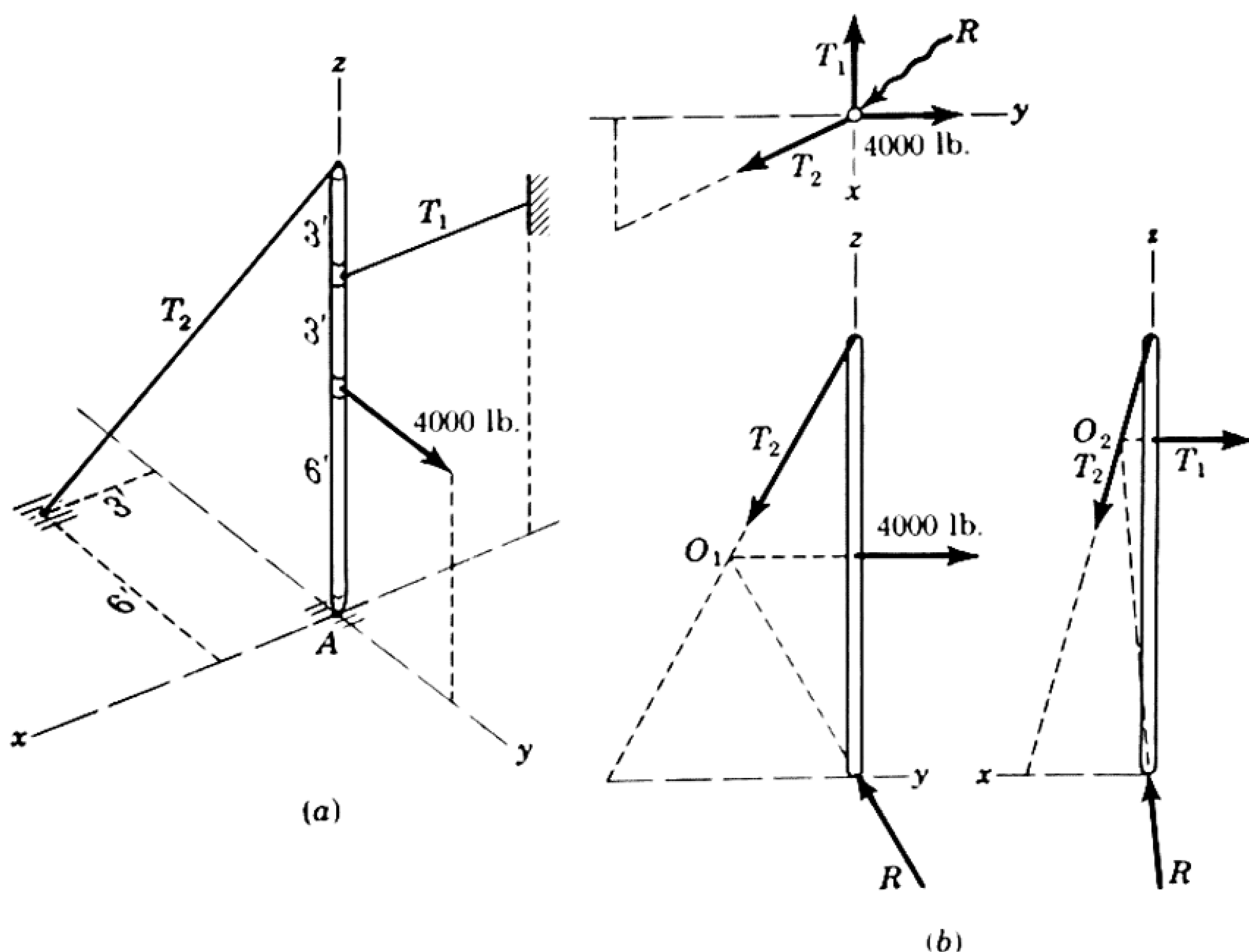
The magnitude of T_1 may be scaled directly from the x–z triangle and is

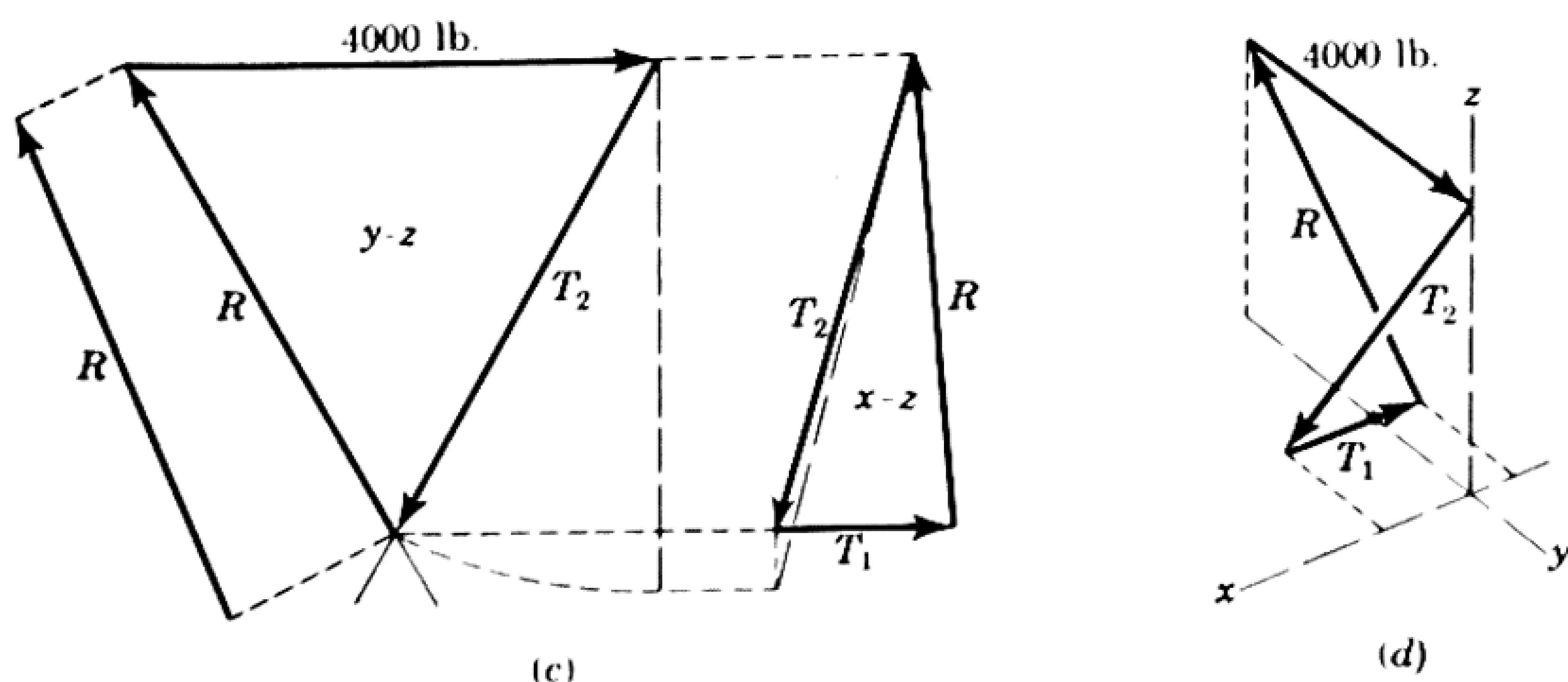
$$T_1 = 1330 \text{ lb.} \qquad \textit{Ans.}$$

The true lengths of R and T_2 must be constructed since neither of the views

shows either one in its true length. The true length of R is constructed in the auxiliary projection and is found to be

$$R = 4490 \text{ lb.} \qquad \textit{Ans.}$$





Prob. 231

The true length of T_2 is found by rotation (to illustrate the second method) and is shown in the dotted line whose magnitude is

$$T_2 = 4580 \text{ lb.} \qquad \textit{Ans.}$$

The space polygon of forces represented initially by its two projections is shown by way of further illustration in the d-part of the figure.

PROBLEMS

Solve the following problems graphically.

232. Prob. 200.
233. Prob. 201.
234. Prob. 204 with $W = 900$ lb. and $b/l = \frac{2}{3}$.
235. Prob. 205.
236. Prob. 206.
237. Prob. 207.
238. Prob. 212.
* **239.** Prob. 208.
* **240.** Prob. 219.
* **241.** Prob. 213 for $W = 4$ lb. and $R = 4r = 8$ in.
* **242.** Prob. 214 if the radius of each sphere is 2 in.

26. Statically Indeterminate Problems. In all the problems and examples of equilibrium treated so far, care has been taken to select only those situations for which the necessary conditions of equilibrium [Eqs. (12) or (16)] were also sufficient conditions. There is a class of problems for which these equations, although necessary, are not sufficient for determining all forces acting on a body in equilibrium. Bodies which have more supports than are necessary to maintain an equilibrium configuration belong to this class and are called *statically indeterminate.* This term means that the statical equations of equilibrium are insufficient to determine all the forces acting.

When any supporting element of a body can be removed without destroying the equilibrium position, the body is statically indeterminate with that member in place. Such a member or supporting element is said to be *redundant.* The more redundant elements present the greater the degree of statical indeterminancy.

Bodies which are statically indeterminate require, in addition to the statical equations of equilibrium, certain relations between the externally applied forces and the resulting strains or movements of the body. These additional relations involve the physical properties of the body and are studied under the heading of strength of materials and other subjects.

Four simple examples of statically indeterminate bodies together with their free-body diagrams are shown in Fig. 34. In the first example there are only two equations of equilibrium for the parallel force system but there are three unknown reactions. The beam will remain in equilibrium if the support at B is removed. The load-deformation properties of the beam must be accounted for if the three reactions are to be computed. In example 2 each pin reaction will have an x- and a y-component for a

given load P. Thus there are four unknowns, one more than the three equations of equilibrium will separate. The components A_y and B_y can be computed by the moment equation applied about either end, but A_x and B_x cannot be computed separately. If either pin connection were

Statically Indeterminate Body	Free-Body Diagram
1 P_1 P_2 A B C	P_1 P_2 A B C
2 P A B	P A_x B_x A_y B_y
3 P A B	P A_x B_x A_y B_y
4 P A B	P A_x B_x A_y B_y

FIG. 34

replaced by a roller, the beam would still retain its equilibrium configuration, and the three unknown forces could be determined. The A-frame in example 3 is statically indeterminate for the same reason as for example 2. If the pin at B is replaced by a roller, the frame would still be in equilibrium, and all external reactions could be computed. Or, if the horizontal supporting member were removed, the frame would not collapse and a solution could be made. In like manner the truss in the last example is statically indeterminate. Removal of the member AB or

substitution of a rocker at B would eliminate the redundant supporting element, and the truss would be statically determinate.

It is essential to be able to recognize when a problem is statically indeterminate; otherwise effort may be wasted in attempting an impossible solution with the aid of the equilibrium equations only. If a body is in equilibrium with the minimum number of supports to maintain its position, the equations of equilibrium are not only necessary but are also sufficient conditions for the determination of all forces external to the body in question.

CHAPTER IV

Structures

27. Trusses. A framework composed of members joined at their ends to form a rigid structure is known as a truss. Bridges, roof supports, derricks, and other such structures are common examples of trusses. Structural members used are I-beams, channels, angles, bars, and special shapes which are fastened together at their ends by riveted connections, welded joints, or large bolts or pins.

The basic element of a truss is the triangle. Three bars joined by pins at their ends, Fig. 35*a*, constitute a rigid frame. Four or more bars pin-

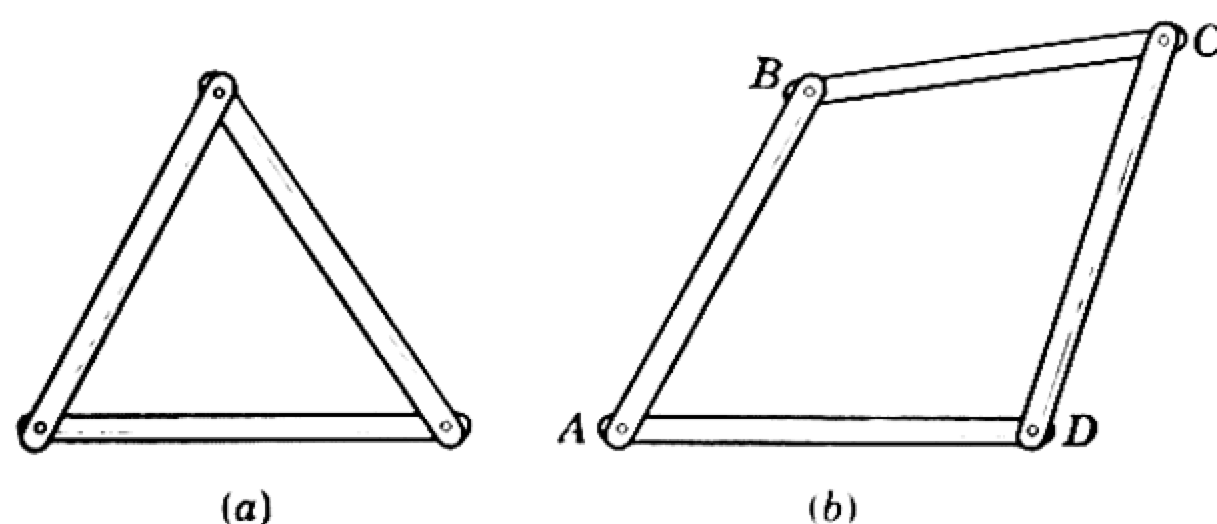


FIG. 35

jointed to form a polygon of as many sides constitute a nonrigid frame. The nonrigid frame in Fig. 35*b* can be made stable or rigid with an additional diagonal bar joining A and C or B and D and thereby forming two triangles. Additional bars may be added to form attached triangles, and the entire structure will remain rigid. The term rigid is used in the sense of noncollapsible and also in the sense of negligible deformation of the members due to induced internal strains. Structures composed of triangular elements in the manner described are known as *simple trusses.* When more members are present than are needed to prevent collapse, the truss is statically indeterminate. A statically indeterminate truss cannot be analyzed by the equations of equilibrium alone. Such additional members or supports which are not necessary for maintaining the equilibrium position are called *redundant.*

The design of a truss involves the determination of the forces in the various members and the selection of appropriate sizes and structural

shapes to withstand the forces. There are several assumptions made in the force analysis of simple trusses. First, all members are assumed to be *two-force members.* A two-force member is one in equilibrium under the action of two forces only. These two forces are applied at the ends of the member and are necessarily equal, opposite, and *collinear* for equilibrium. The member may be in tension or compression, as shown in Fig. 36. It should be noted that in representing the equilibrium of a

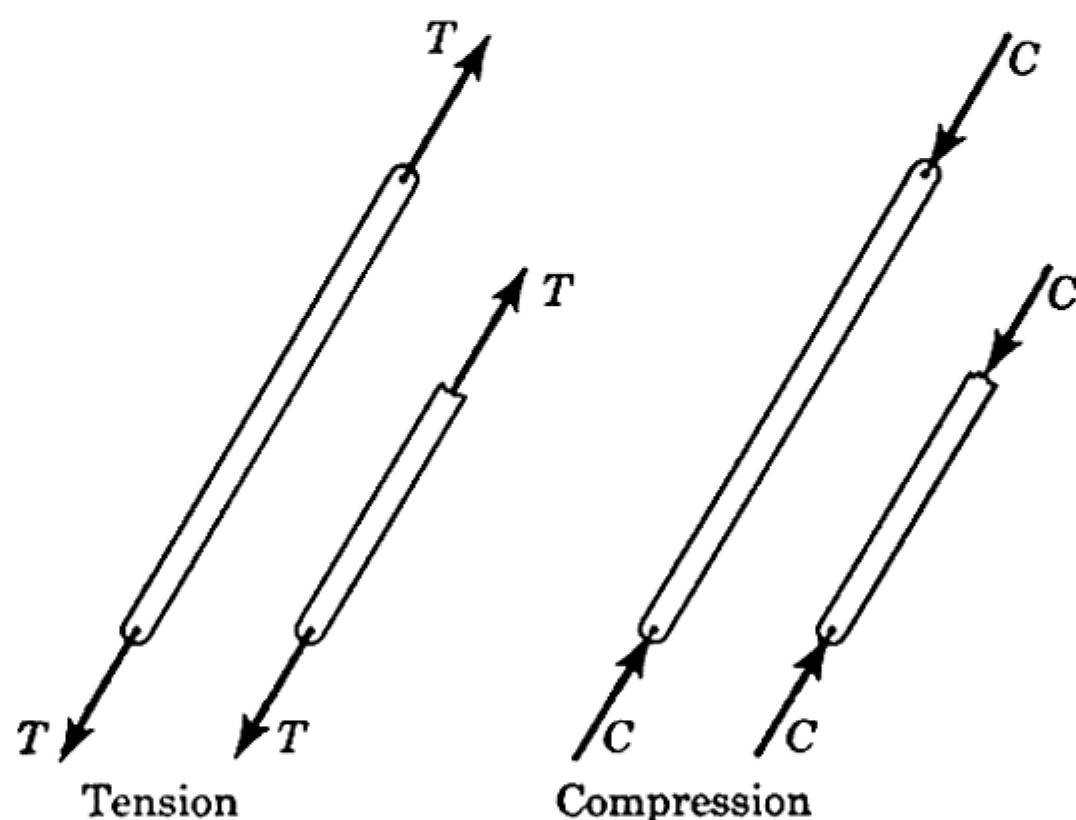


Fig. 36

portion of a two-force member the tension T or compression C acting on the cut section is the same for all sections. It is assumed here that the weight of the member is small compared with the force it supports. If such is not the case or if the small effect of the weight is to be accounted for, the weight W of the member, if uniform, may be assumed to be replaced by two forces, each $W/2$, acting at each end of the member. These forces, in effect, are treated as loads externally applied to the pin connections. Accounting for the weight of a member in this way gives the correct result for the average tension or compression along the member but will not account for the effect of bending of the member.

When riveted or welded connections are used to join structural members, the assumption of a pin-jointed connection is usually satisfactory if the centerlines of the members are concurrent at the joint as in Fig. 37.

It is furthermore assumed in the analysis of simple trusses that all external forces are applied at the pin connections. This condition is satisfied in most trusses. In bridge trusses the deck is usually laid on cross beams that are supported at the joints.

Provision for expansion and contraction due to temperature changes and for deformations resulting from applied loads is usually made at one of the supports for large trusses. A roller, rocker, or some kind of slip

joint is provided. Trusses and frames wherein such provision is not made are statically indeterminate, as explained in Art. 26.

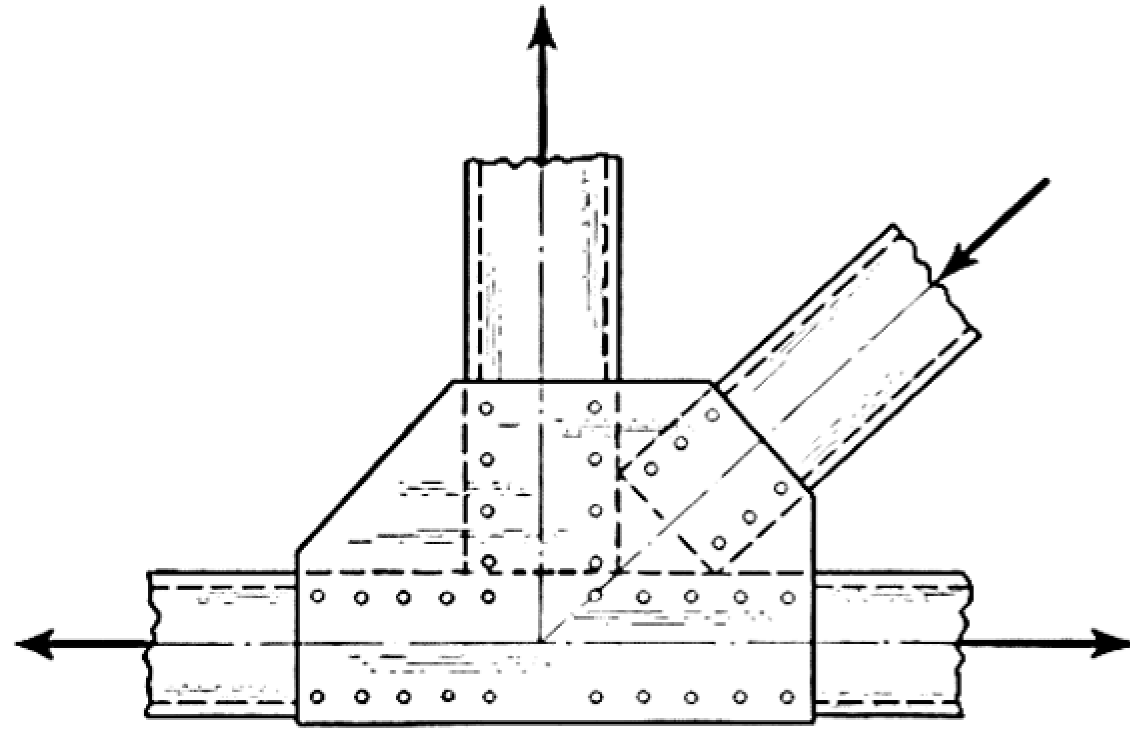


FIG. 37

Three methods for the force analysis of simple trusses will be given, and reference will be made to the simple truss shown in Fig. 38*a* for each of the three methods. The free-body diagram of the truss as a whole is shown in Fig. 38*b*, and for this example a system of Bow's notation is used here to designate each member and all forces. The spaces external to the truss between points of application of the external forces are labeled with small letters in a clockwise order. Next the triangular spaces within the truss are labeled in any order. The applied load L may now be designated by the corresponding capital letters AB, and the support reactions

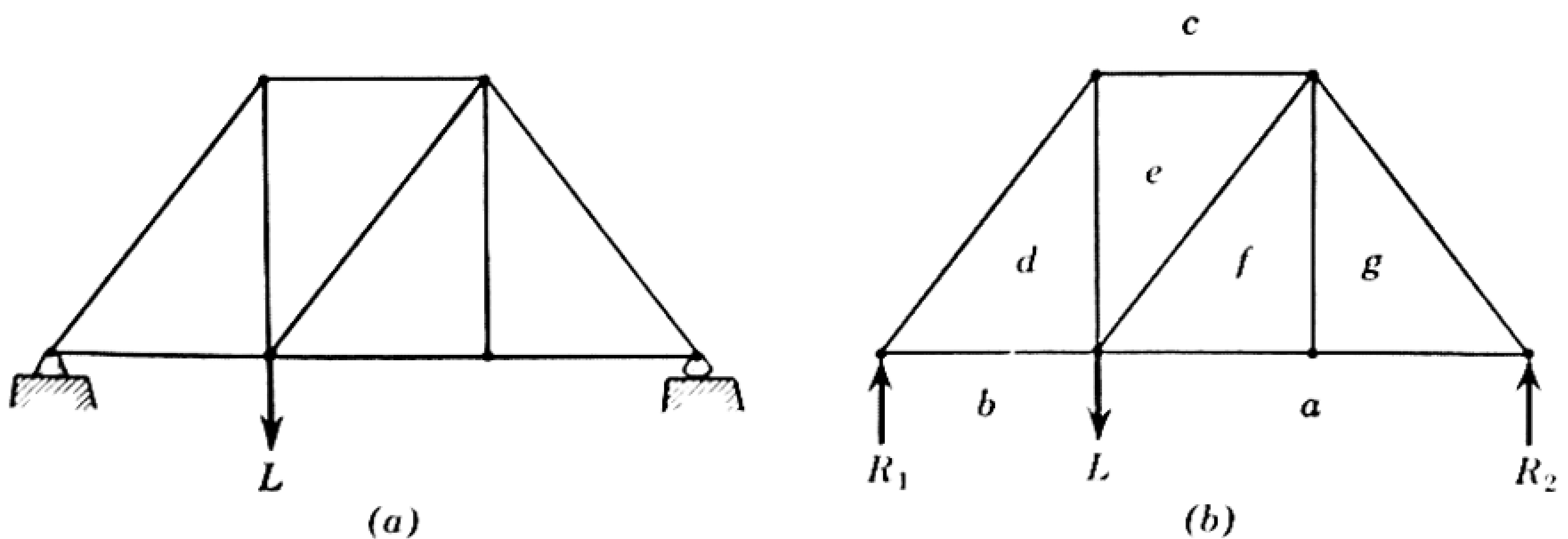


FIG. 38

R_1 and R_2 are designated as BC and CA, respectively. The external reactions are usually determined by computation from the equilibrium equations applied to the truss as a whole before proceeding with the force analysis of the remainder of the truss.

28. Method of Joints. Although any convenient notation may be used to designate each member and joint in a simple truss, such as the labeling of each joint by a letter or number, Bow's notation is employed

for illustration here by reason of its convenience for the graphical analysis to be explained in the next article for this same truss. With Bow's notation each joint is uniquely designated by the clockwise sequence of letters surrounding the joint. Thus the pin connection at the left support is designated as joint *bcd*, and the joint at which the load L is applied is denoted by *abdef*. The *method of joints* consists of satisfying the conditions of equilibrium for the forces acting on the connecting pin of each joint. The method therefore deals with the equilibrium of concurrent forces, and only two independent equilibrium equations are involved. The analysis is begun with any joint where at least one known load exists and where not more than two unknown forces are present. Solution may be started with the pin at the left end, and its free-body diagram is shown in Fig. 39. The proper directions of the forces should be evident

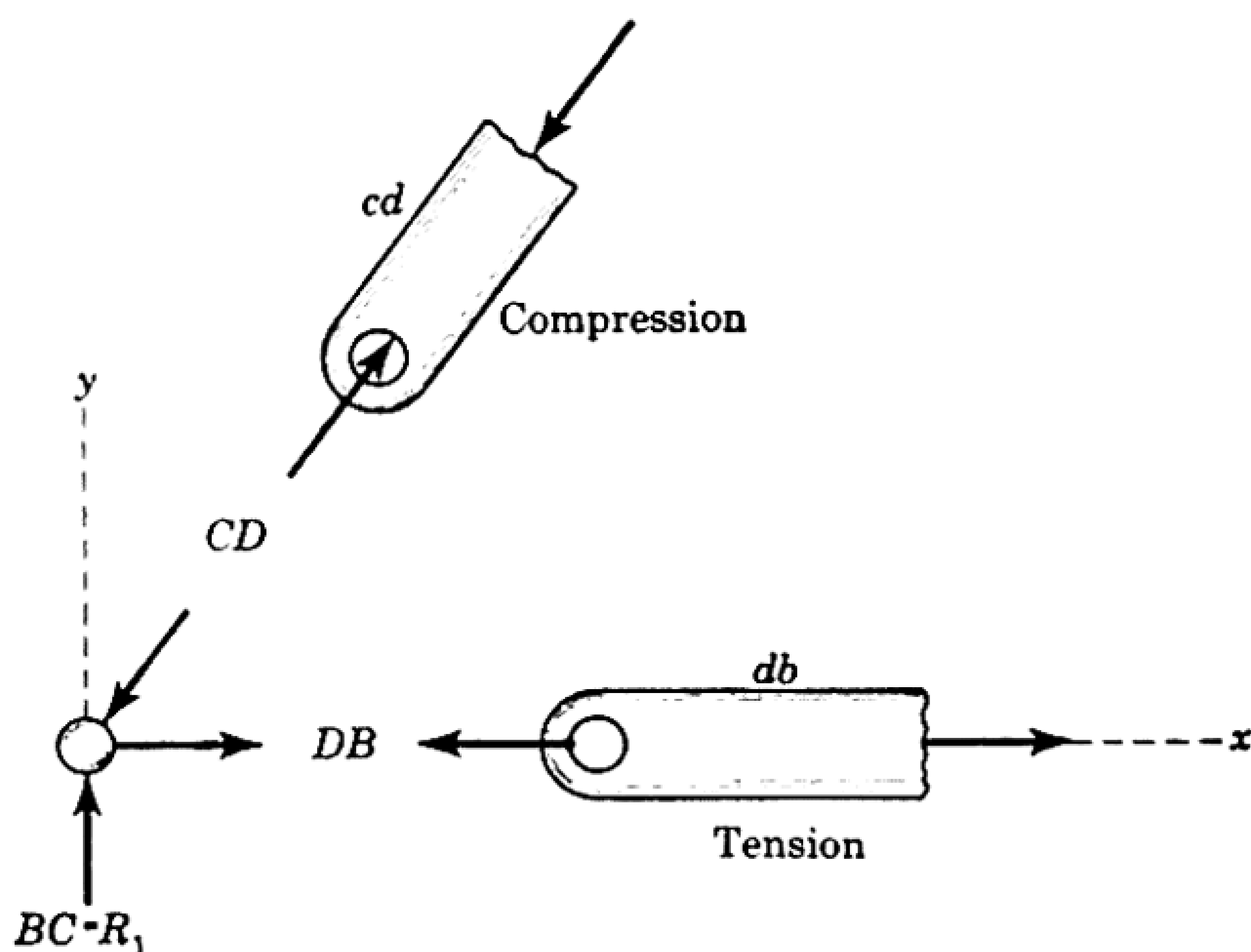


FIG 39

for this simple case by inspection. The free-body diagrams of portions of members *cd* and *db* are also shown to indicate clearly the mechanism of the action and reaction. The member *db* actually makes contact on the left side of the pin, although the force DB is drawn from the right side and is shown acting away from the pin. Thus, if the force arrows are consistently drawn on the *same* side of the pin as the member, then tension (such as DB) will always be indicated by an arrow *away* from the pin, and compression (such as CD) will always be indicated by an arrow *toward* the pin. The magnitude of CD is obtained from the equation $\Sigma F_y = 0$, and DB is then found from $\Sigma F_x = 0$.

Joint *dce* must be analyzed next since it now contains only two unknowns CE and ED. Joints *bdefa, afg, cgfe,* and *agc* are subsequently

analyzed in that order. The free-body diagram of each joint and its corresponding force polygon which represents graphically the two equilibrium conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are shown in Fig. 40. The

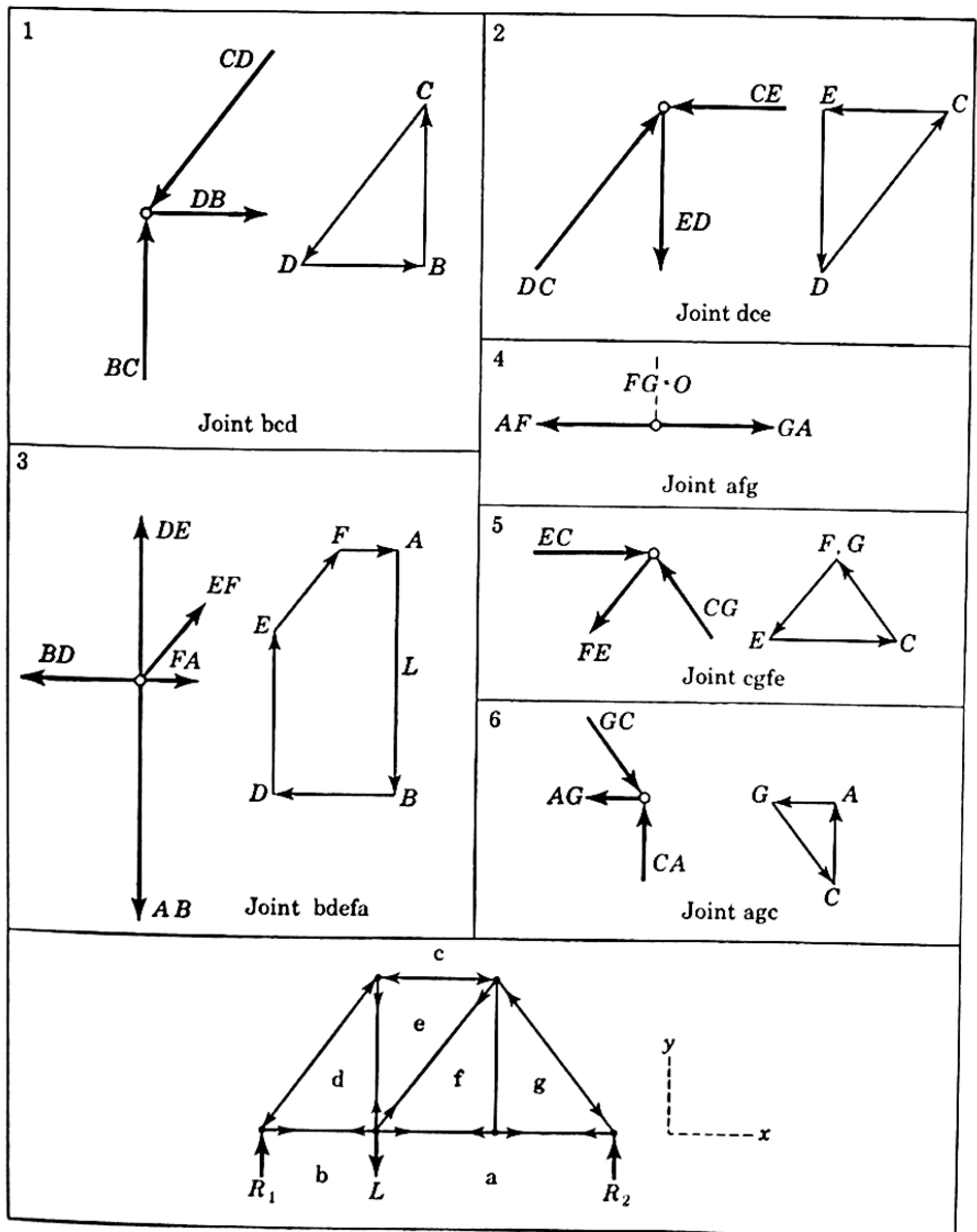


Fig. 40

numbers indicate the order in which the joints are analyzed. The force polygons shown have a particular significance which will be described in the next article. It should be noted that, when joint *agc* is finally

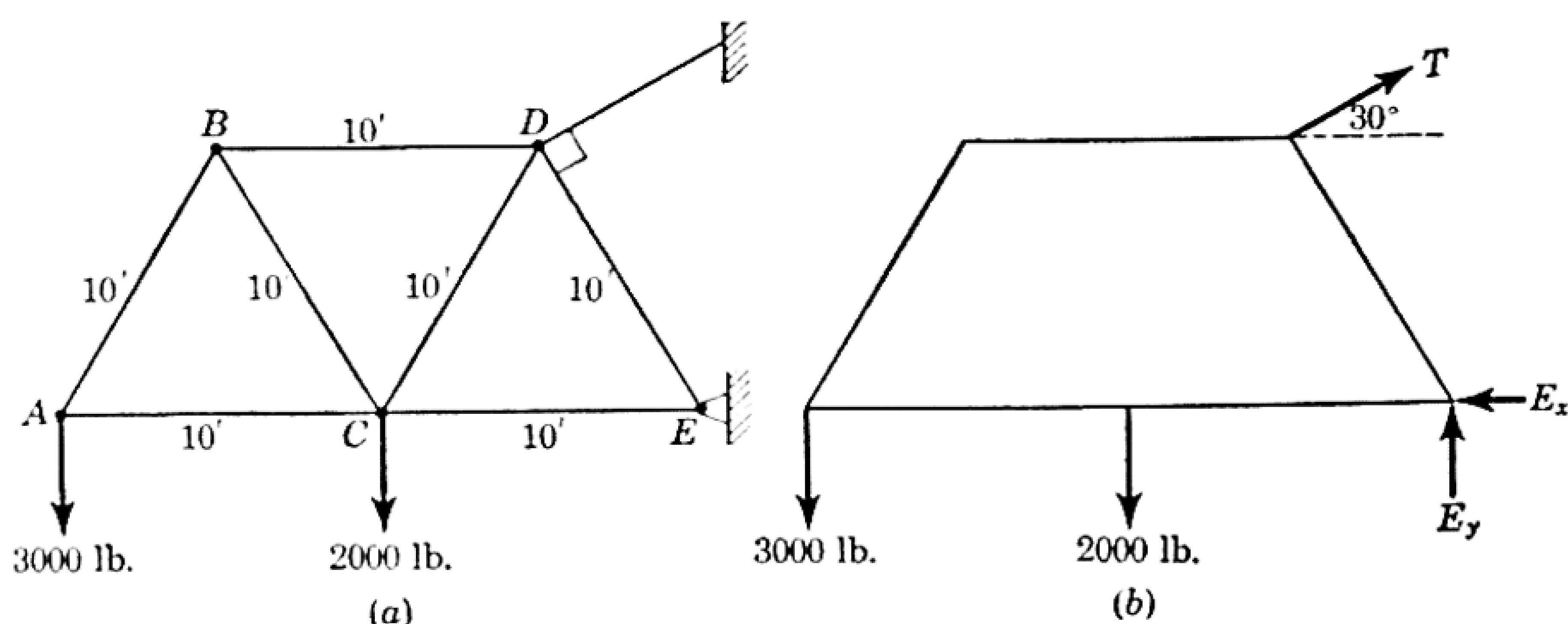
reached, the computed reaction R_2 must be in equilibrium with the forces in members *ag* and *gc*, determined previously from the two neighboring joints. This requirement will provide a check on the correctness of the work. It should also be noted that isolation of joint *afg* quickly discloses the fact that the force in *fg* is zero when the equation $\Sigma F_y = 0$ is applied. The force in this member would not be zero, of course, if an external load were applied at *afg*.

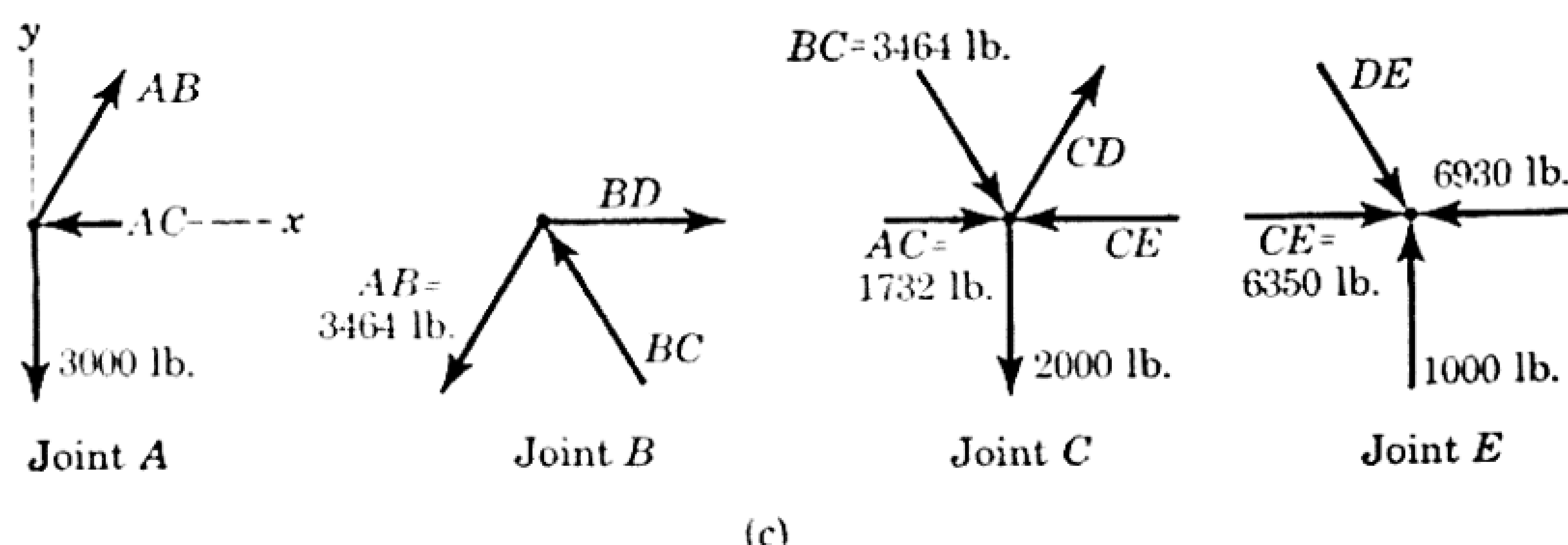
It is often convenient to indicate the tension and compression of the various members directly on the original truss diagram by drawing arrows away from the pins for tension and toward the pins for compression. This designation is illustrated at the bottom of Fig. 40.

In some instances it is not possible to assign initially the correct direction of one or both of the unknown forces acting on a given pin. In this event an arbitrary assignment may be made. A negative value from the computation indicates that the assumed direction should be reversed.

SAMPLE PROBLEM

243. Compute the forces in each member of the loaded cantilever truss.





PROB. 243

Solution: With the joints indicated by letters the force in each member will be designated by the two letters defining the ends of the member. If it were not desired to calculate the external reactions at D and E, the analysis for a cantilever truss could begin with the joint at the loaded end. However, this truss will be solved completely, so that the first step will be to compute the external forces at D and E from the free-body diagram of the truss as a whole in the b-part of the figure. The equations of equilibrium give

$$T = 8000 \text{ lb.,} \qquad E_x = 6930 \text{ lb.,} \qquad E_y = 1000 \text{ lb.}$$

In the c-part of the figure are drawn the free-body diagrams showing the forces acting on each of the connecting pins. The correctness of the assigned directions of the forces is verified when each joint is considered in sequence. There should be no question as to the correct direction of the forces on joint A. Equilibrium requires

$[\Sigma F_y = 0]$ $\quad 0.866AB - 3000 = 0, \quad AB = 3464$ lb. T, *Ans.*

$[\Sigma F_x = 0]$ $\quad AC - 0.5 \times 3464 = 0, \quad AC = 1732$ lb. C. *Ans.*

Joint B must be analyzed next since there are more than two unknown forces on joint C. The force BC must provide an upward component in which case BD must balance the force to the left. Again the forces are obtained from

$[\Sigma F_y = 0]$ $\quad 0.866BC - 0.866 \times 3464 = 0, \quad BC = 3464$ lb. C, *Ans.*

$[\Sigma F_x = 0]$ $\quad BD - 0.5 \times 2 \times 3464 = 0, \quad BD = 3464$ lb. T. *Ans.*

Joint C now contains only two unknowns, and these are found as before:

$[\Sigma F_y = 0]$ $\quad 0.866CD - 0.866 \times 3464 - 2000 = 0,$

$CD = 5774$ lb. T, *Ans.*

$[\Sigma F_x = 0]$ $\quad CE - 1732 - 0.5 \times 3464 - 0.5 \times 5774 = 0,$

$CE = 6350$ lb. C. *Ans.*

Lastly from joint E there results

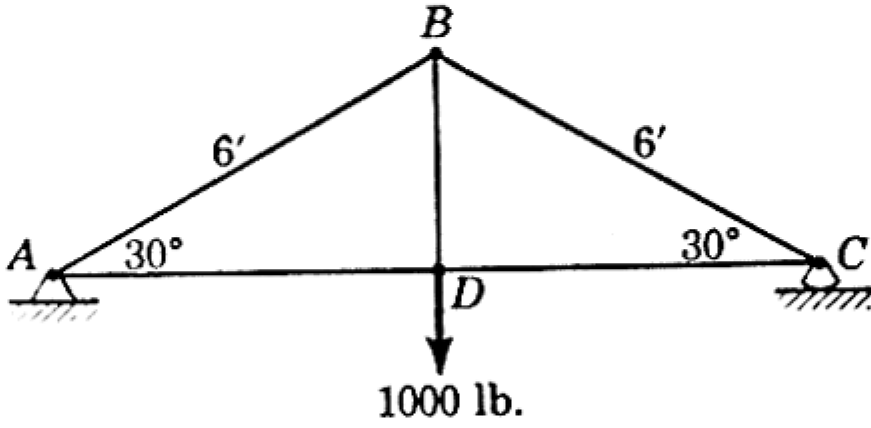
$[\Sigma F_y = 0]$ $\quad 0.866DE = 1000, \quad DE = 1154$ lb. C, *Ans.*

and the equation $\Sigma F_x = 0$ checks.

PROBLEMS

244. Determine the forces in each member of the truss.

Ans. $AB = BC = 1000$ lb. C, $AD = DC = 866$ lb. T, $BD = 1000$ lb. T

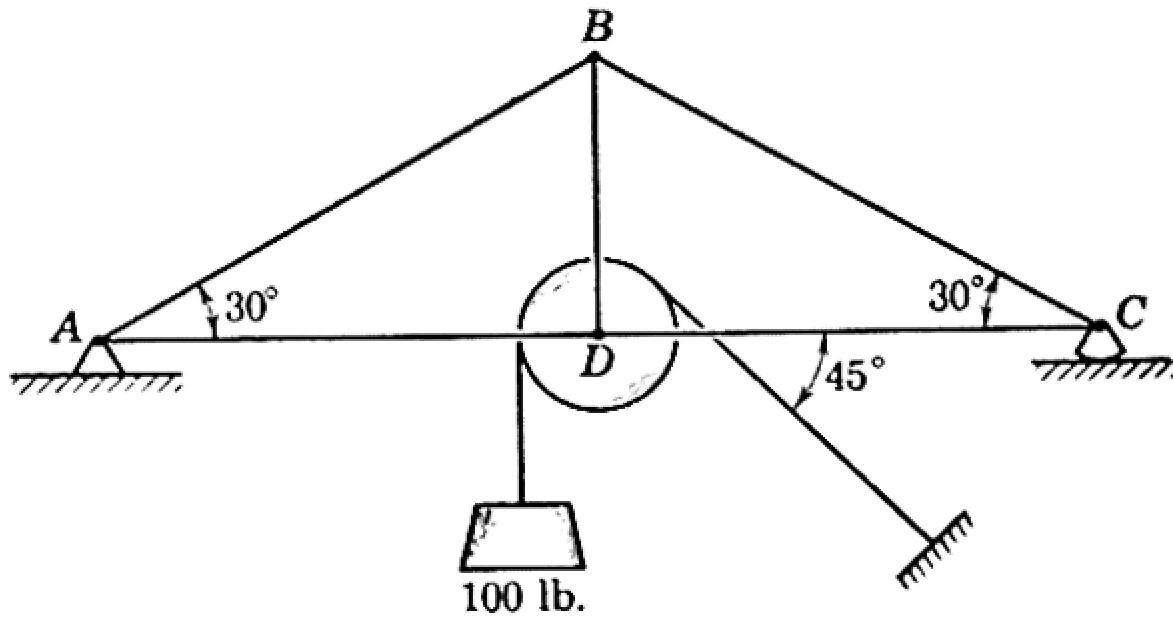


PROB. 244

245. If the 1000 lb. load in Prob. 244 is applied at B instead of D, determine the forces in the members. What would be the effect of applying the 1000 lb. load directly to the member BD at any location between the joints?

246. Find the force in each member of the hoisting truss.

Ans. $AB = BC = 170.7$ lb. C, $BD = 170.7$ lb. T, $AD = 219$ lb. T, $DC = 147.8$ lb. T

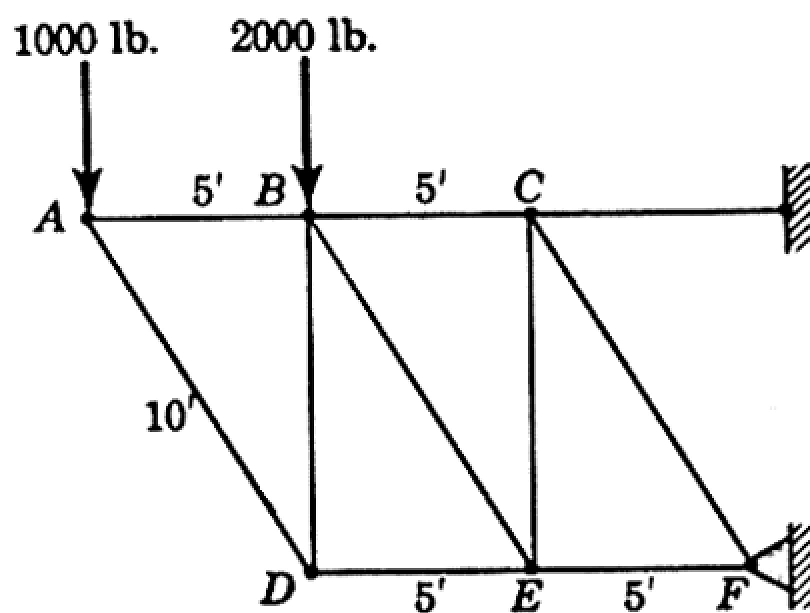


PROB. 246

247. If each member of the truss of Prob. 244 weighs 5 lb. per foot of length, determine the tensions and compressions in the members resulting from their weights alone.

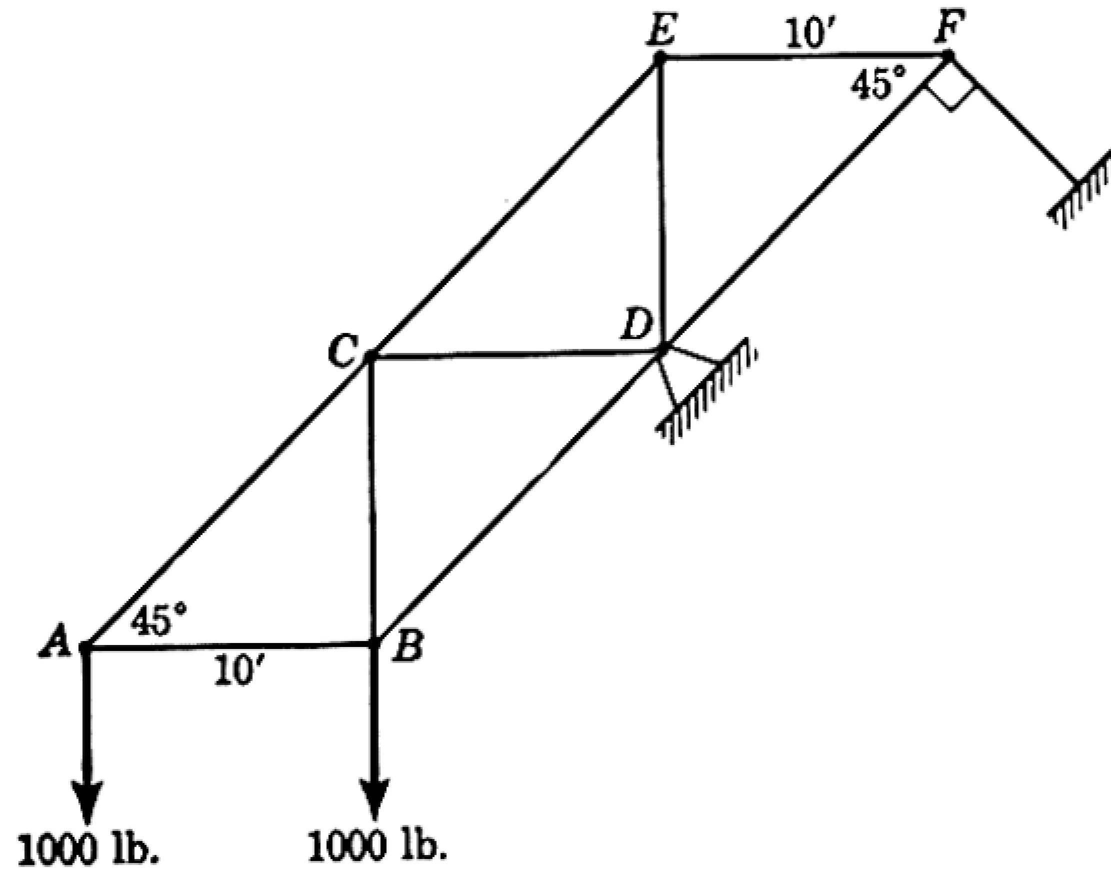
Ans. $AB = BC = 71.0$ lb. C, $AD = CD = 61.5$ lb. T, $BD = 33.5$ lb. T

248. Determine the forces in members BE and EF of the cantilever truss.



PROB. 248

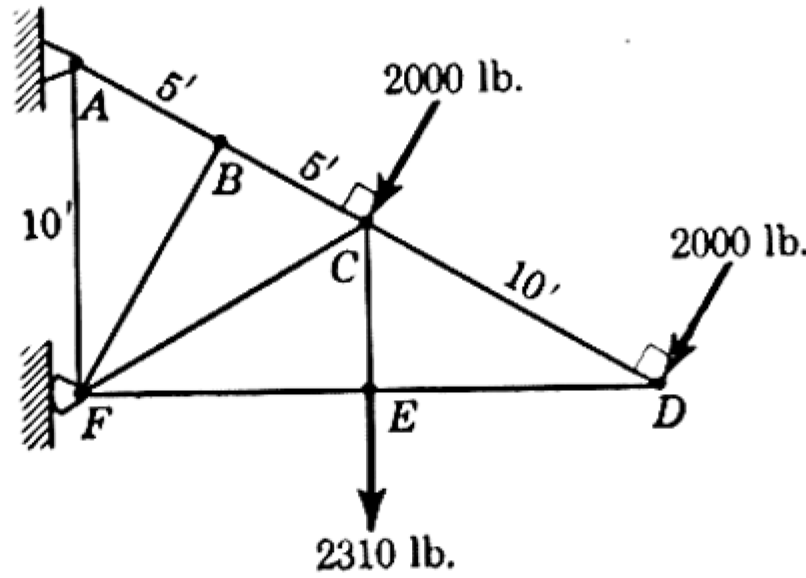
249. Determine the forces in members CD, CE, and DE of the loaded truss shown.



Prob. 249

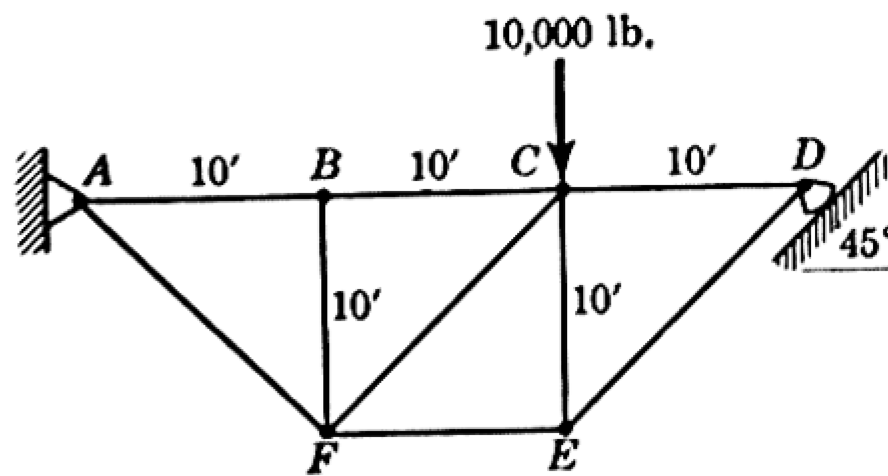
250. Determine the forces in members AB, CF, and EF of the cantilever truss.

Ans. AB = 6930 lb. T, CF = 4620 lb. C, EF = 4000 lb. C



Prob. 250

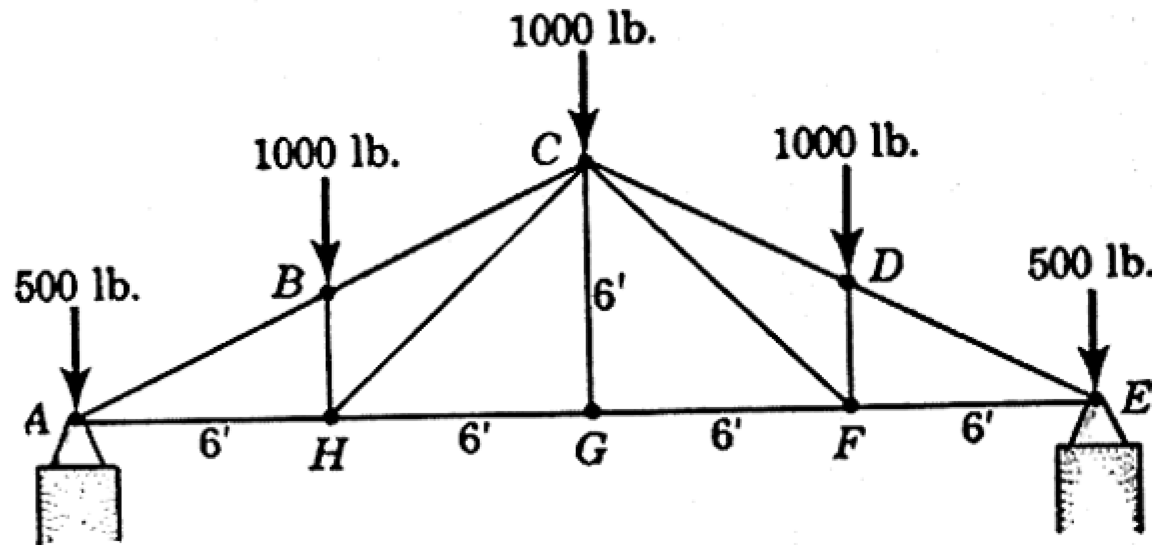
251. Determine the forces in members BC, CF, and FE of the truss.



Prob. 251

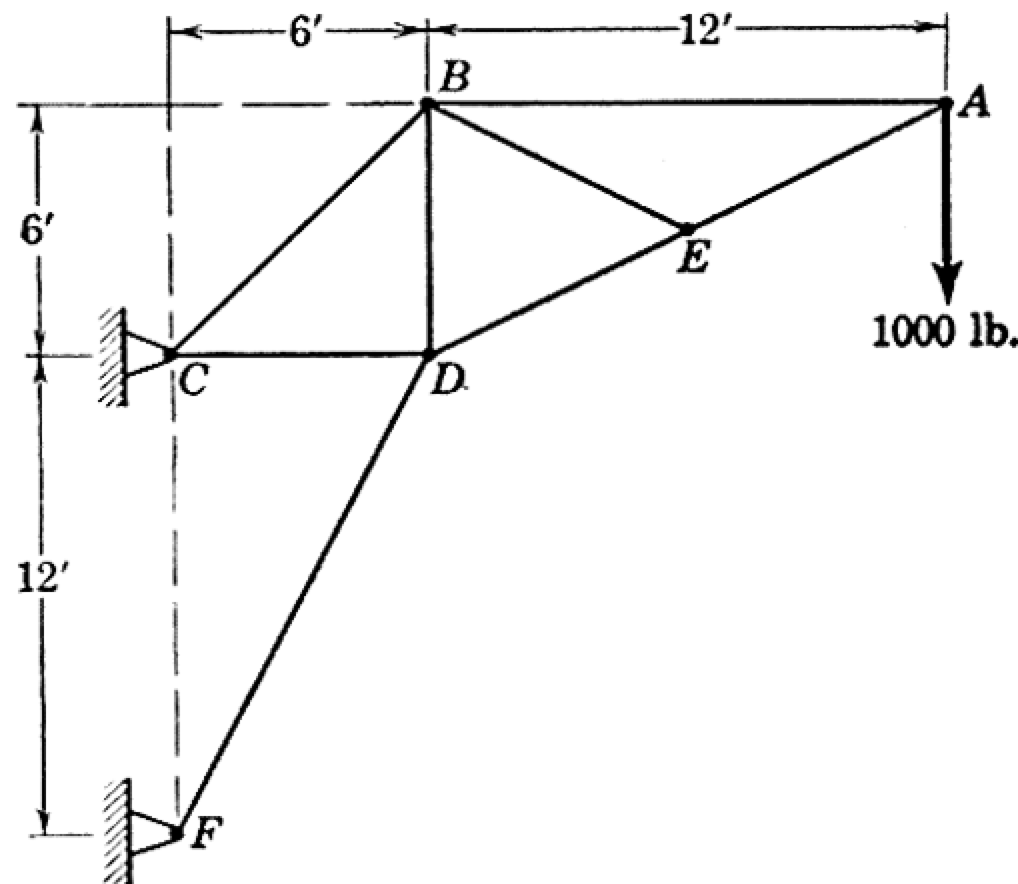
252. The weight of snow transfers the forces shown to each of the upper joints of the Pratt roof truss. Determine the forces in members *BH*, *BC*, and *CH*. Neglect any horizontal reactions at the supports.

Ans. $BH = 1000$ lb. *C*, $BC = 3350$ lb. *C*, $CH = 1414$ lb. *T*



PROB. 252

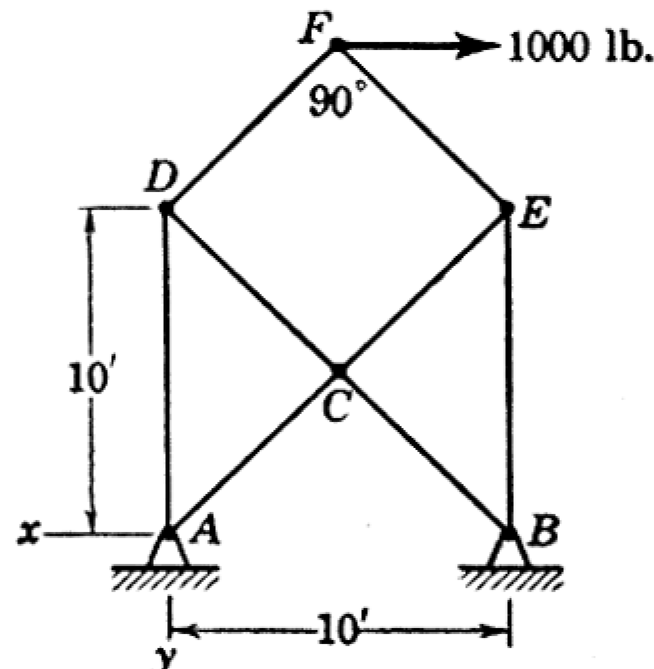
253. Find the force in member *BD* of the cantilever truss.



PROB. 253

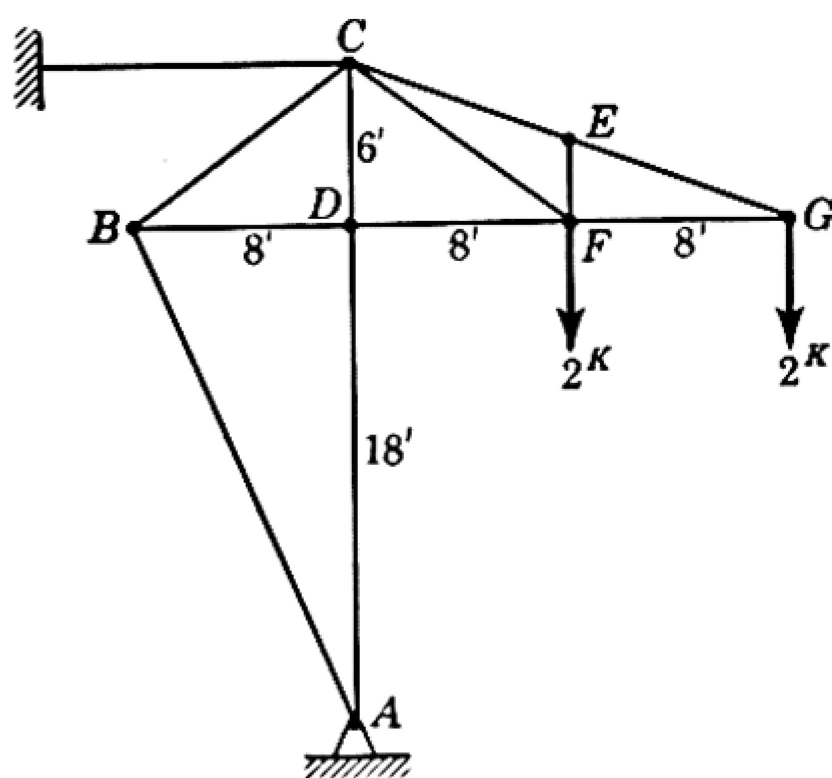
254. Calculate the components of the reactions at *A* and *B* acting on the truss.

Ans. $A_x = B_x = 500$ lb. left, $A_y = 1500$ lb. down, $B_y = 1500$ lb. up



PROB. 254

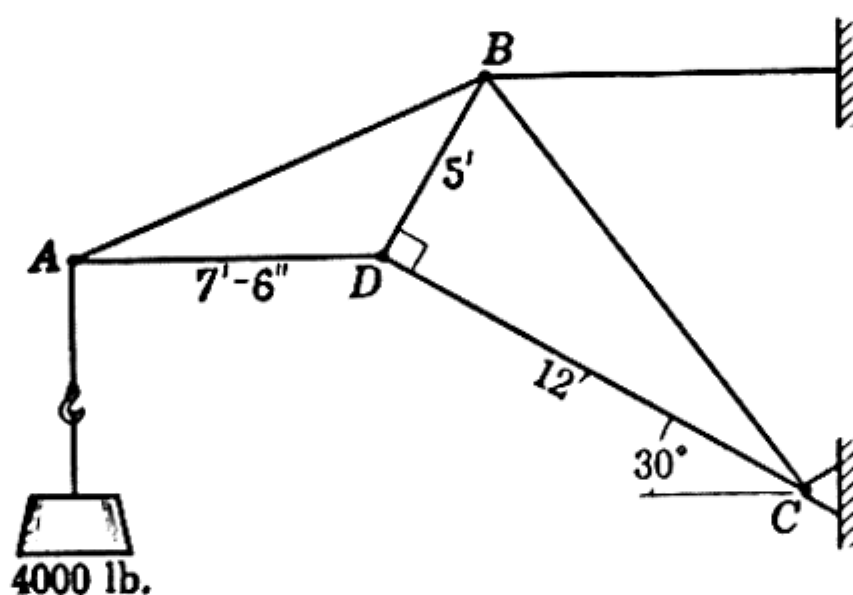
255. Determine the forces in members *BC* and *CF* for the crane truss.



Prob. 255

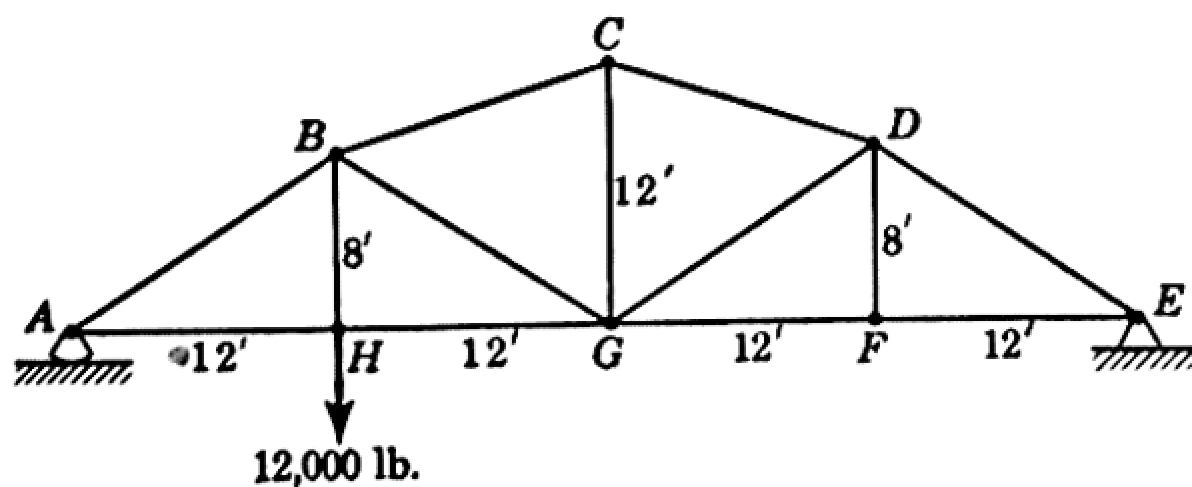
256. Find the force in each member of the loaded crane.

Ans. $AD = 9240$ lb. C, $AB = 10{,}070$ lb. T, $BC = 0$, $DC = 8000$ lb. C, $DB = 4620$ lb. C



Prob. 256

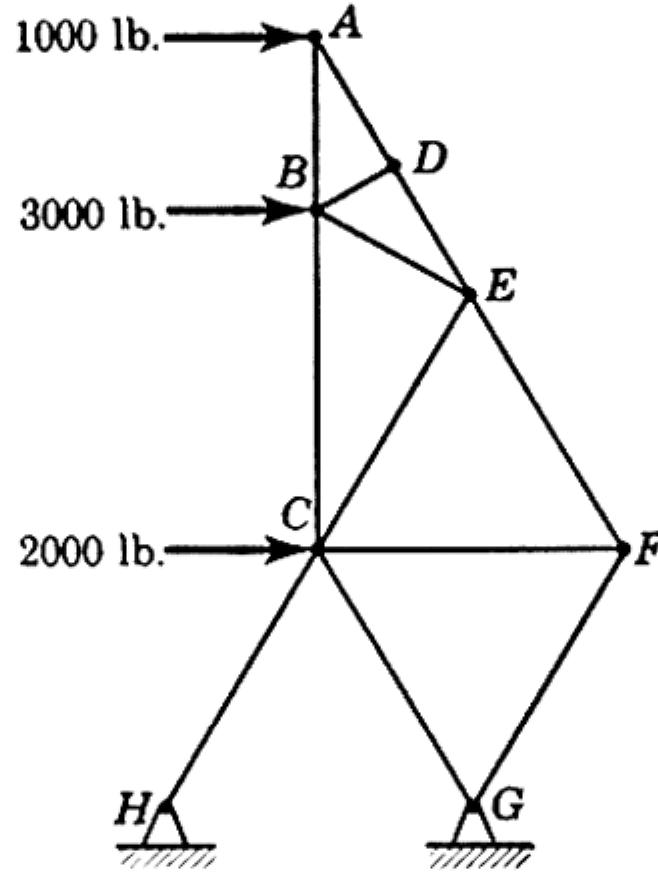
257. Determine the forces in members *BC*, *BG*, and *CG* of the truss due to the single 12,000 lb. load.



Prob. 257

258. A signboard truss is subjected to the wind loads shown. Find the resulting forces in members *BE, CE,* and *CF.* The panels are composed of equilateral and 30–60 deg. triangles.

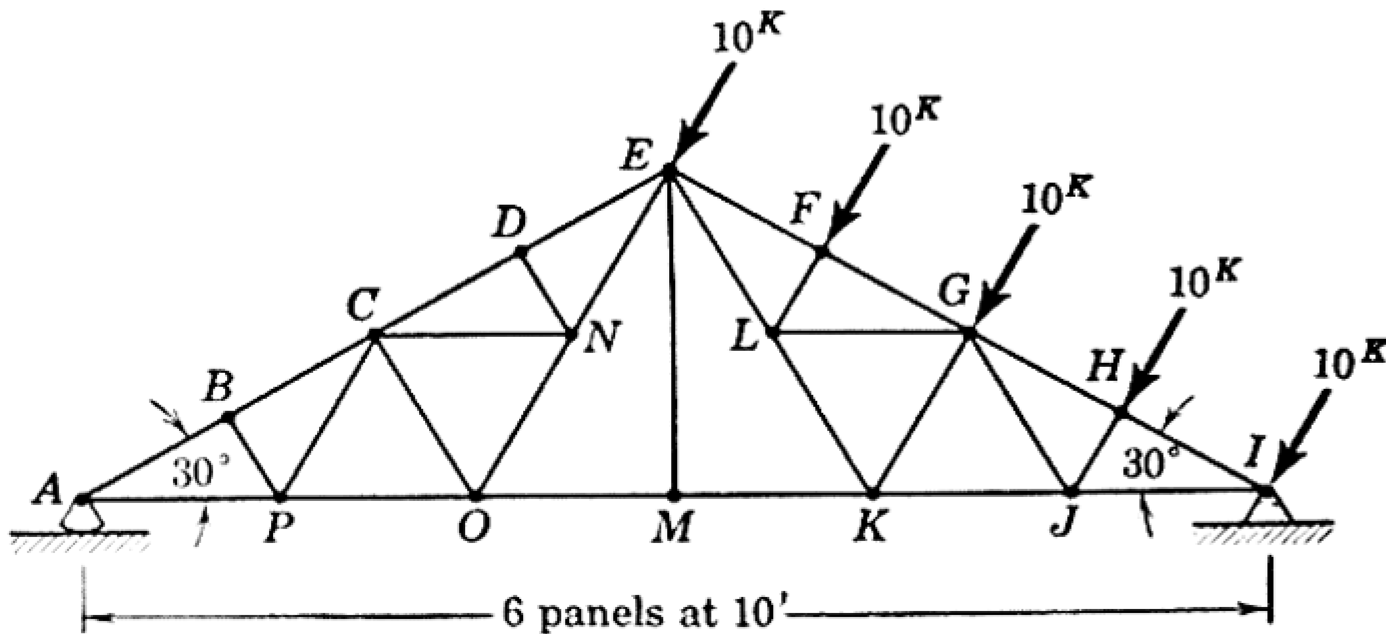
Ans. $BE = 3464$ lb. C, $CE = 2000$ lb. T, $CF = 6000$ lb. T



Prob. 258

259. Find the forces in members *EF, KL,* and *GL* for the Fink truss shown. (*Hint:* Note that the forces in *BP, PC, DN,* etc., are zero.)

Ans. $EF = 40.4$ kips C, $KL = 20.0$ kips T, $GL = 10.0$ kips T



Prob. 259

29. Graphical Solution; Maxwell Diagram. A graphical procedure of wide use for determining the tensions and compressions in the various members of a simple truss will now be explained, using the truss of Fig. 38 for illustration. The procedure is based on the method of joints, and for this reason the explanation in the previous article should be thoroughly understood.

The force polygons which represent the equilibrium of each joint of the truss of Fig. 38 are shown in Fig. 40. These polygons have been formed by vector addition of the forces consistently taken in a clockwise order around each respective joint. This consistent order makes it possible to superimpose all the polygons on a single diagram as shown in Fig. 41*a*. The resulting figure is called the *Maxwell diagram* * for this truss,

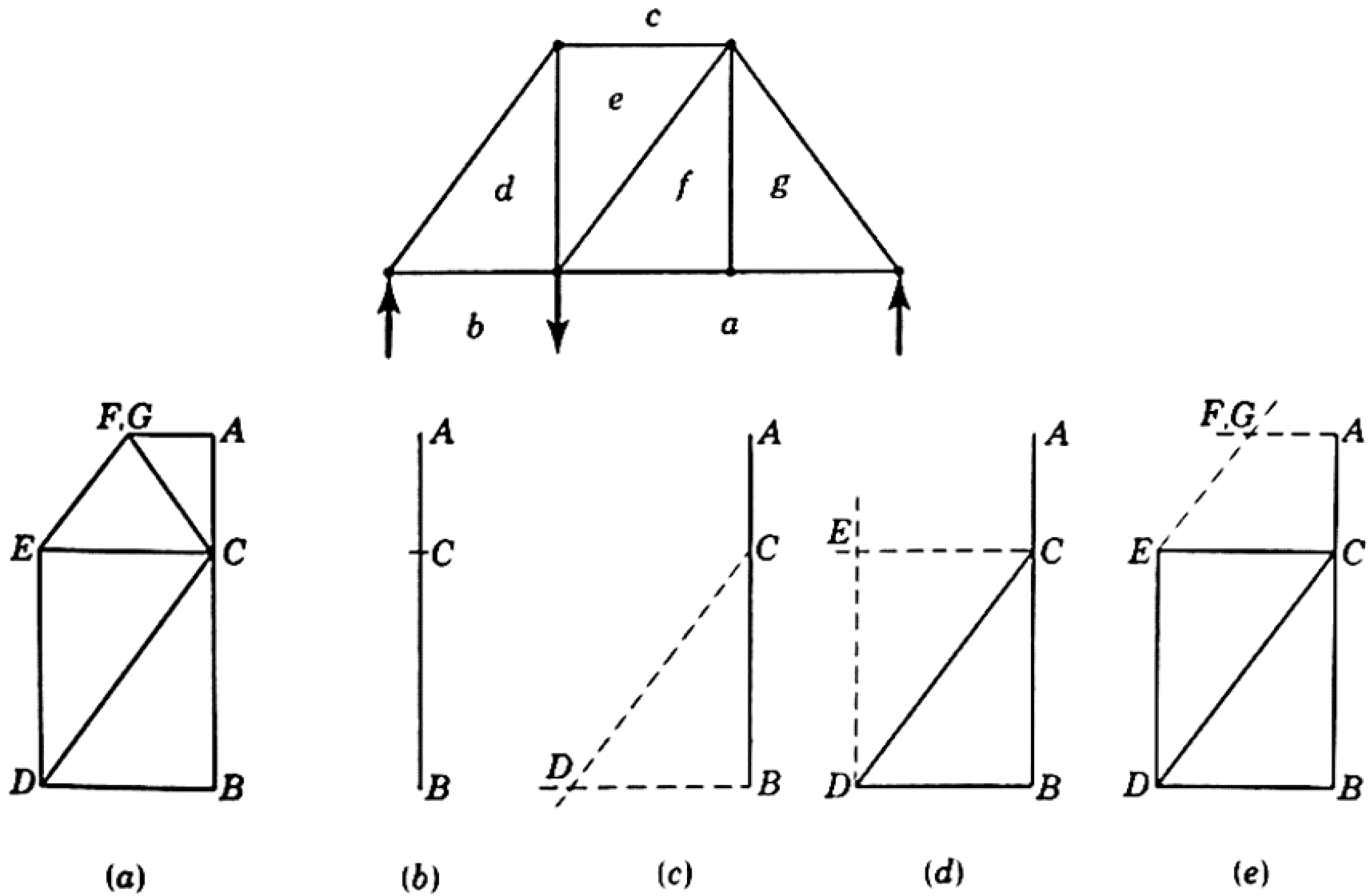


Fig. 41

and it contains all the force polygons for each joint. The arrowheads have been omitted since the direction of the force in a member will depend on which of two joints is considered.

The Maxwell diagram may be constructed directly from the original truss without the necessity for prior calculation of the forces in the internal members. The diagram requires the use of Bow's notation, and a clockwise order for labeling the external forces on the truss is generally used. As a first step the external reactions are determined, usually by calculation. Next the equilibrium polygon of forces external to the truss as a whole is drawn. If the loads and reactions are all parallel, this polygon becomes a line, shown in Fig. 41*b* for the present truss.

The force polygon of a joint where only two unknown forces act, such as *bcd*, is first drawn. This is accomplished, Fig. 41*c*, by merely constructing a line through *B* with the known direction of *db* and a line through *C* with the known direction of *cd*. The intersection of these

* The method was published by James Clerk Maxwell in 1864.

two lines locates point *D*, and the resulting triangle *BCD* is the force polygon for joint *bcd*, previously shown in step 1 of Fig. 40.

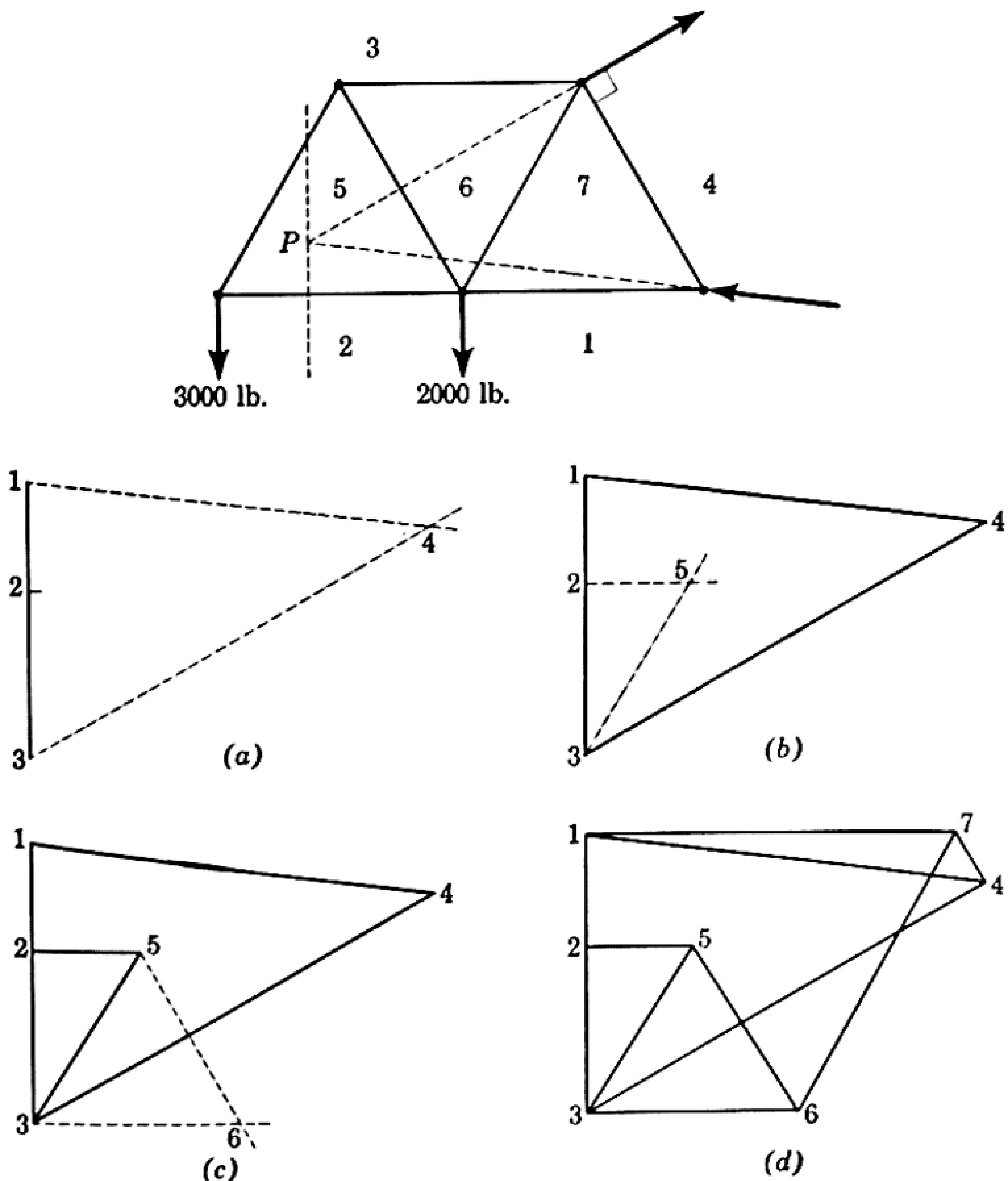
The adjacent joint *dce* upon which only two unknown forces act is analyzed next. A line is now drawn through *D*, Fig. 41*d*, with the direction of member *de* on the truss, and another line is constructed through *C* with the direction of member *ce*. The intersection gives point *E* and completes the force polygon for joint *dce*, obtained earlier in step 2 of Fig. 40. Point *F* on the diagram, Fig. 41*e*, is similarly found from the intersection of a line through *E* with the direction of *ef* and a line through *A* with the direction of *fa*. The resulting polygon *ABDEF* (step 3, Fig. 40) represents the equilibrium of forces on joint *abdef*. From joint *afg* it is seen that *G* must lie on a horizontal line through *A* and on a vertical line through *F*. Thus *G* is coincident with *F*, which checks the fact that the force in member *fg* is zero. Finally the equilibrium of joint *agc* gives the polygon *AGC*. Since these three points are already obtained it is necessary only to connect *G* with *C*, and thus the completed polygon of Fig. 41*a* is obtained. If the construction has been accurately carried out, the direction of *GC* will coincide with the direction of the member *gc*, thus providing a check on the correctness of the diagram.

Once the Maxwell diagram is constructed, it is a simple matter to obtain the magnitude of the force in any member and determine whether it is tension or compression. The force which the member *ef* of the sample truss exerts on the joint *abdef*, for instance, is the vector from *E* to *F* on the Maxwell diagram. The direction is *away* from the joint so that the member is in *tension*. The force which the same member exerts on the joint *cgfe* is the vector from *F* to *E* and likewise indicates tension since it is away from the joint. It should be carefully noted that the proper sequence of letters for the force vector is obtained from a *clockwise* travel about the joint in question. A counterclockwise diagram could, of course, be used with equal success, provided this direction was consistently used in the notation and the interpretation. Such a diagram would result in an upside-down mirror image of the clockwise construction but is not used in practice.

Although the load line (or polygon) of external forces is normally the first step in the construction of the Maxwell diagram, in some cases it may be just as convenient or even necessary to begin the construction without the initial determination of the external reactions.

SAMPLE PROBLEM

260. Draw the Maxwell diagram for the truss of Sample Prob. 243.



PROB. 260

Solution: The truss is redrawn in the accompanying figure, and Bow's notation with numbers is used to avoid confusion with the letters previously employed to designate each joint. The external reactions may be obtained graphically from the concurrency principle for three forces by locating the intersection P of the line of action of the force in member 3–4 with the line of the resultant of the two applied loads. The reaction 4–1 must pass through this point. The force polygon representing the equilibrium of the truss as a whole is next drawn as shown in the a-part of the illustration.

The Maxwell diagram is constructed on the force polygon. By starting with the left end of the truss, point 5 is seen to lie on a horizontal line through point 2 and on a line through point 3 having the direction of 3–5 as illustrated in the

b-part of the drawing. The triangle 2–3–5 represents in that order the force polygon for joint 2–3–5. The same three forces are described in the figure for Prob. 243 for joint *A*. Point 6 is located in similar manner in the *c*-part of the figure. Point 7 may be obtained from joint 1–7–4 and the diagram completed by joining points 6 and 7 as shown in the *d*-part of the illustration. The line joining points 6 and 7 is parallel to the member 6–7, which forms a check on the work.

In interpreting the diagram to obtain the force in member 6–5, for example, it is first necessary to consider one of the two joints to which this member is attached. If joint 3–6–5 is chosen, the polygon 3–6–5 is traced on the Maxwell diagram in that order, and the force which member 6–5 exerts on the pin is up and to the left (from 6 to 5). This is a compressive force since it is toward the pin. Scaling the diagram gives a magnitude for the line 6–5 in agreement with that already obtained. If joint 1–2–5–6–7 had been considered, the force would be designated 5–6 (clockwise order around the joint). The direction of the force is from 5 to 6 on the Maxwell diagram, which is toward the joint and is likewise compression.

PROBLEMS

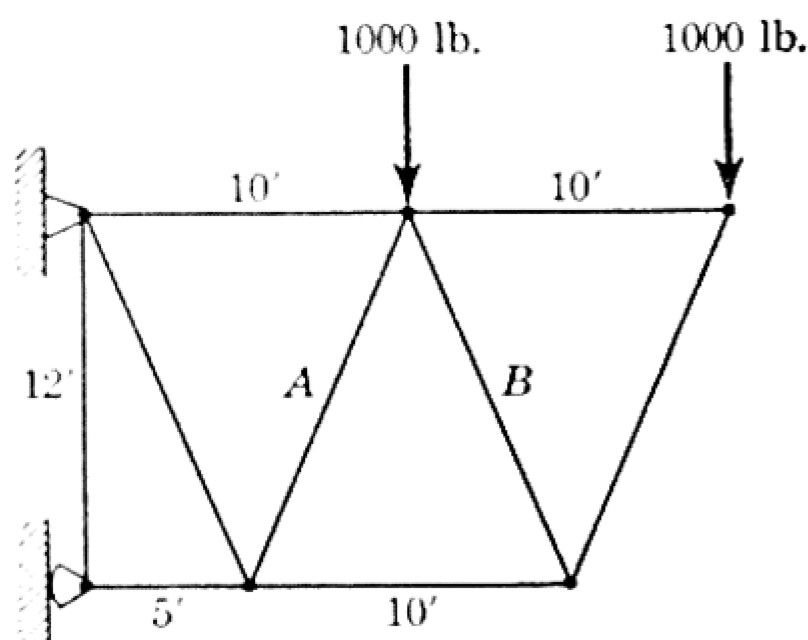
Solve the following problems by drawing the complete Maxwell diagram. Relabel as needed with Bow's notation.

261. Prob. 244.

262. Prob. 251.

263. Find the forces in members *A* and *B* for the cantilever truss.

Ans. $A = 2170$ lb. C, $B = 1085$ lb. T



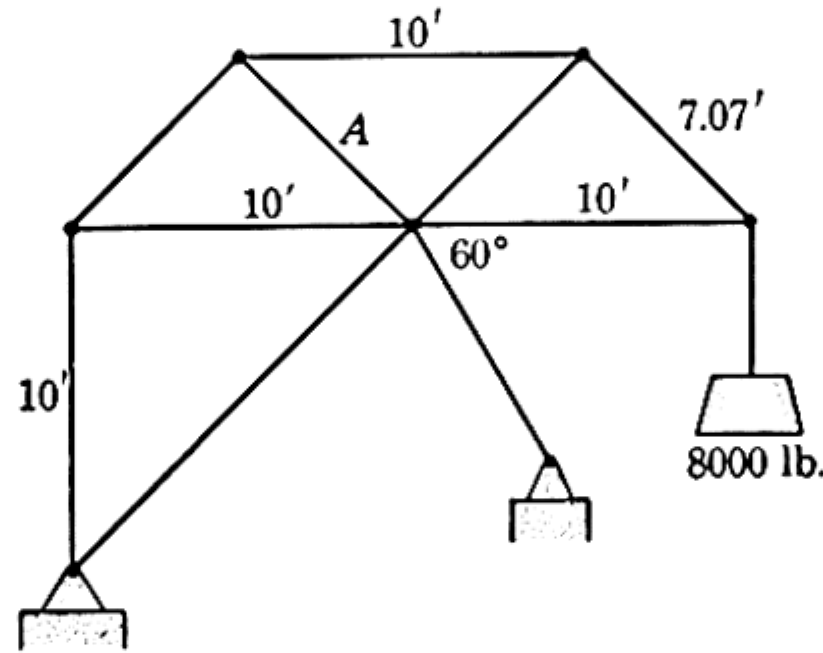
Prob. 263

264. Prob. 248.

265. Make a free-hand sketch of the Maxwell diagram for the crane truss in Prob. 256 and estimate the force in member *BD*.

266. Prob. 249.

267. Make a free-hand sketch of the Maxwell diagram for the loaded structure and determine the force in member A.



PROB. 267

268. Find the force in member A for the truss shown. All angles are 30 or 60 deg.

Ans. $A = \frac{P}{3}$ tension

269. Prob. 254.

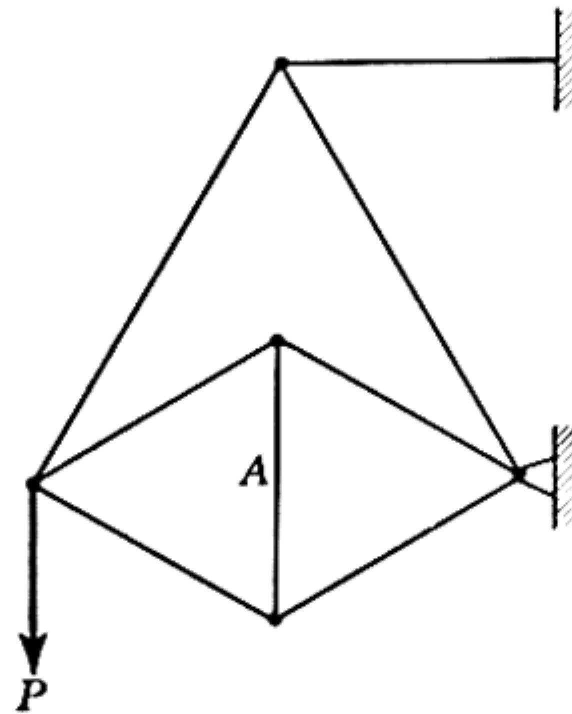
270. Prob. 257.

271. Prob. 252.

272. Prob. 255.

273. Prob. 258.

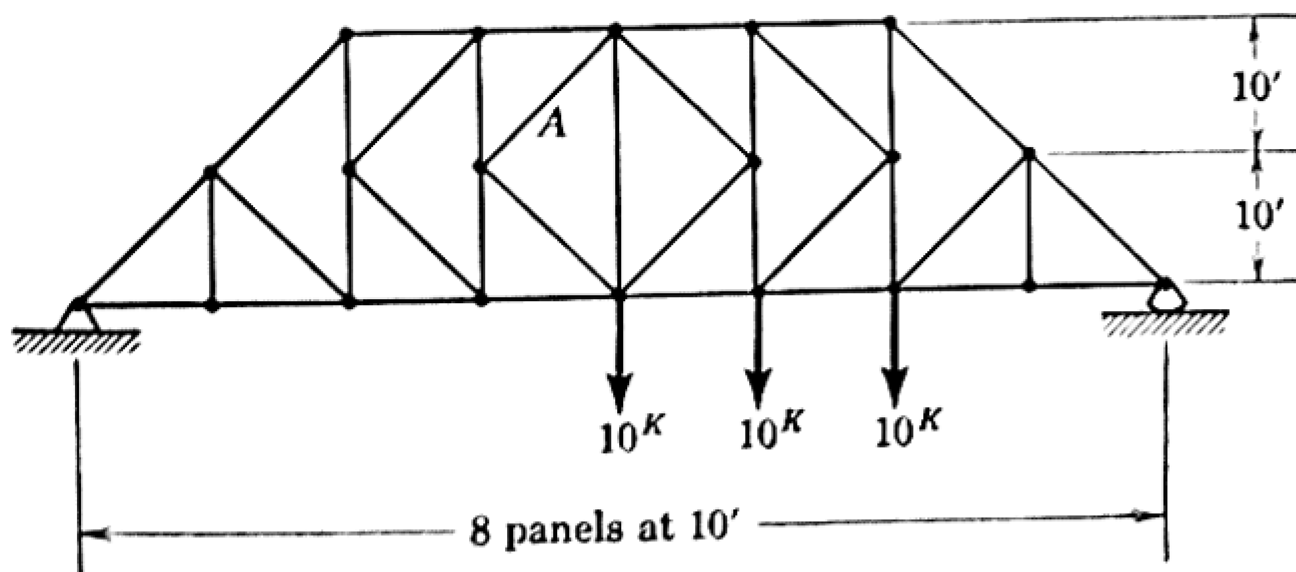
274. Prob. 259.



PROB. 268

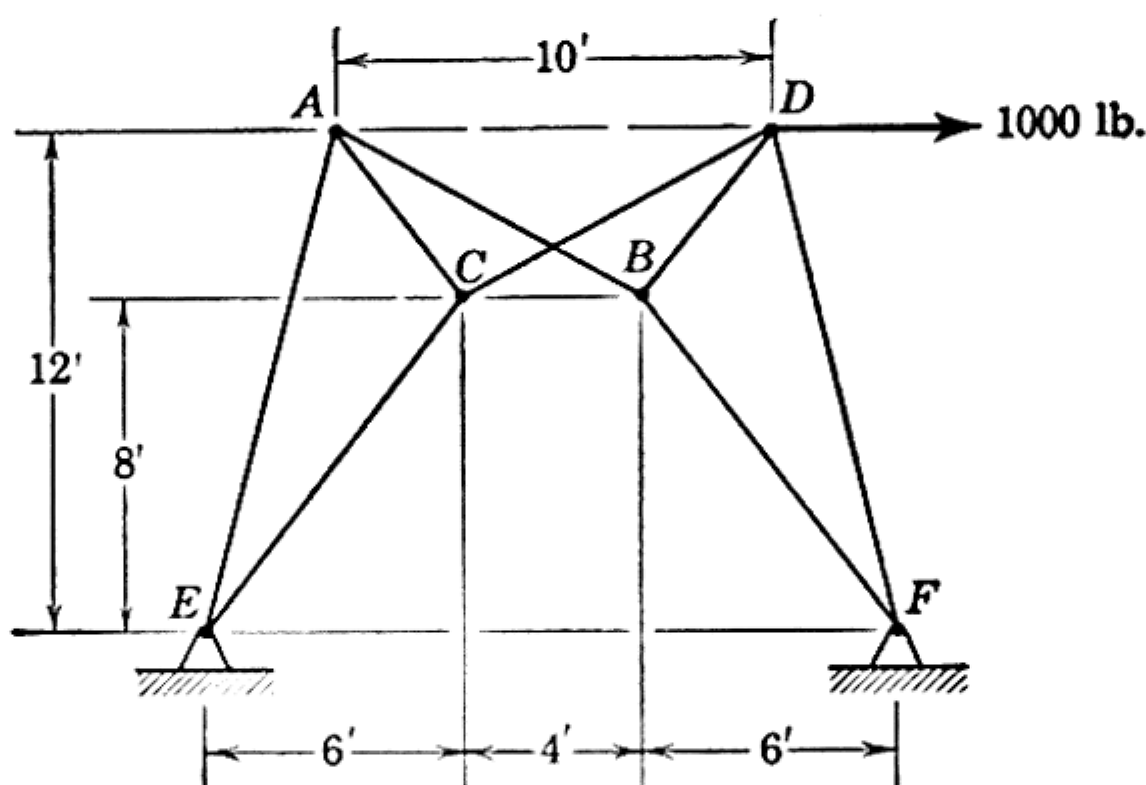
* **275.** Find the force in member A from the Maxwell diagram of the K-truss.

Ans. $A = 7.96$ kips C



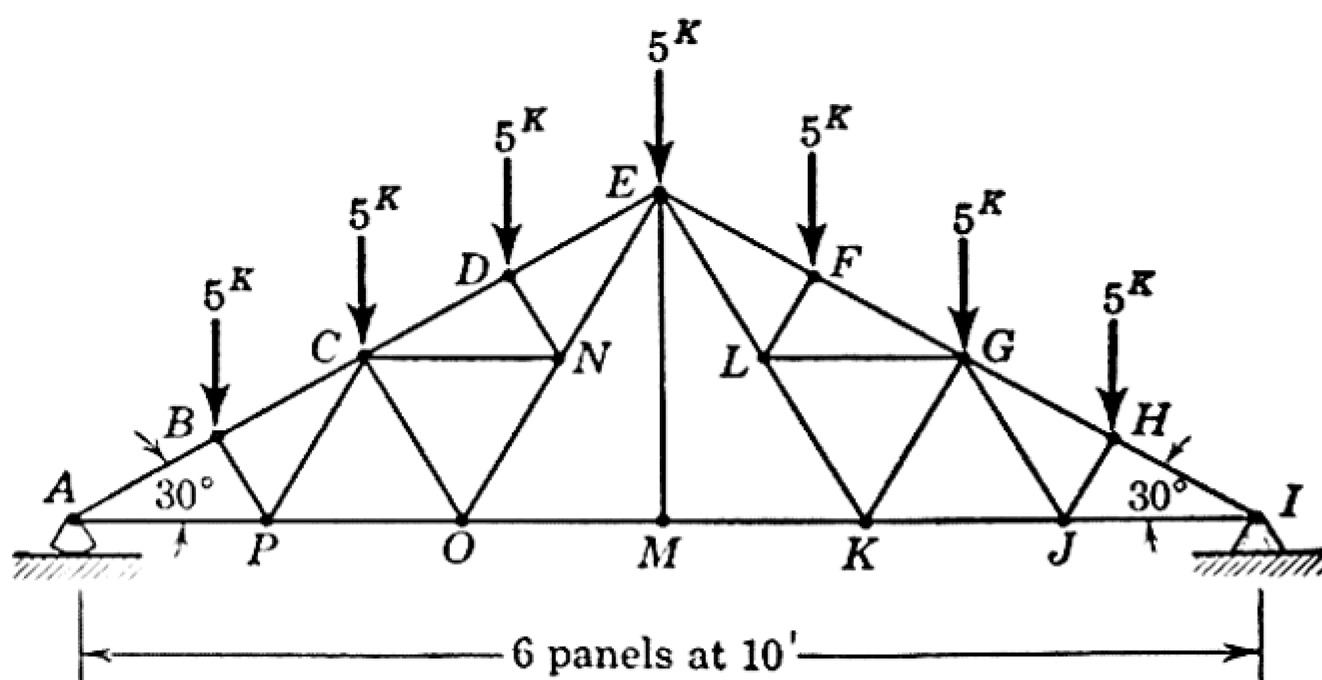
PROB. 275

* **276.** The hinged frames ACE and DFB are connected by two hinged bars AB and CD which cross without being connected. Draw the Maxwell diagram and determine the force in AB. *Ans.* $AB = 377$ lb. C



PROB. 276

* **277.** Draw the Maxwell diagram for the Fink truss of Prob. 259 loaded in the manner shown here. Find the force in member LG. (*Hint:* For the right side of the truss the force in EF is unaffected by replacing members FL and GL by a member from F to K. The diagram can be completed with the temporary member and modified to account for the resubstitution of the original two members. By reason of symmetry only one half of the diagram need be drawn.) *Ans.* $LG = 4.33$ kips T



PROB. 277

30. Method of Sections. In the method of joints and in the graphical method advantage is taken of only two of the three equilibrium equations since the procedures involve concurrent forces at each joint. The third equilibrium principle may be used to advantage by considering an entire section of the truss as a free body in equilibrium under the action of a nonconcurrent system of forces. This *method of sections* has the basic

advantage that the force in almost any desired member may be found directly from an analysis of a section which has cut that member. Thus it is not necessary to proceed with the calculation from joint-to-joint until the member in question has been reached. In choosing a section of the truss it should be noted that in general not more than three members whose forces are unknown may be cut since there are only three available equilibrium relations which are independent.

The method of sections will now be illustrated for the truss in Fig. 38, which was used in the explanation of the two previous methods. The truss is shown again in Fig. 42*a* for ready reference. The external re-

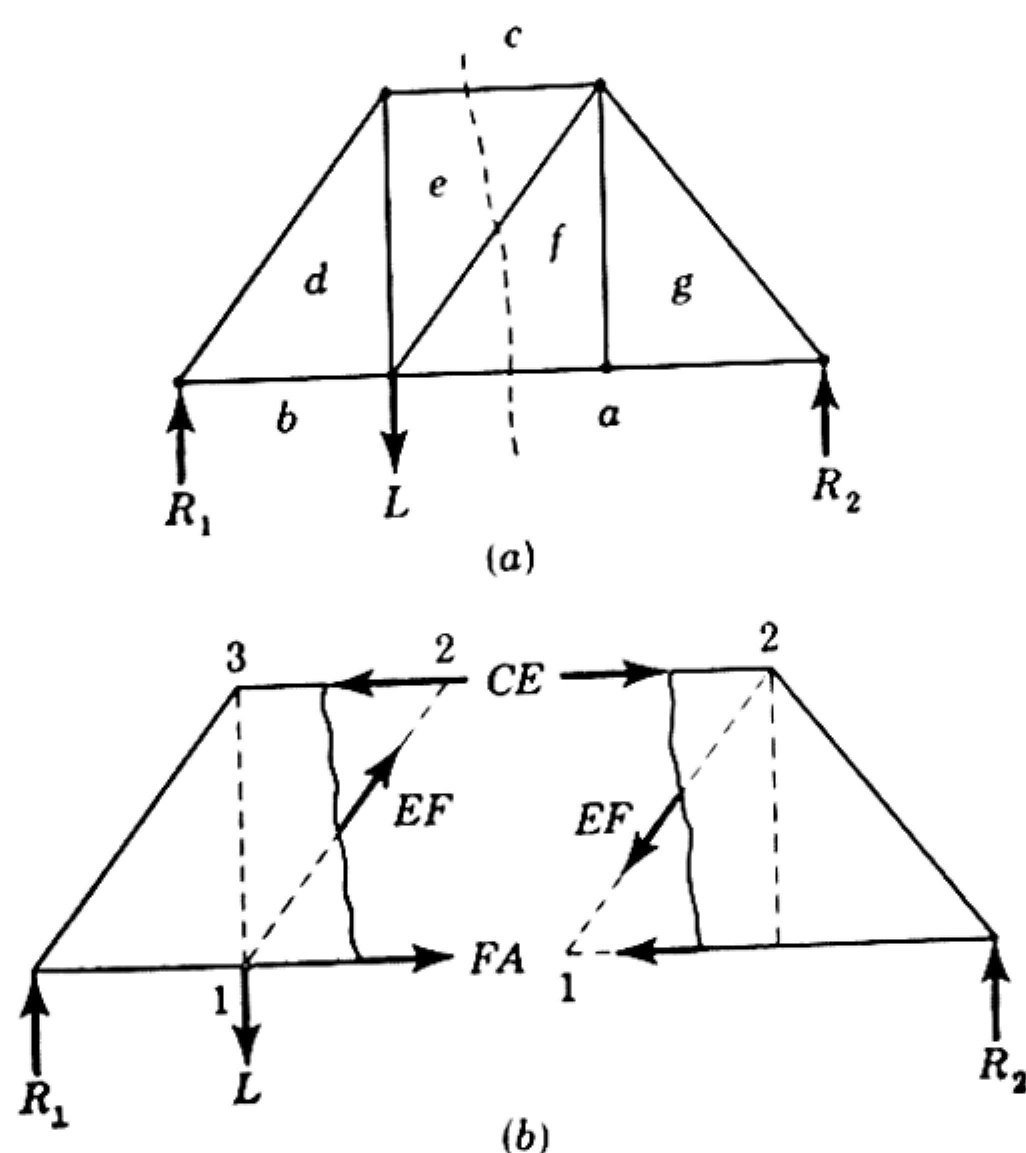


FIG. 42

actions are first computed as before, considering the truss as a whole. Now let it be desired to determine the force in the member *ef*. An imaginary section, indicated by the dotted line, is passed through the truss cutting it into two parts, Fig. 42*b*. This section has cut three members whose forces are initially unknown. In order that the portion of the truss on each side of the section will remain in equilibrium it is necessary to apply to each cut member the force which was exerted on it by the member cut away. These forces, either tension or compression, will always be in the direction of the respective members for simple trusses composed of two-force members. The left-hand section is in equilibrium under the action of the applied load L, the end reaction R_1, and the three forces exerted on the cut members by the right-hand section which has been re-

moved. The forces may usually be drawn with their proper senses by a visual approximation of the equilibrium requirements. Thus in balancing the moments about point 1 the force *CE* is clearly to the left, which makes it compression since it acts toward the cut section of member *ce*. The load *L* is greater than the reaction R_1 so that the force *EF* must be up and to the right to supply the needed upward component for vertical equilibrium. Force *EF* is therefore tension since it acts away from the cut section. With the approximate magnitudes of R_1 and *L* in mind the balance of moments about point 2 requires that *FA* be to the right. A casual glance at the truss should lead to the same conclusion when it is realized that the lower horizontal member will stretch under the tension caused by bending. The equation of moments about joint 1 eliminates three forces from the relation, and *CE* may be determined directly. The force *EF* is calculated from the equilibrium equation for the *y*-direction. Lastly *FA* may be determined by balancing moments about point 2. In this way each of the three unknowns has been determined independently of the other two.

The right-hand section of the truss, Fig. 42*b*, is in equilibrium under the action of R_2 and the same three forces in the cut members applied in the directions opposite to those for the left section. The proper sense for the horizontal forces may easily be seen from the balance of moments about points 1 and 2.

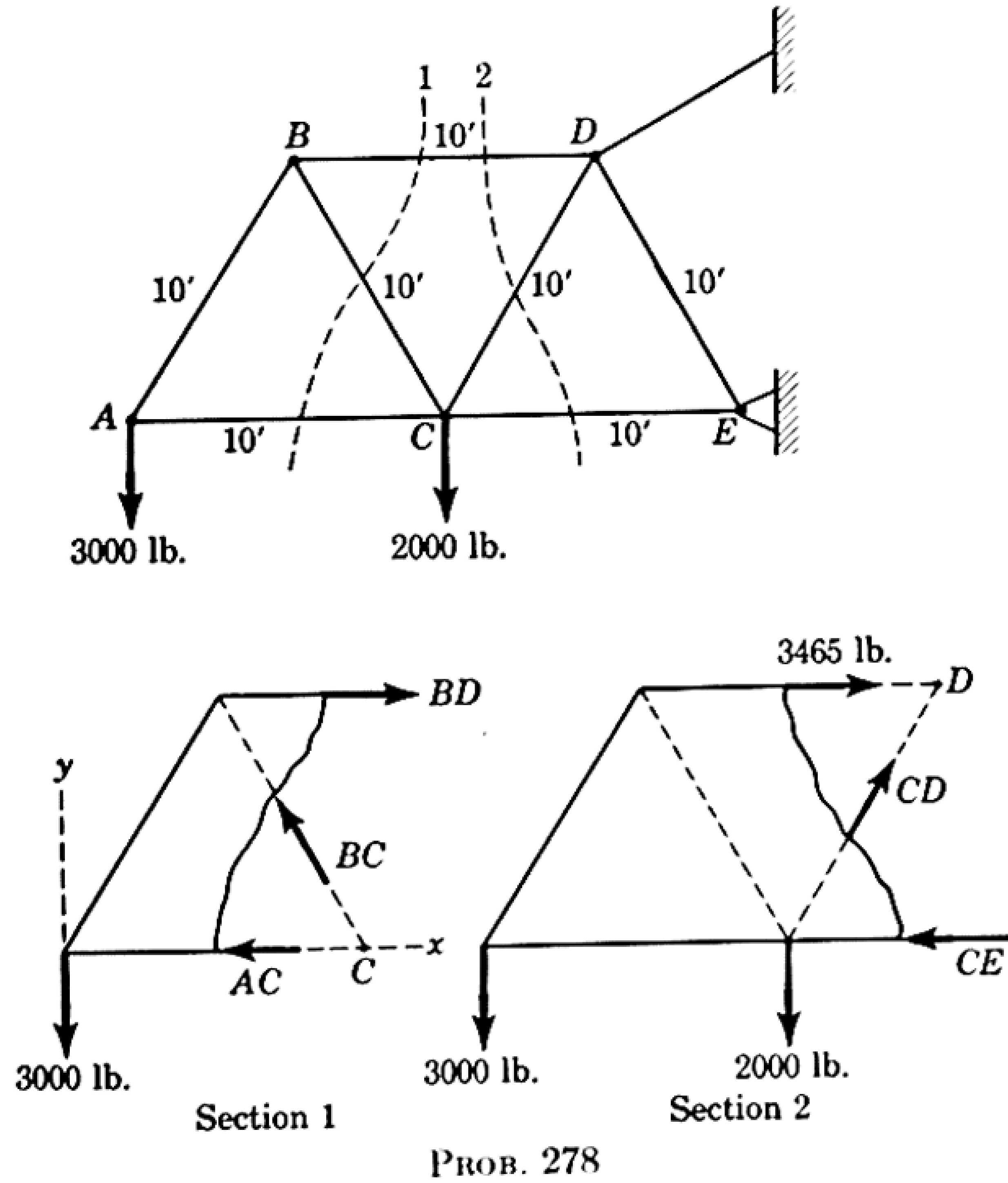
It is essential to understand that in the method of sections an entire portion of the truss is considered as a single body in equilibrium. Thus the forces in members internal to the section are not involved in the analysis of the section as a whole. In order to clarify the free body and the forces acting externally on it, the section is preferably passed through the members and not the joints.

Either section of a truss may be used for the calculations, but the one involving the least number of forces will usually yield the simpler solution.

The moment equations may be used to great advantage in the method of sections, and a moment center through which as many forces pass as possible should be chosen. It is not always possible to assign an unknown force in the proper sense when the free-body diagram of a section is drawn. With an arbitrary assignment made, a positive answer will verify the assumed sense and a negative result will indicate that the force is in the sense opposite to that assumed. Any system of notation desired may be used, although usually it is found convenient to letter the joints and designate a member and its force by the two letters defining the ends of the member.

SAMPLE PROBLEMS

278. Calculate the forces in members BC, BD, and CE for the truss of Probs. 243 and 260, which is relabeled with this present problem.



PROB. 278

Solution: Section 1 cuts only three members whose forces are unknown, and the free-body diagram of the portion of the truss to the left of the section is drawn. A force summation in the y-direction gives

$[\Sigma F_y = 0]$ $\quad 0.866BC - 3000 = 0, \quad BC = 3464$ lb. C. *Ans.*

Using C as a moment center gives

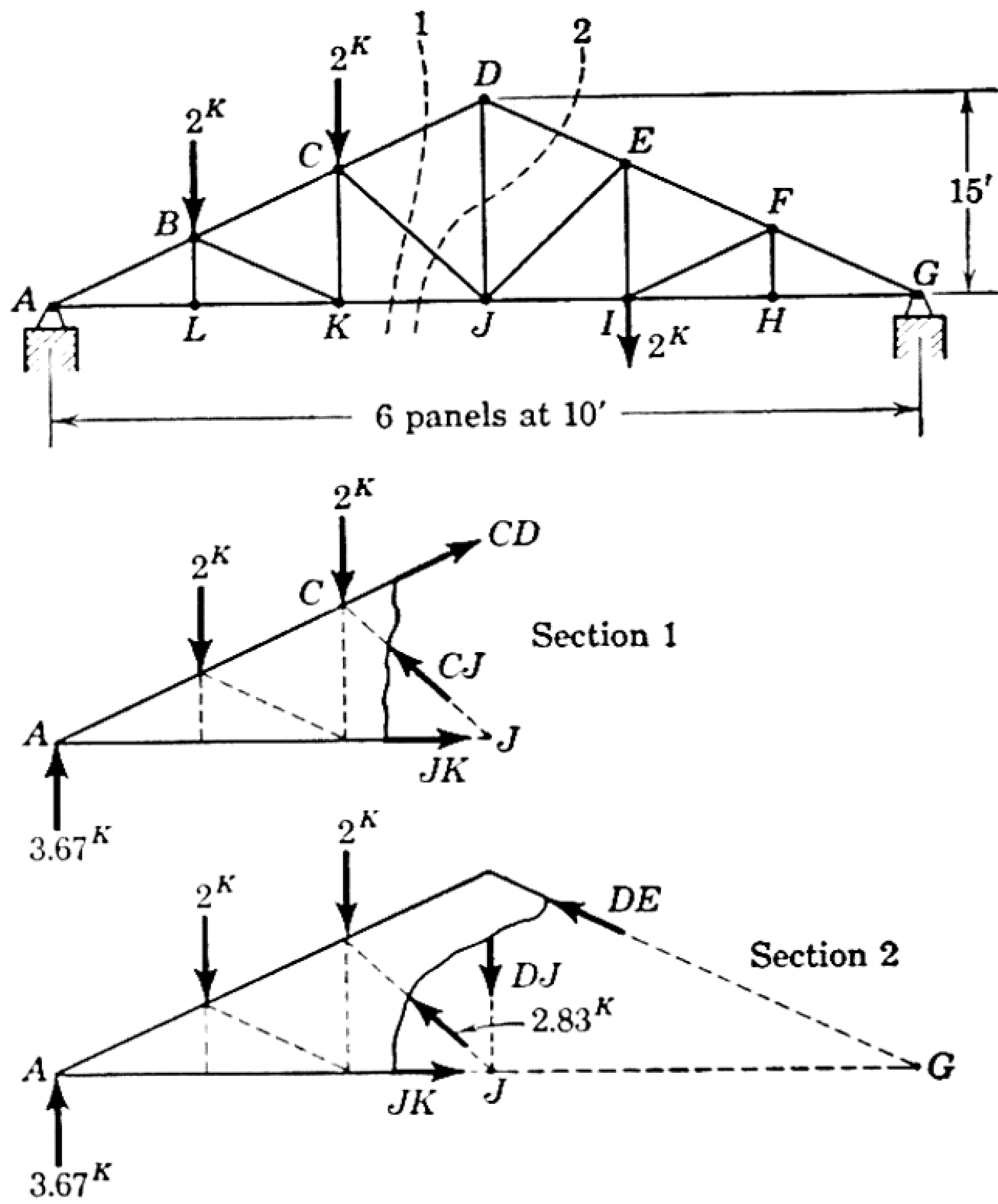
$[\Sigma M_C = 0]$ $\quad 8.66BD - 3000 \times 10 = 0, \quad BD = 3464$ lb. T. *Ans.*

Section 2 may be used to find the force in CE. From the free-body diagram of the portion of the truss to the left of section 2 is obtained

$[\Sigma M_D = 0]$ $\quad 8.66CE - 2000 \times 5 - 3000 \times 15 = 0, \quad CE = 6350$ lb. C. *Ans.*

For a loaded cantilever truss, such as illustrated in this problem, it should be noted that the forces in the members may be obtained without calculation of the support reactions.

279. Calculate the force in member DJ of the roof truss illustrated. Neglect any horizontal components of force at the supports.



Prob. 279

Solution: It is not possible to pass a section through DJ without cutting four members whose forces are unknown. Although three of these cut by section 2 are concurrent at J and therefore the moment equation about J could be used to obtain the fourth, DE, the force in DJ cannot be obtained from the remaining two equilibrium principles. It is necessary to consider first the adjacent section 1 before considering section 2.

The free-body diagram for section 1 is drawn and includes the reaction of 3.67 kips at A, which is previously calculated from the equilibrium of the truss as a whole. In assigning the proper directions for the forces acting on the three cut members a balance of moments about A eliminates the effects of CD and JK and clearly requires that CJ be up and to the left. A balance of moments about C eliminates the effect of the three forces concurrent at C and indicates that JK must be to the right to supply sufficient counterclockwise moment. Again it should be fairly obvious that the lower chord is under tension due to the bending tendency of the truss. Although it should also be apparent that the top chord is under compression, the force in CD will be arbitrarily assigned as ten-

sion. There is no harm in assigning one or more of the forces in the wrong direction as long as the calculations are consistent with the assumption. A negative answer will show the need for reversing the direction of the force.

By the analysis of section 1, CJ is obtained from

$$[\Sigma M_A = 0] \quad (0.707CJ)30 - 2 \times 10 - 2 \times 20 = 0, \qquad CJ = 2.83 \text{ kips } C.$$

In this equation the moment of CJ is calculated by considering its horizontal and vertical components acting at point J. Equilibrium of moments about J requires

$$[\Sigma M_J = 0] \quad (0.894CD)15 + 3.67 \times 30 - 2 \times 10 - 2 \times 20 = 0,$$

$$CD = -3.73 \text{ kips.}$$

The moment of CD about J is calculated here by considering its two components as acting through D. The minus sign indicates that CD was assigned in the wrong direction. Thus

$$CD = 3.73 \text{ kips } C. \qquad \textit{Ans.}$$

If desired, the direction of CD may be changed on the free-body diagram and the algebraic sign of CD reversed in the calculations, or else the work may be left as it stands with a note stating the proper direction.

From the free-body diagram of section 2, which now includes the known value of CJ, a balance of moments about G is seen to eliminate DE and JK. Thus

$$[\Sigma M_G = 0] \quad 30DJ + 2 \times 40 + 2 \times 50 - 3.67 \times 60 - 2.83 \times 0.707 \times 30 = 0,$$

$$DJ = 3.33 \text{ kips } T. \qquad \textit{Ans.}$$

Again the moment of CJ is determined from its components considered as acting at J. The answer for DJ is positive so that the assumed tension direction was correct. An analysis of the joint D alone also verifies this conclusion.

Although the procedure used here is undoubtedly the shortest for obtaining DJ, the student should consider other possibilities. It may be observed that, if for some other problem the 2 kip load at I were not present but the other loads remained the same, then the forces in IE and JE (as well as in HF and IF) would be zero. In this event a section through members CD, DJ, JE, and IJ would involve only three unknown forces, and a solution for DJ could be obtained with a single moment equation about A.

PROBLEMS

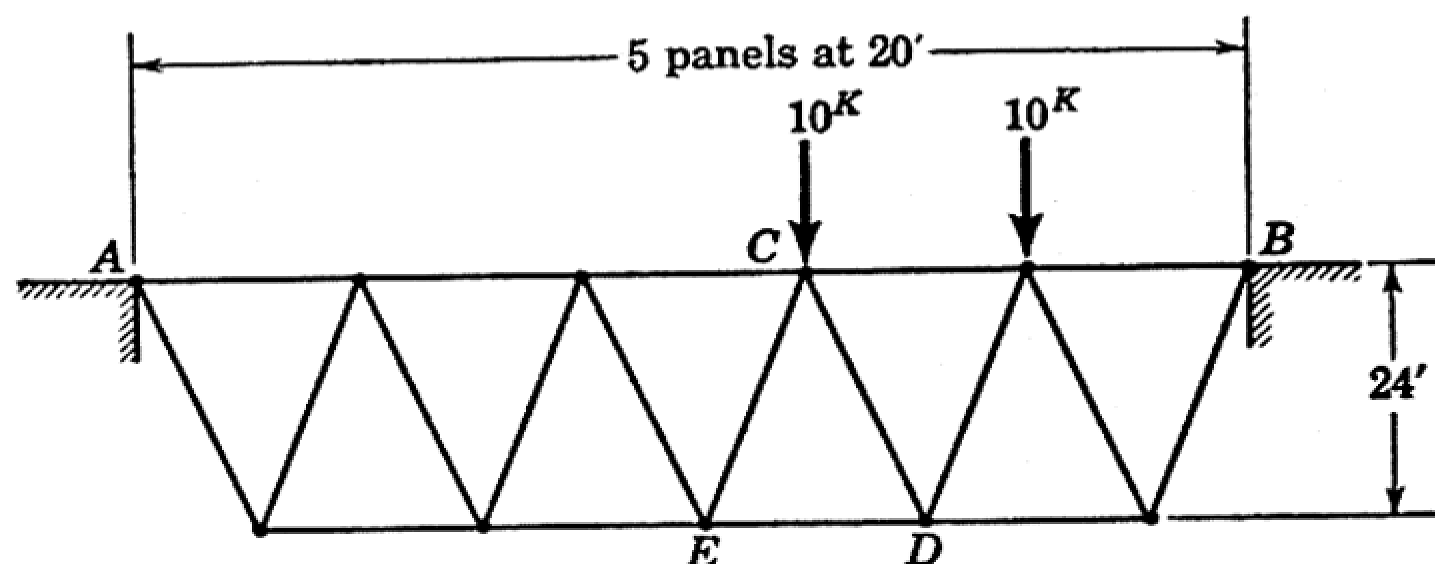
Solve the following problems by the method of sections.

280. Determine the force in members BE and DE in the truss of Prob. 248.

281. Calculate the force in member CD of the truss in Prob. 249 from just one equation.

282. Determine the forces in members CD and DE of the bridge truss. Neglect any horizontal components of the reactions at A and B.

Ans. CD = 4.33 kips C, DE = 15 kips T



PROB. 282

283. Determine the forces in members DF, CE, and BE for the truss of Prob. 254. Solve for each force from an equation which eliminates the other two forces.

284. After analyzing joint G for the truss in Prob. 252, determine the forces in members CH and FG from only one section of the truss.

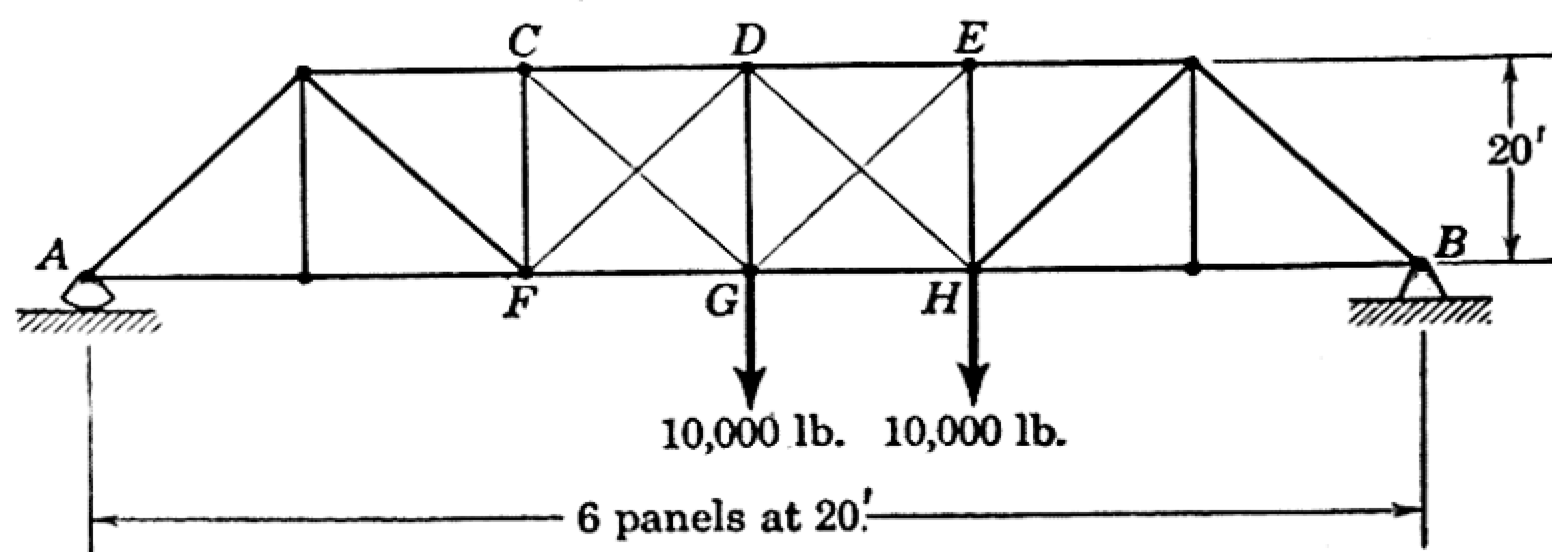
285. Calculate the forces in members BC, CE, and EF for the signboard truss of Prob. 258. Solve for each force from an equation which eliminates the other two.

286. Calculate the force in BD for the truss of Prob. 253 from just one equation of equilibrium.

287. Find the forces in members CF and EF of the truss in Prob. 250.

288. The crossed members in the two center panels of the Pratt truss are slender tie rods incapable of supporting compression. Retain the two rods which are under tension and find the magnitudes of their tensions.

Ans. GE = 2360 lb. T, CG = 11,780 lb. T

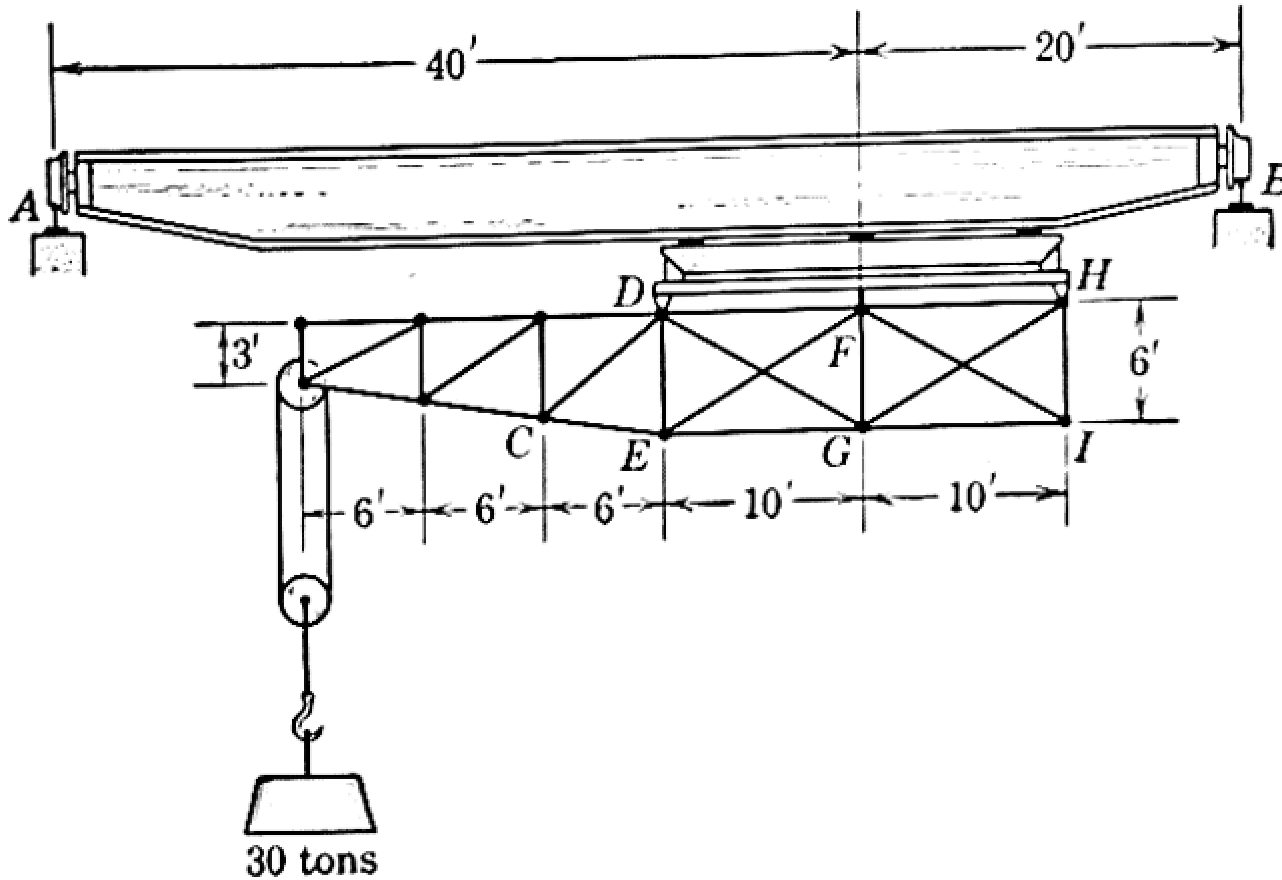


PROB. 288

289. Calculate the forces in members BC, CD, and DF for the truss of Prob. 255.

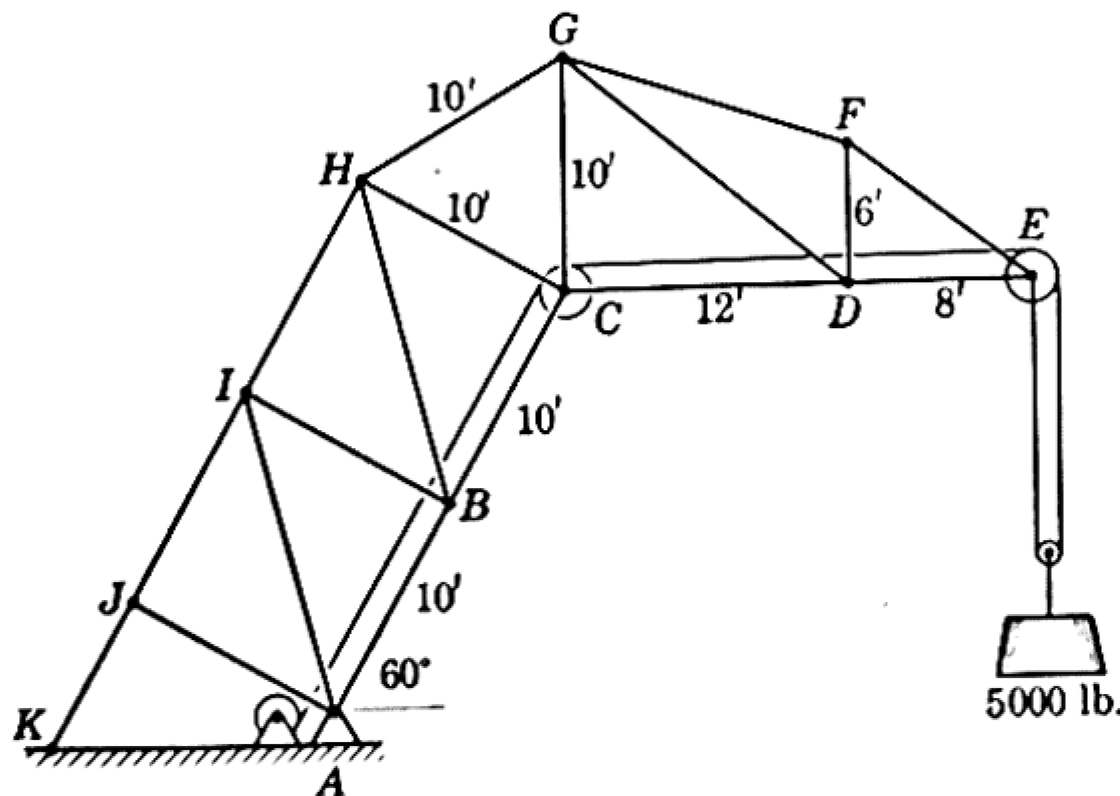
290. Determine the forces in members BC and CG for the truss of Prob. 257.

291. Determine the vertical components of the rail reactions at A and B and the forces in members CD, DG, and EF due to the action of the applied load for the combination traveling and rotating jib crane. When the crane is under load, the reaction at H and the horizontal components of the forces at D and F are negligible. Members DG, EF, FI, and GH are slender rods unable to support compression. *Ans.* $A = 24$ tons, $B = 6$ tons, $CD = 23.5$ tons T, $DG = 105$ tons T, $EF = 0$



Prob. 291

* **292.** Determine the forces in members JK, HC, and DG for the crane truss. *Ans.* $JK = 15{,}000$ lb. T, $HC = 3270$ lb. C, $DG = 4340$ lb. T

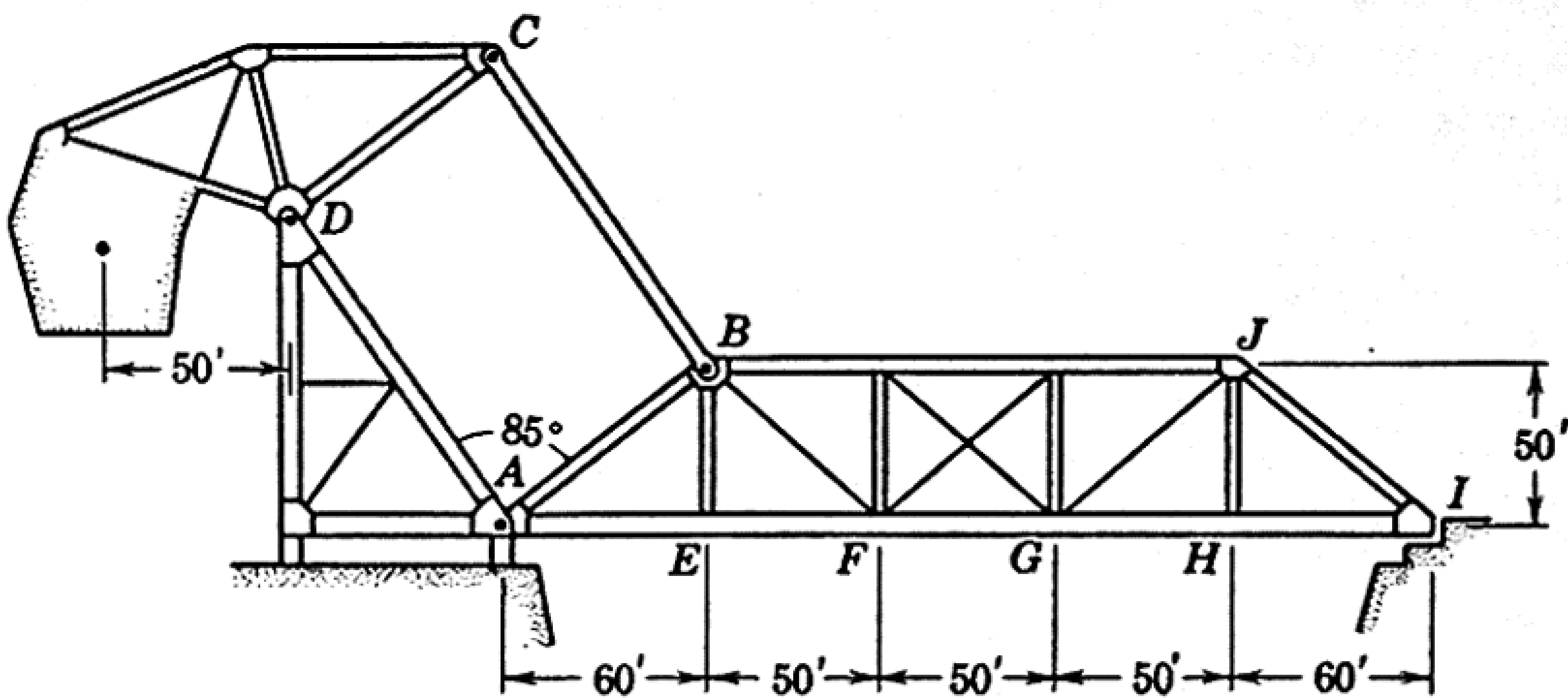


Prob. 292

* **293.** Prob. 276.

* **294.** The single-leaf bascule bridge is exactly balanced with a counterweight of 1800 tons located as shown. Determine the force in JG as the bridge starts to lift, assuming that the weights of the members are negligible compared with the weight of the uniform roadway. Treat all intersections of the truss members

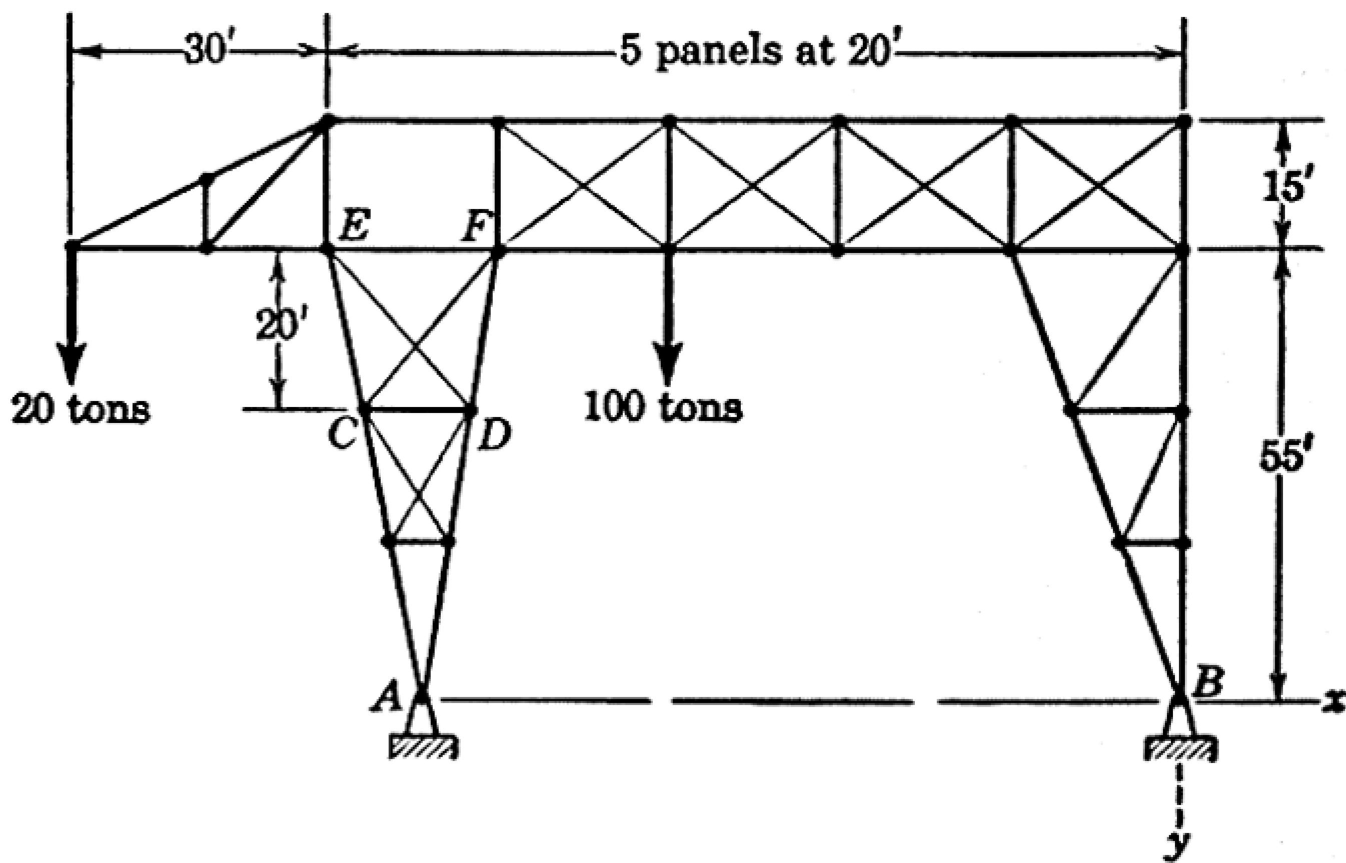
as pin joints connecting two-force members and assume each of the joints A, E, F, G, H, I carries half the weight of the roadway on each side of the joint. Figure $ABCD$ is a parallelogram. *Ans.* $JG = 297$ tons C



PROB. 294

* **295.** In the traveling bridge crane shown all crossed members are slender tie rods incapable of supporting compression. Determine the force in members DF and EF and find the horizontal reaction on the truss at A. Show that if $CF = 0$, then $DE = 0$ also.

Ans. $DF = 76.8$ tons C, $EF = 36.4$ tons C, $A_x = 10.1$ tons to the right



PROB. 295

31. Space Trusses. All members of the trusses described in the preceding four articles have been in a single plane so that the equilibrium problem has been two-dimensional. There are, however, many types of construction which utilize a system of two-force members in space joined together at their ends to form a rigid structure. Such a configuration of bars is known as a *space truss*.

In a plane truss it has already been seen that a triangle of pin-connected bars forms a rigid, noncollapsible unit. A space truss, on the other hand, requires six bars joined at their ends to form the edges of a tetrahedron for the fundamental noncollapsible unit. In Fig. 43*a* the two bars AD and BD joined at D require a third support CD to keep the

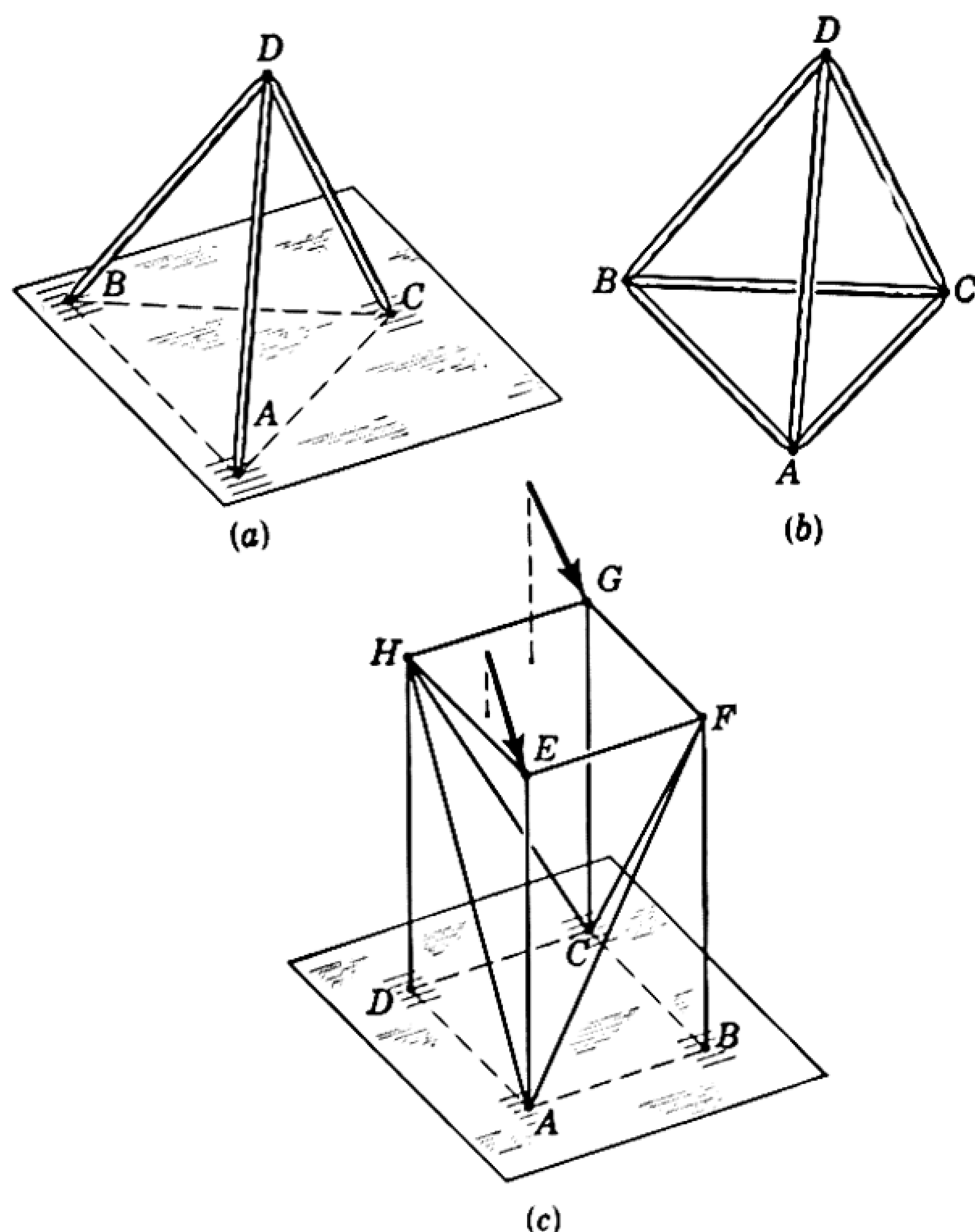


FIG. 43

triangle ABD from rotating about AB. In Fig. 43*b* the supporting base is replaced by three more bars AB, BC, and AC to form a tetrahedron independent of the foundation for its own rigidity. Added units to the structure may be formed with three additional concurrent bars whose ends are attached to three fixed joints on the existing structure. Thus in Fig. 43*c* the bars AF, BF, and CF are attached to the foundation and therefore fix point F in space. Likewise point H is fixed in space by the bars AH, DH, and CH. The three additional bars CG, FG, and HG are attached to the three fixed points C, F, and H and therefore fix G in space. Point E is similarly established. The structure is entirely rigid, and the two applied loads shown will result in forces in all of the members.

Ideally there must be point support, such as represented by a ball and socket joint, for the connections of a space truss in order that there be no bending in the members. Again, as in riveted and welded connections for plane trusses, if the centerlines of joined members intersect at a point, the assumption of two-force members under simple tension and compression may be justified.

If a space truss is loaded and supported at the joints in such a way that the entire truss is not statically indeterminate, the forces in all members may be found by the methods of equilibrium already developed. The method of joints may be used in a manner similar to that employed with plane trusses except that the *three* equilibrium equations

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0, \qquad \Sigma F_z = 0$$

must be satisfied at each joint. It is necessary to start at some joint where at least one known force acts and not more than three unknown forces are present. Adjacent joints upon which not more than three unknown forces act may be analyzed in turn. A method of sections could also be employed wherein all six of the equations of equilibrium would be used, but this method is tedious by reason of the labor of computing moment arms in space.

32. Frames and Machines. Structures and mechanisms composed of joined members any one of which has more than two forces acting on it cannot be analyzed by the methods developed for simple trusses. Such members are multi-force members (three or more forces), and in general the forces will *not* be in the direction of the members. In the previous chapter the equilibrium of multi-force bodies was discussed and illustrated, but attention was focused on the equilibrium of a *single* rigid body. In this present article attention is focused on the equilibrium of *interconnected* rigid bodies which contain multi-force members. The forces acting on each member of a connected system are found by isolating the member with a free-body diagram and applying the established equations of equilibrium. The *principle of action and reaction* must be carefully observed when representing the forces of interaction on the separate free-body diagrams. If the structure contains more members or supports than are necessary to prevent collapse, then, as in the case of trusses, the problem is statically indeterminate, and the principles of equilibrium although necessary are not sufficient for solution.

If the frame or machine constitutes a rigid unit by itself, the analysis is best begun by establishing all the forces external to the structure considered as a single rigid body. The structure is then dismembered and the equilibrium of each part is considered. The equilibrium equations for the several parts will be related through the terms involving the

forces of interaction. If the structure is not rigid by itself but depends on its external supports for rigidity, then it is usually necessary to consider first the equilibrium of a portion of the system which itself is inherently rigid.

It will be found that in most cases the analysis of frames and machines is facilitated by representing the forces in terms of their rectangular components. This is particularly so when the dimensions of the parts are given in mutually perpendicular directions. The advantage of this representation is that the calculation of moment arms is accordingly simplified. It is not always possible to assign all forces or their components in the proper sense when drawing the free-body diagrams, and it becomes necessary to make an arbitrary assignment. In this event it is *absolutely necessary* that a force be *consistently* represented on the diagrams for interacting bodies which involve the force in question. Thus for two bodies which are connected by the pin A, Fig. 44a, when separated the components must be consistently represented in the *opposite* directions, Fig. 44b. The assigned directions may prove to be wrong

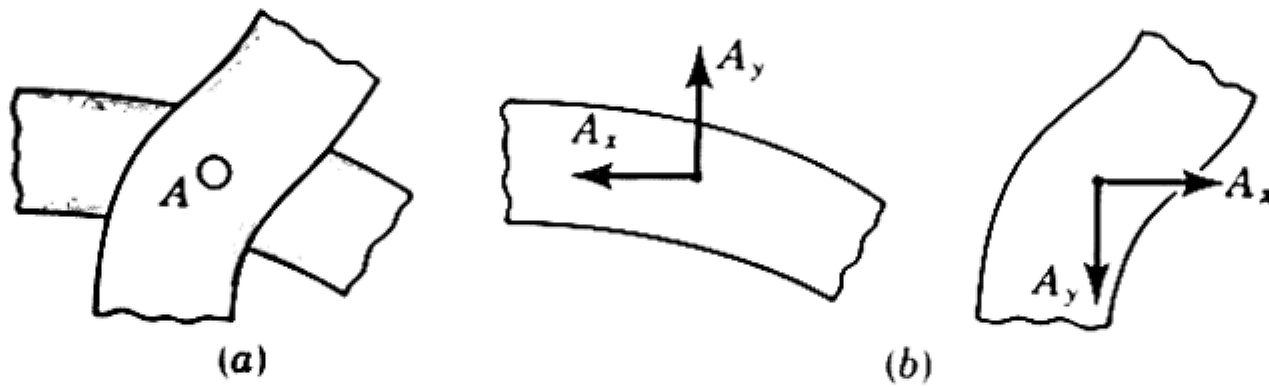


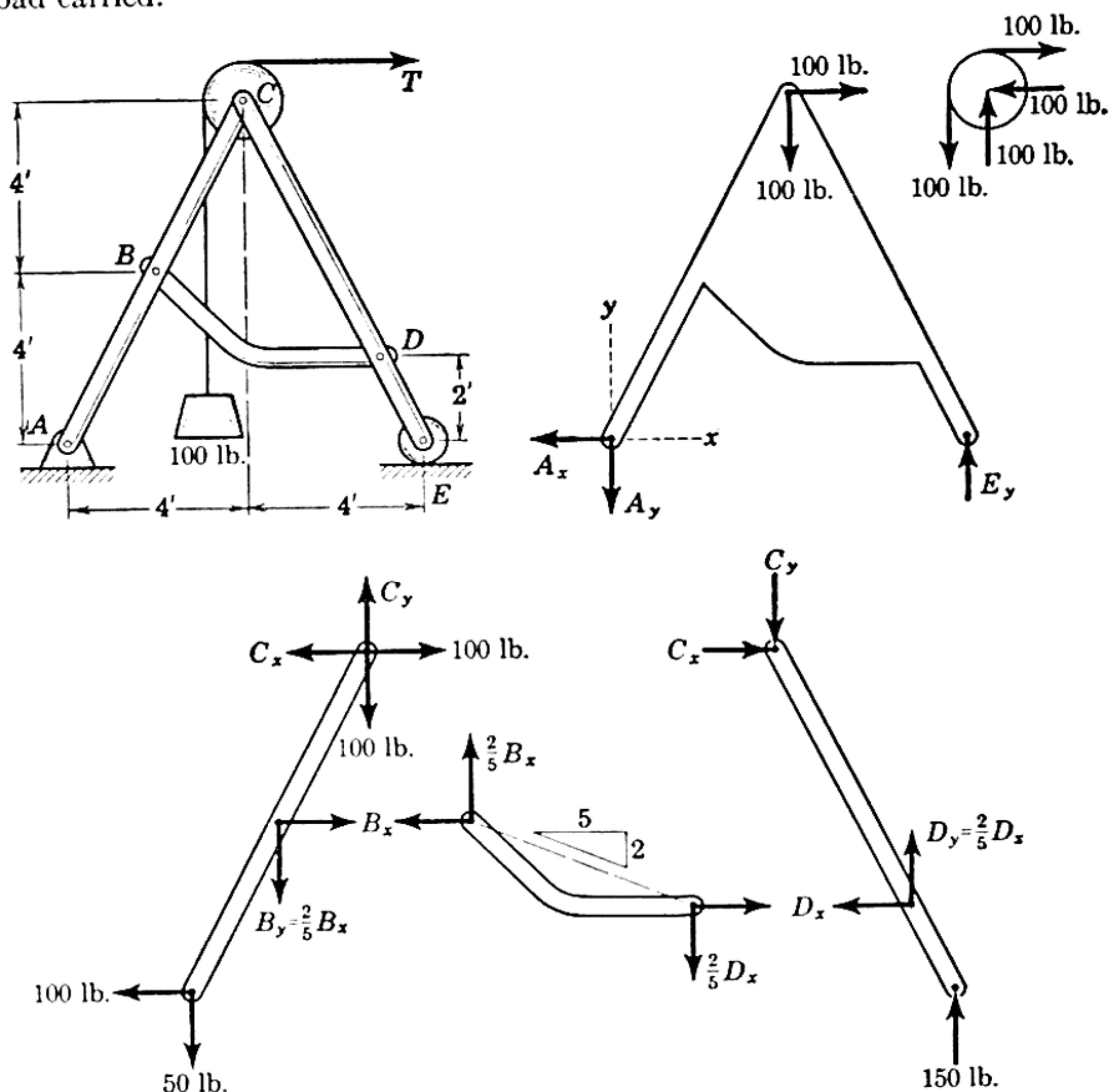
FIG. 44

when the algebraic signs of the components are determined upon calculation. If A_x, for instance, should turn out to be negative, then it is actually acting in the direction opposite to that originally represented. Accordingly it would be necessary to reverse the direction of the force on *both* members and to reverse the sign of this force term in the equations. Or the representation may be left as originally made, and the proper sense of the force will be understood from the negative sign.

Finally, situations occasionally arise where it is necessary to solve two or more equations simultaneously in order to separate the unknowns. In most instances, however, simultaneous solutions may be avoided by careful choice of the member or group of members for the free-body diagram and by a careful choice of moment centers which will eliminate undesired terms from the equations. The method of solution described in the foregoing paragraphs is illustrated in the following sample problem.

SAMPLE PROBLEM

296. Determine the x- and y-components of all forces acting on each member of the hoisting frame. The weights of the parts are small compared with the load carried.



PROB. 296

Solution: The frame is a noncollapsible unit if the external supports are removed, so the reactions external to the frame as a whole will be computed first. The free-body diagram of the entire frame minus the pulley is shown in the figure. The forces acting at C are obtained from the free body of the pulley shown separately. The equations of equilibrium for the frame give

$[\Sigma M_A = 0]$ $\qquad 8E_y - 4 \times 100 - 8 \times 100 = 0, \qquad E_y = 150$ lb.,

$[\Sigma F_y = 0]$ $\qquad A_y + 100 - 150 = 0, \qquad A_y = 50$ lb.,

$[\Sigma F_x = 0]$ $\qquad A_x = 100$ lb.

The members are now separated and *all* forces acting on each member are represented. The diagrams are best arranged in their approximate relative position to aid in designating the common forces of interaction. The forces exerted by the pulley on the frame can be considered as applied to either of the two members and are shown acting on AC in this solution. Or if desired a separate free-body diagram of the pulley shaft which connects the members could be drawn to account for the action of the applied loads and the reactions of the two members at C. It is suggested that the equivalence of this alternate representation be studied. It is somewhat difficult to visualize the proper direction of the components of the interactions at C, so C_x and C_y are arbitrarily but consistently assigned as shown. The member BD is a two-force member when its weight is neglected. Therefore the components at B and D must be in a ratio such that their resultants are in the direction of the line joining points B and D. Since the slope of line BD is 2 to 5, the ratio of B_y to B_x and D_y to D_x is also 2 to 5. It should be noted that the shape of the actual member between B and D is of no consequence in considering its equilibrium.

For member AC the moment equation about C will yield B_x. Thus the equilibrium of AC requires

$[\Sigma M_C = 0]$ $\quad 2 \times \frac{2}{5}B_x + 4B_x + 4 \times 50 - 8 \times 100 = 0, \qquad B_x = 125 \text{ lb.},$ *Ans.*

$[\Sigma F_x = 0]$ $\quad 100 + 125 - 100 - C_x = 0, \qquad C_x = 125 \text{ lb.},$ *Ans.*

$[\Sigma F_y = 0]$ $\quad C_y - 100 - \frac{2}{5} \times 125 - 50 = 0, \qquad C_y = 200 \text{ lb.}$ *Ans.*

The values of C_x and C_y are positive, so that the directions were correctly guessed on the free-body diagrams.

From member BD it is clear that

$[\Sigma F_x = 0]$ $\quad D_x = B_x = 125 \text{ lb.},$ *Ans.*

from which

$$D_y = B_y = \tfrac{2}{5} \times 125 = 50 \text{ lb.} \qquad \textit{Ans.}$$

All forces acting on CE have been found from the two other members. As a check on the correctness of the work the equilibrium of CE may be checked. Thus

$[\Sigma M_C = 0]$ $\quad 3 \times \frac{2}{5} \times 125 + 4 \times 150 - 125 \times 6 = 0,$ *(Check)*

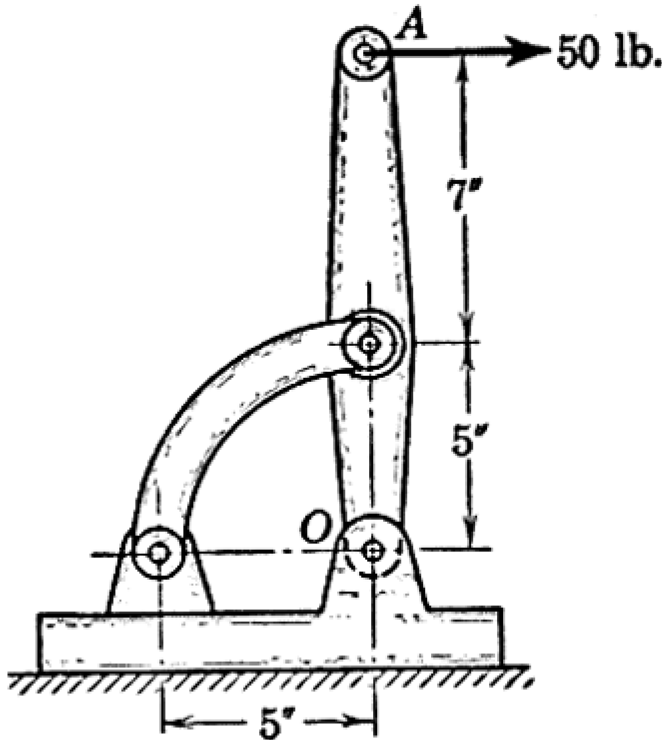
$[\Sigma F_x = 0]$ $\quad 125 - 125 = 0,$ *(Check)*

$[\Sigma F_y = 0]$ $\quad 150 + \frac{2}{5} \times 125 - 200 = 0.$ *(Check)*

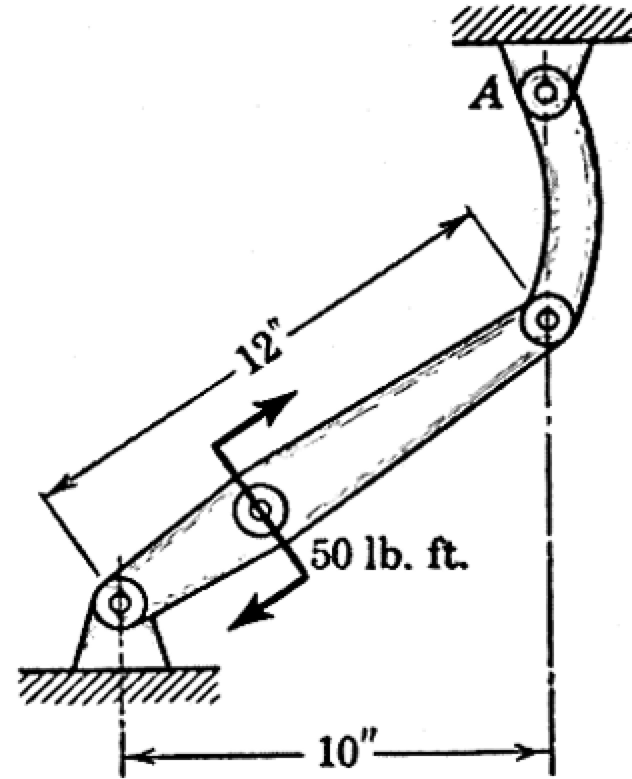
If desired the total reactions at each connection may be obtained by combining the two force components.

PROBLEMS

297. Determine the resultant force exerted by the pin at O on the lever OA.
Ans. $O = 139$ lb.

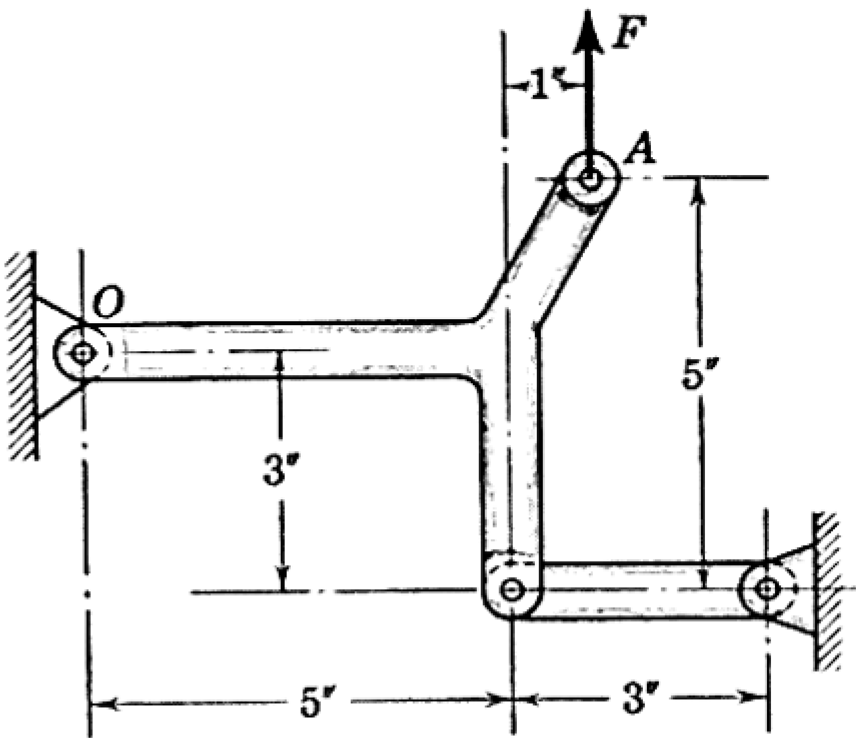


Prob. 297



Prob. 299

298. Solve Prob. 297 if the 50 lb. force is replaced by a counterclockwise couple of 600 lb. in. applied to a small pin at A fixed to the lever.

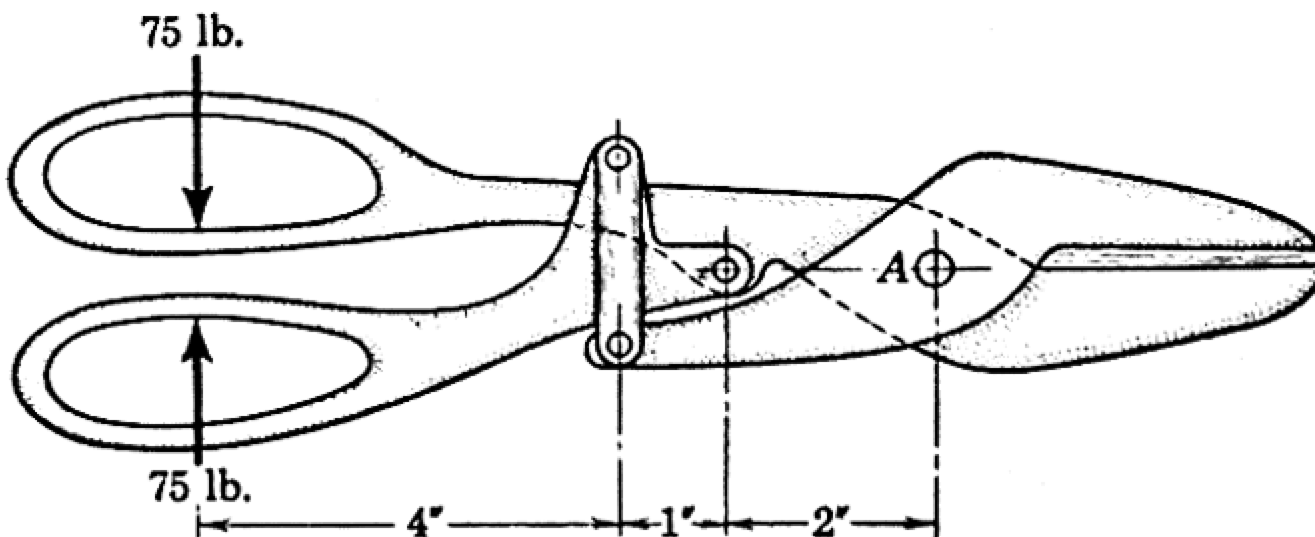


Prob. 300

299. Determine the force acting on pin A when the inclined link is subjected to the pure torque shown. The weights of the two links are negligible.
Ans. $A = 60$ lb.

300. Find the greatest force F which can be applied to the pin at A such that the reaction at O does not exceed 500 lb.
Ans. $F = 224$ lb.

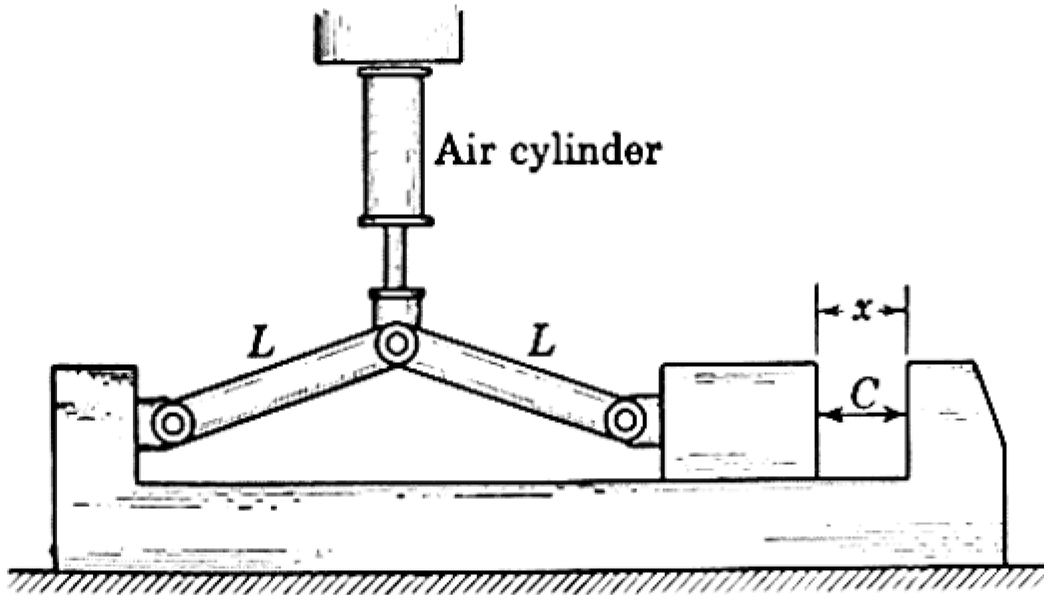
301. Compound lever snips, shown in the figure, are often used in place of regular tinners' snips when large cutting forces are required. For the gripping force of 75 lb. what is the cutting force P at a distance of 1 in. along the blade from the pin at A?



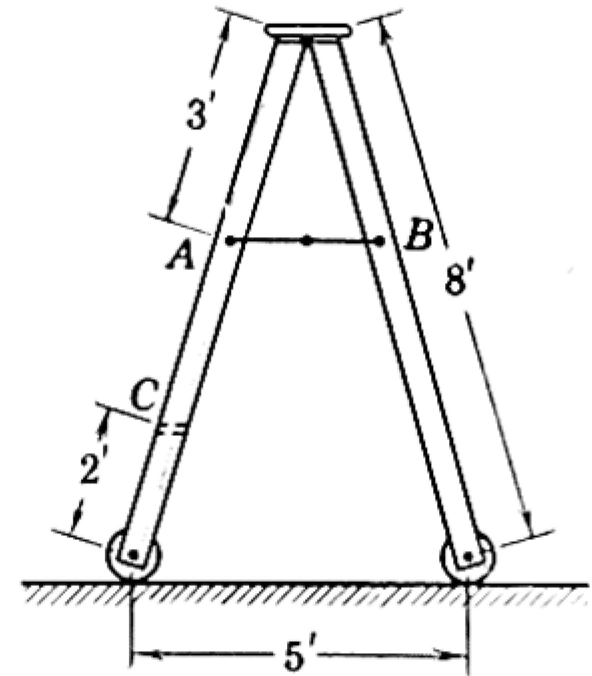
Prob. 301

302. A quick-acting vise, used as a production fixture, works by action of the toggle controlled by an air cylinder. Determine the clamping force C as a function of the vise opening x for a given air pressure p if the cylinder remains in a vertical position. The piston area is A, and the jaws are just closed when the toggle links are horizontal.

$$Ans. \quad C = pA\frac{L - \frac{x}{2}}{\sqrt{4Lx - x^2}}$$



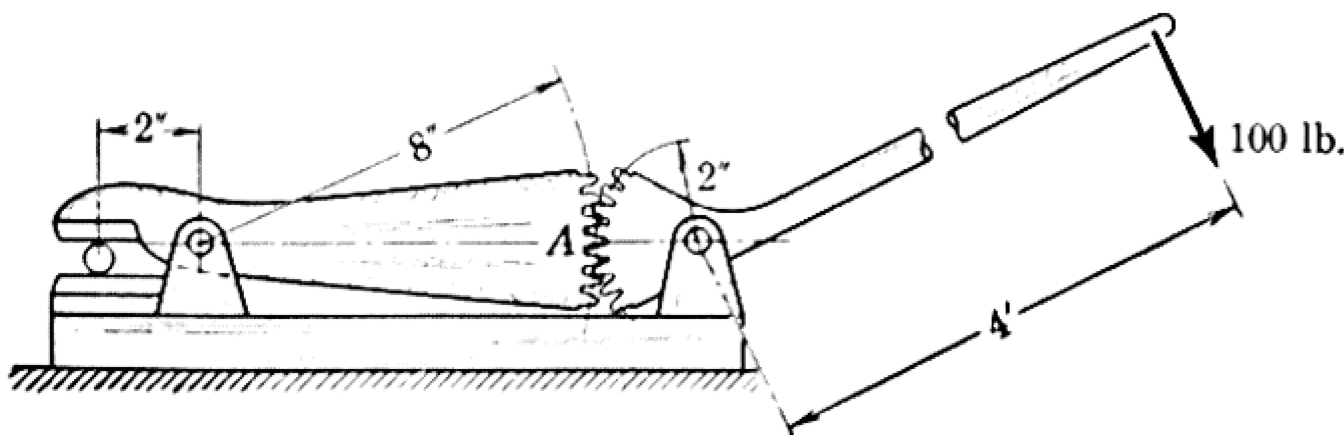
PROB. 302

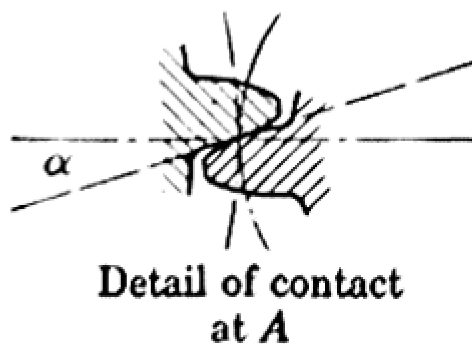


PROB. 303

303. A symmetrical 50 lb. stepladder is equipped with rollers so that it may be moved easily. Determine the tension in the connecting link AB when a 150 lb. man is standing on the step at C. The top hinge may be assumed to be on the center lines of the legs. *Ans.* $AB = 27.4$ lb.

304. A hand shear for cutting small steel bars is shown in the figure together with a detail of the tooth action. Determine the shearing force Q which the operator can apply at the shear if he is capable of exerting a 100 lb. force on the lever.

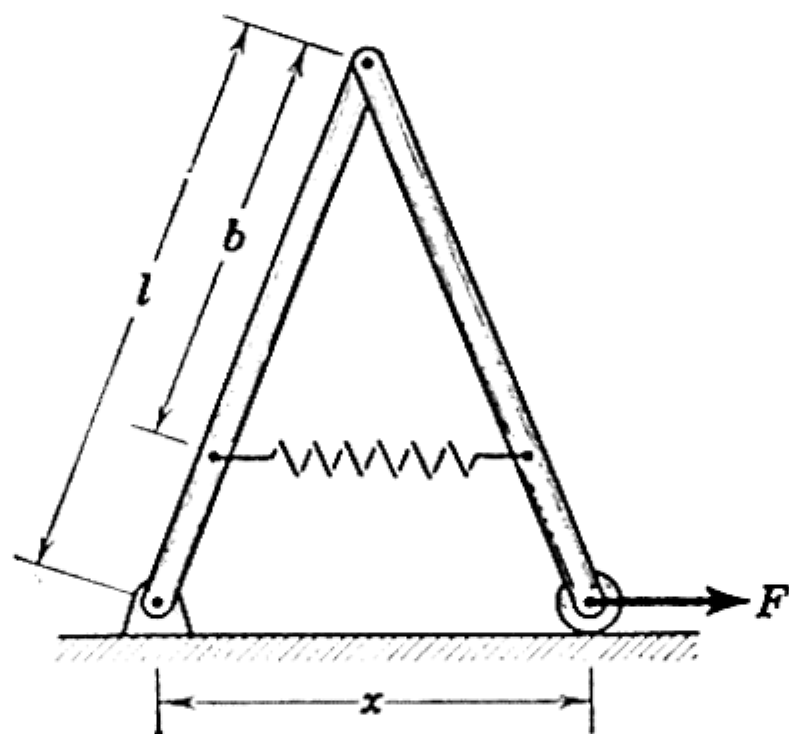




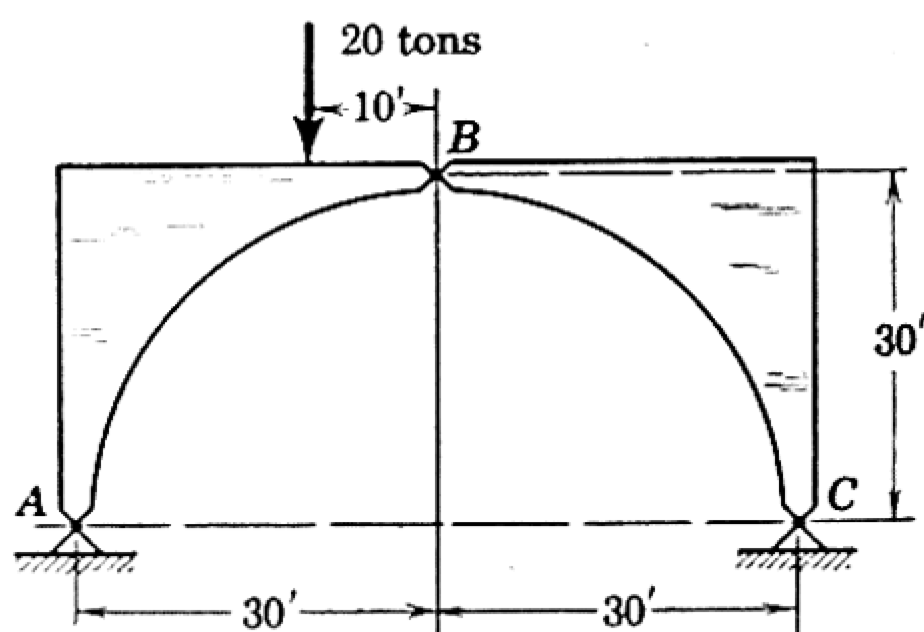
Detail of contact at A

PROB. 304

305. Determine the equilibrium value of x for the spring-connected members under the action of the force F. The spring has a stiffness k (spring modulus or spring constant) and has zero tension when $x = a$. *Ans.* $x = a + \frac{Fl^2}{kb^2}$



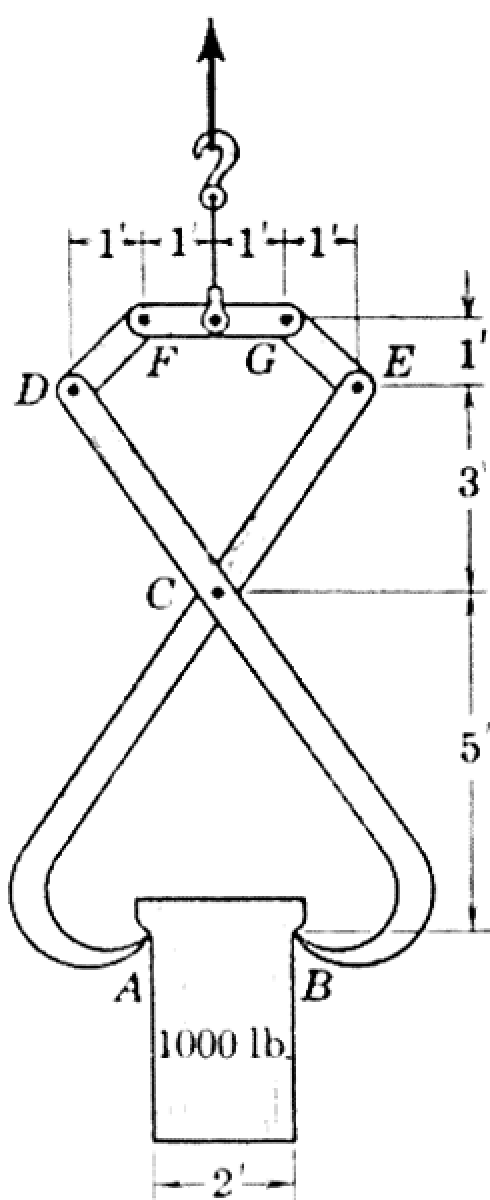
Prob. 305



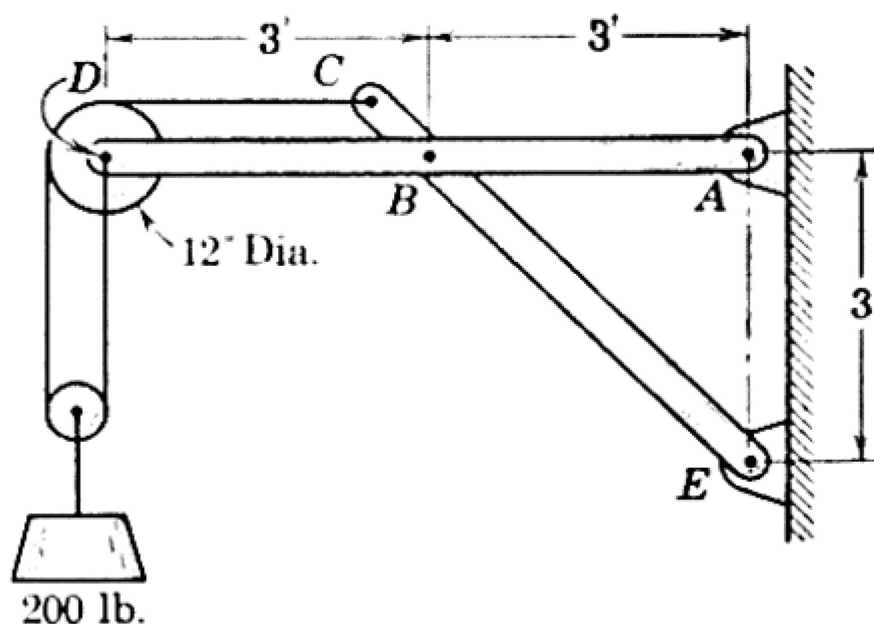
Prob. 306

306. Neglect the weight of the three-hinged arch compared with the applied load and compute the total hinge reaction at A. *Ans.* $A = 14.9$ tons

307. Find the force acting on the pin at C for the lifting tongs.



Prob. 307

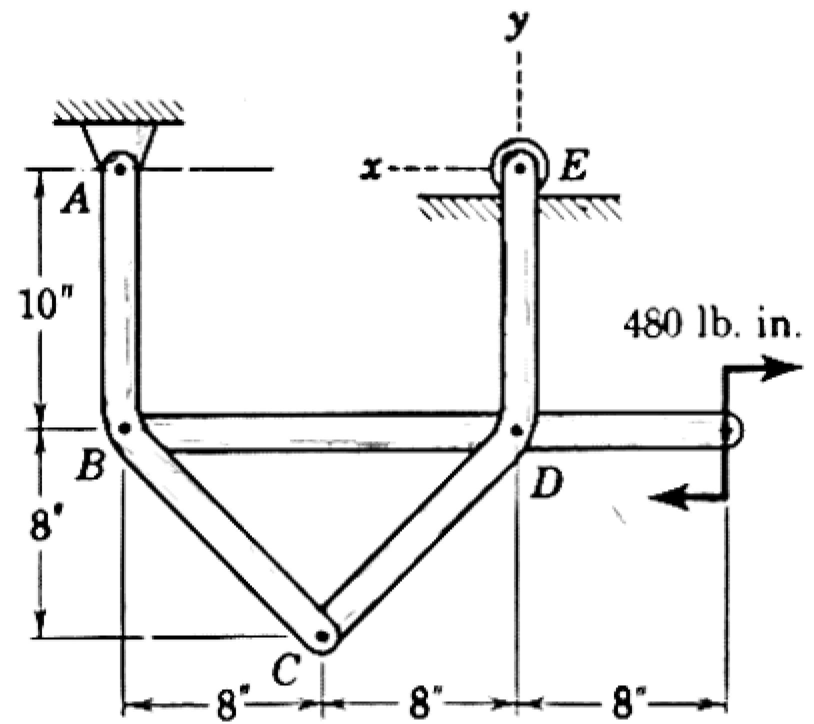


Prob. 308

308. Find the x- and y-components of the pin reactions at B and E.

Ans. $B_x = 517$ lb., $B_y = E_y = 400$ lb., $E_x = 417$ lb.

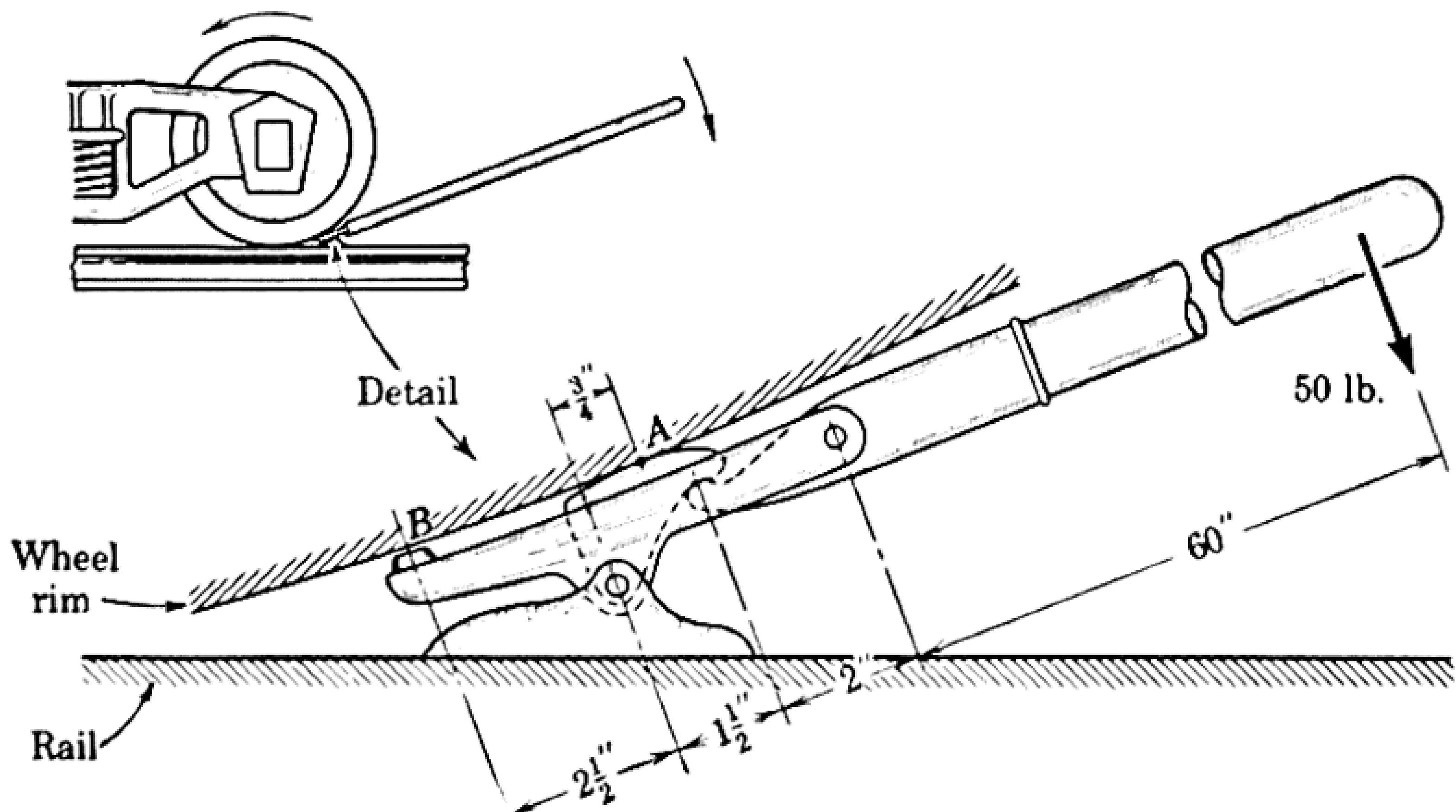
309. Determine the x- and y-components of all forces acting on each member. The frame has negligible weight.



Prob. 309

310. A car mover for rolling freight cars consists of a long hand-operated lever with toggle action as shown in the figure. For a 50 lb. force applied to the handle find the forces F_A and F_B exerted on the wheel by the mover. Assume that the upper shoe contact is at A and that the lower shoe does not slip on the rail. Neglect the curvature of the wheel rim between A and B.

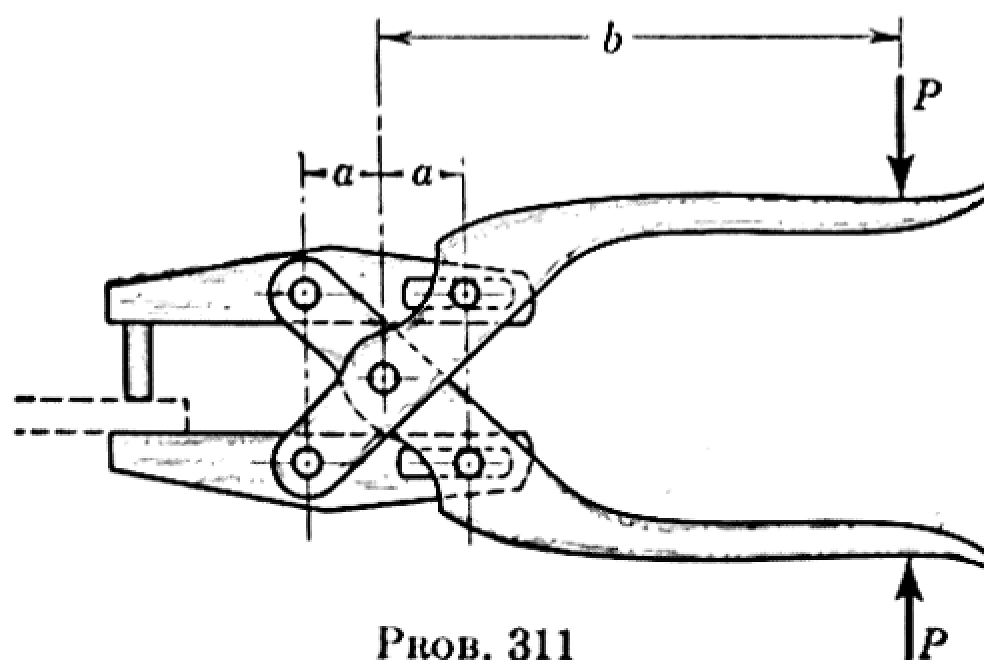
Ans. $F_A = 3000$ lb., $F_B = 2170$ lb.



Prob. 310

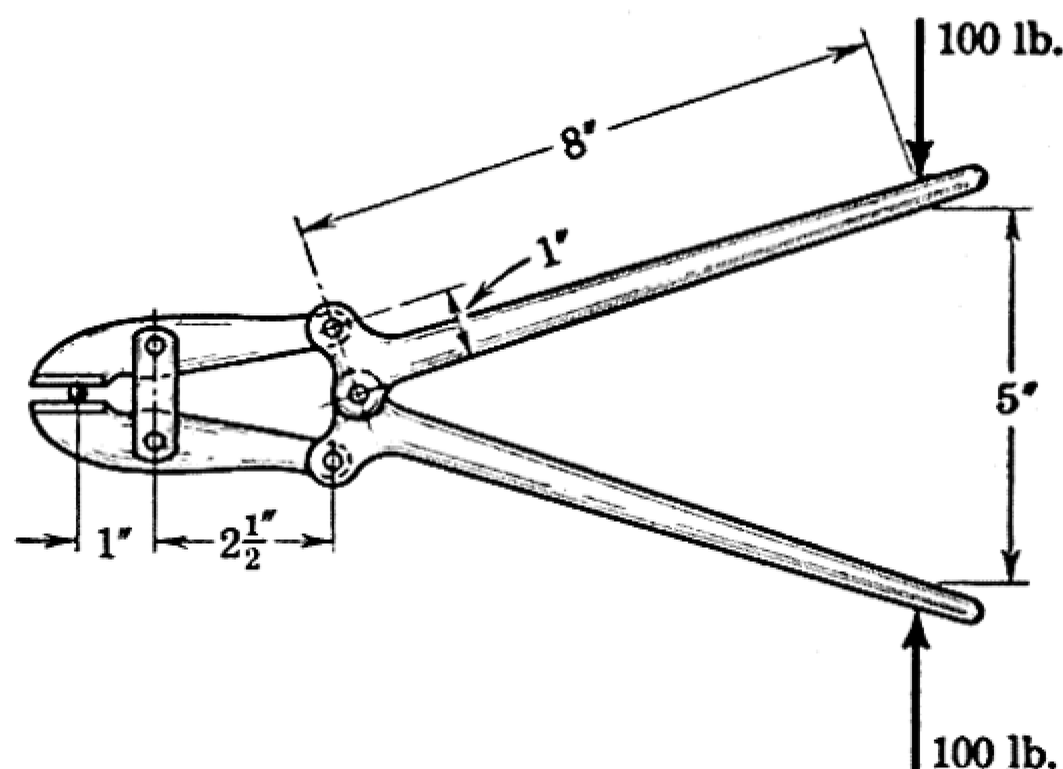
311. For the paper punch shown find the punching force Q for a hand grip of P.

Ans. $Q = P\dfrac{b}{a}$



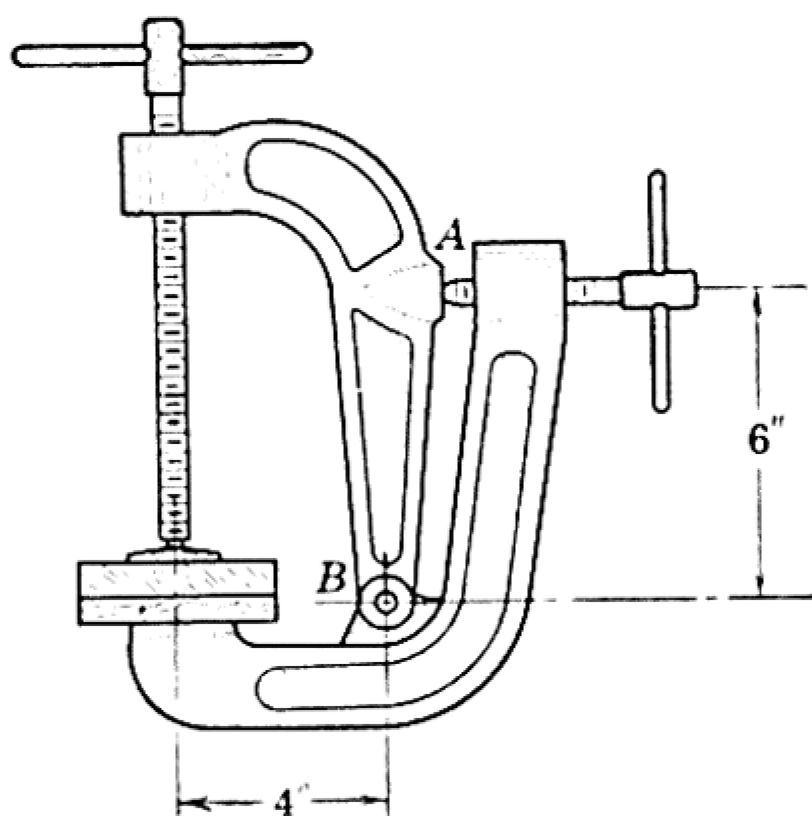
Prob. 311

312. Determine the force Q exerted on each side of the bolt by the cutters if the handles are subjected to loads of 100 lb.

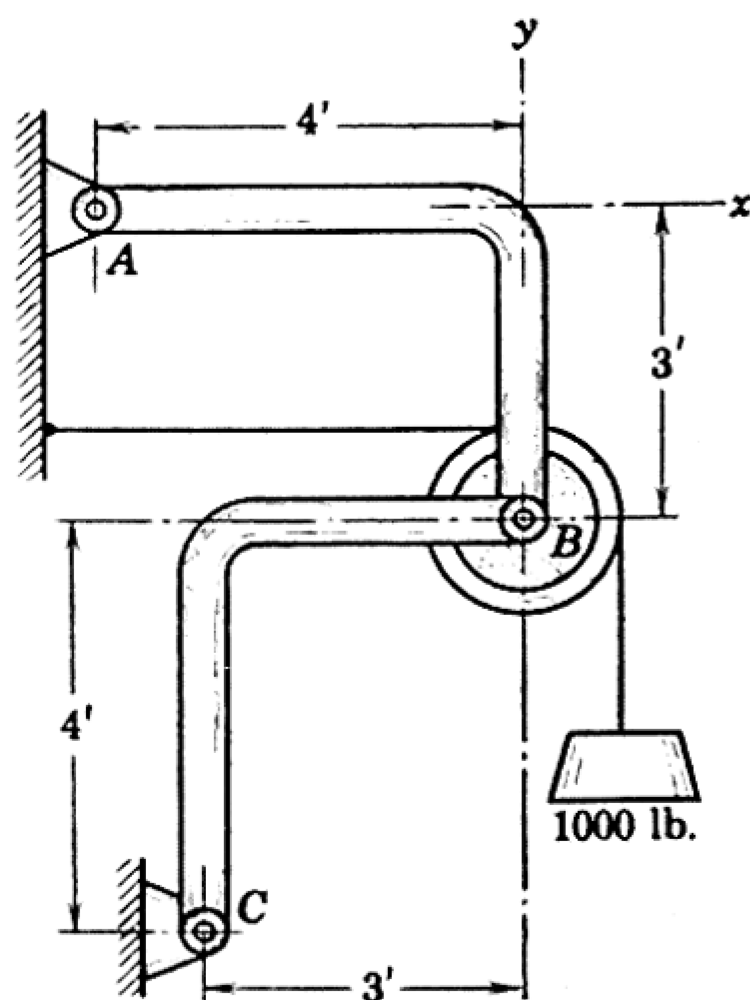


PROB. 312

313. A dual-grip clamp shown in the figure is used to provide added clamping force with a positive action. If the vertical screw is tightened to produce a clamping force of 1000 lb. and then the horizontal screw is tightened until the force at A is doubled, find the total clamping force P and the reaction R on the pin at B. *Ans.* $P = 2000$ lb., $R = 2400$ lb.



PROB. 313

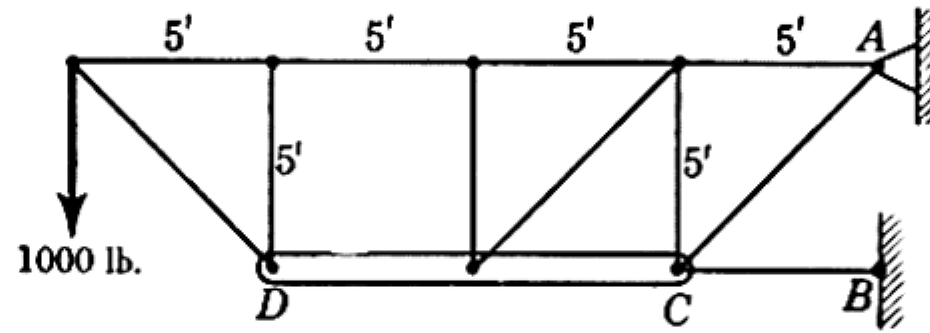


PROB. 314

314. Find the x- and y-components of all forces acting on each member of the frame. Neglect the weights of the pulley and members.

315. Calculate the total force exerted on the member CD at C.

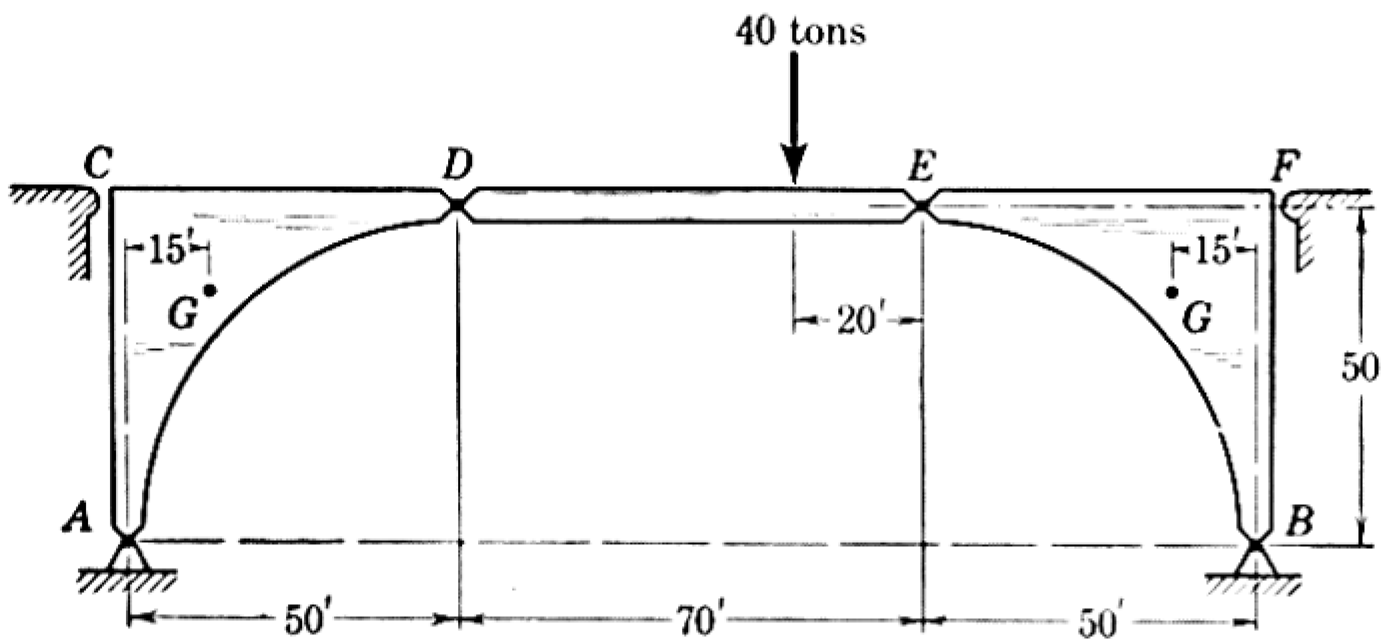
Ans. $C = 3160$ lb.



Prob. 315

316. A bridge consists of three sections hinged at A, B, D, and E. The end sections are prevented from rocking by horizontal support either at C or at F but not at both simultaneously. The midsection is uniform and weighs 50 tons, and each end section weighs 100 tons with center of gravity at G. Determine the reaction at C or F and the total force on the hinge at D.

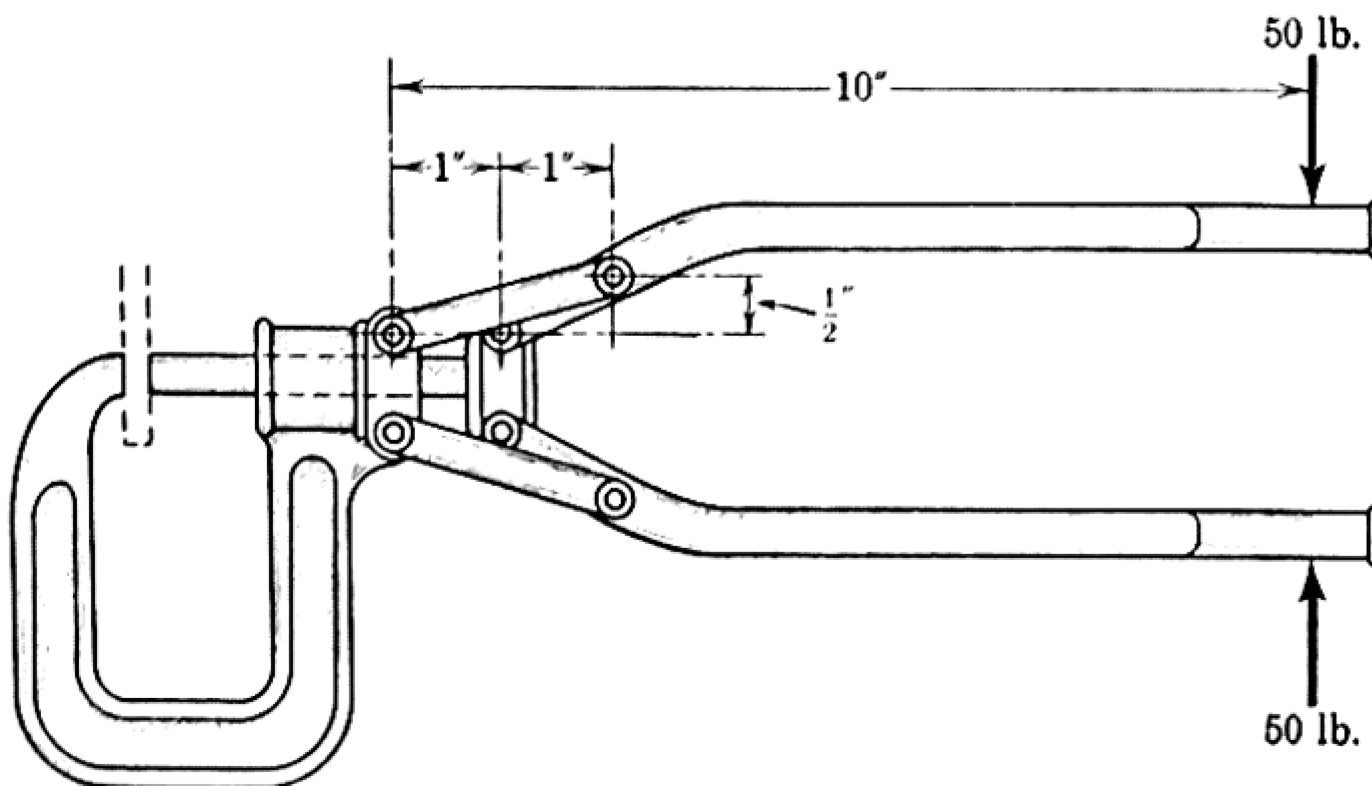
Ans. $C_x = 17.2$ tons, $D = 91.2$ tons



Prob. 316

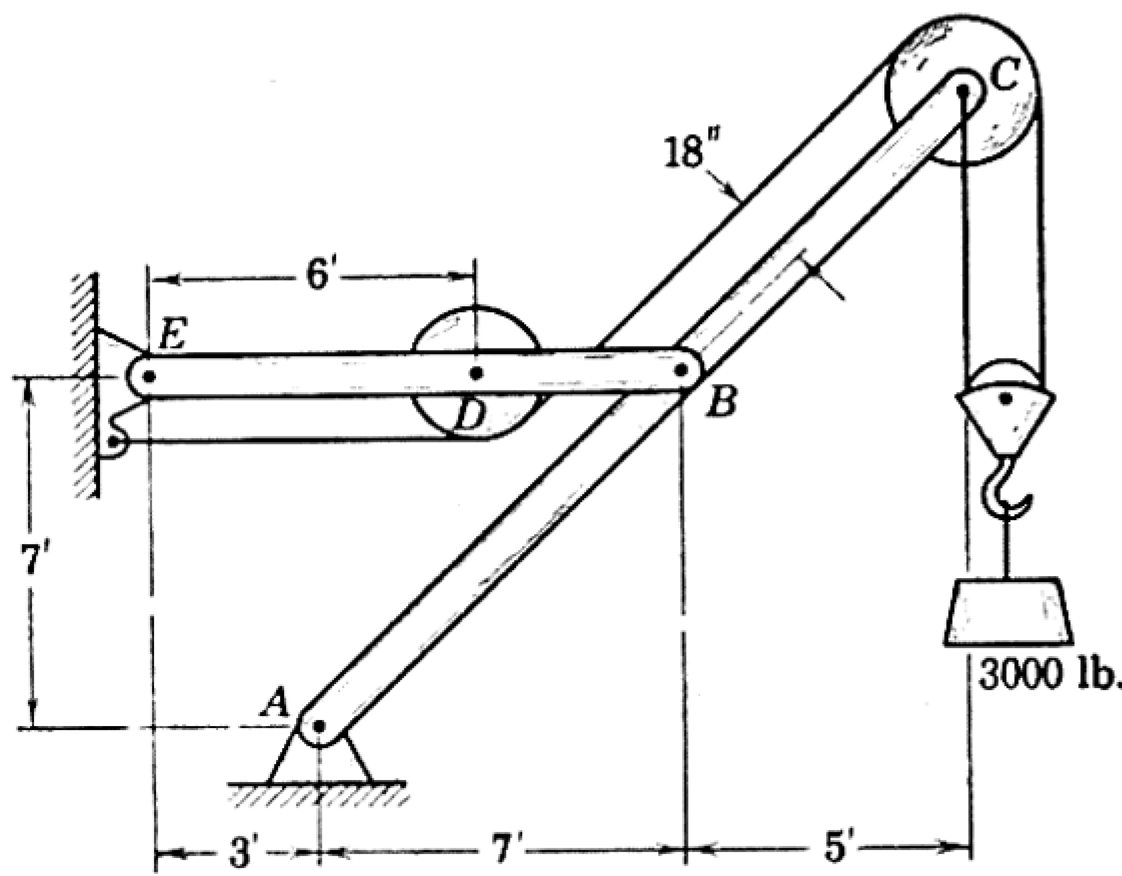
317. Determine the punching force P for the aircraft rivet squeezer shown if a 50 lb. grip is used.

Ans. $P = 3600$ lb.



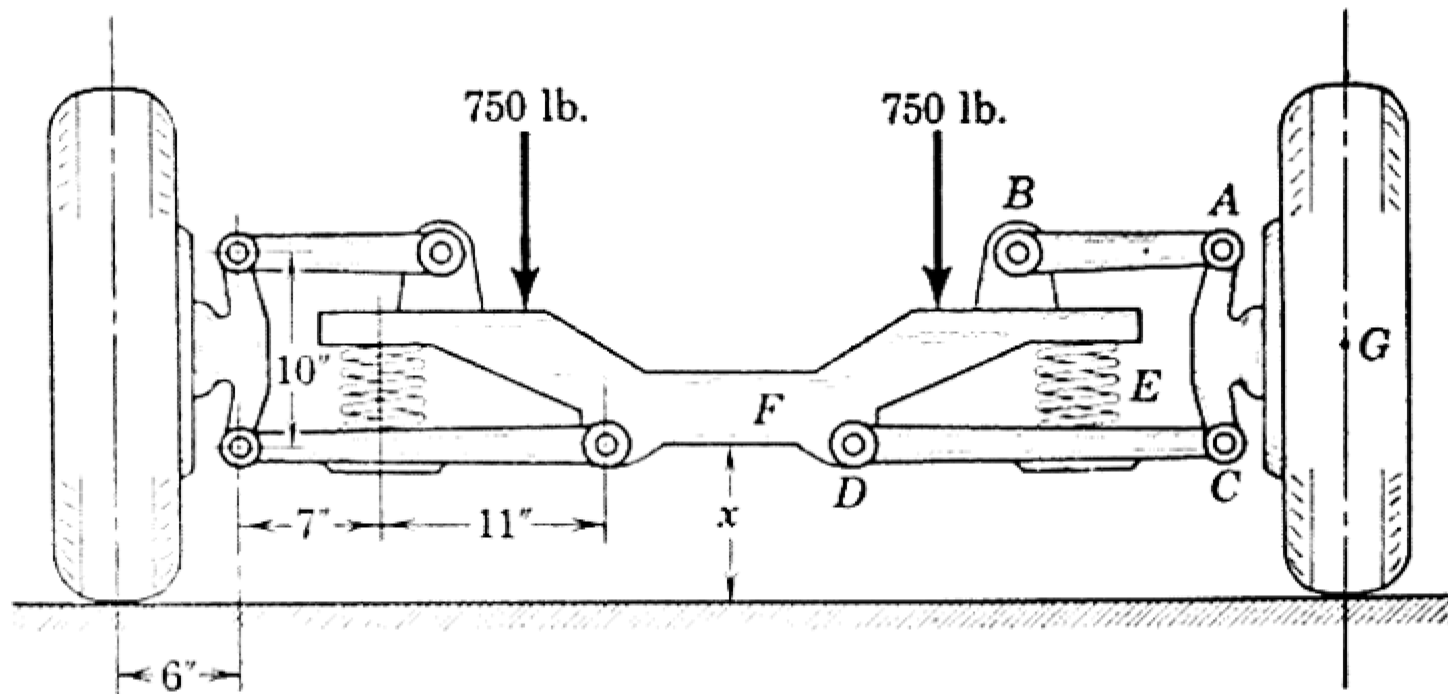
Prob. 317

318. Calculate the total reaction on the pin at A for the hoisting frame. Neglect the weights of the members.



PROB. 318

319. A passenger-car front-end suspension is shown in the sketch. The weight of the frame F is included in the vertical loading of 1500 lb. The weights of the links AB and DC and the springs E are small compared with the applied load and may be neglected. The weight of each wheel, brake, and steering



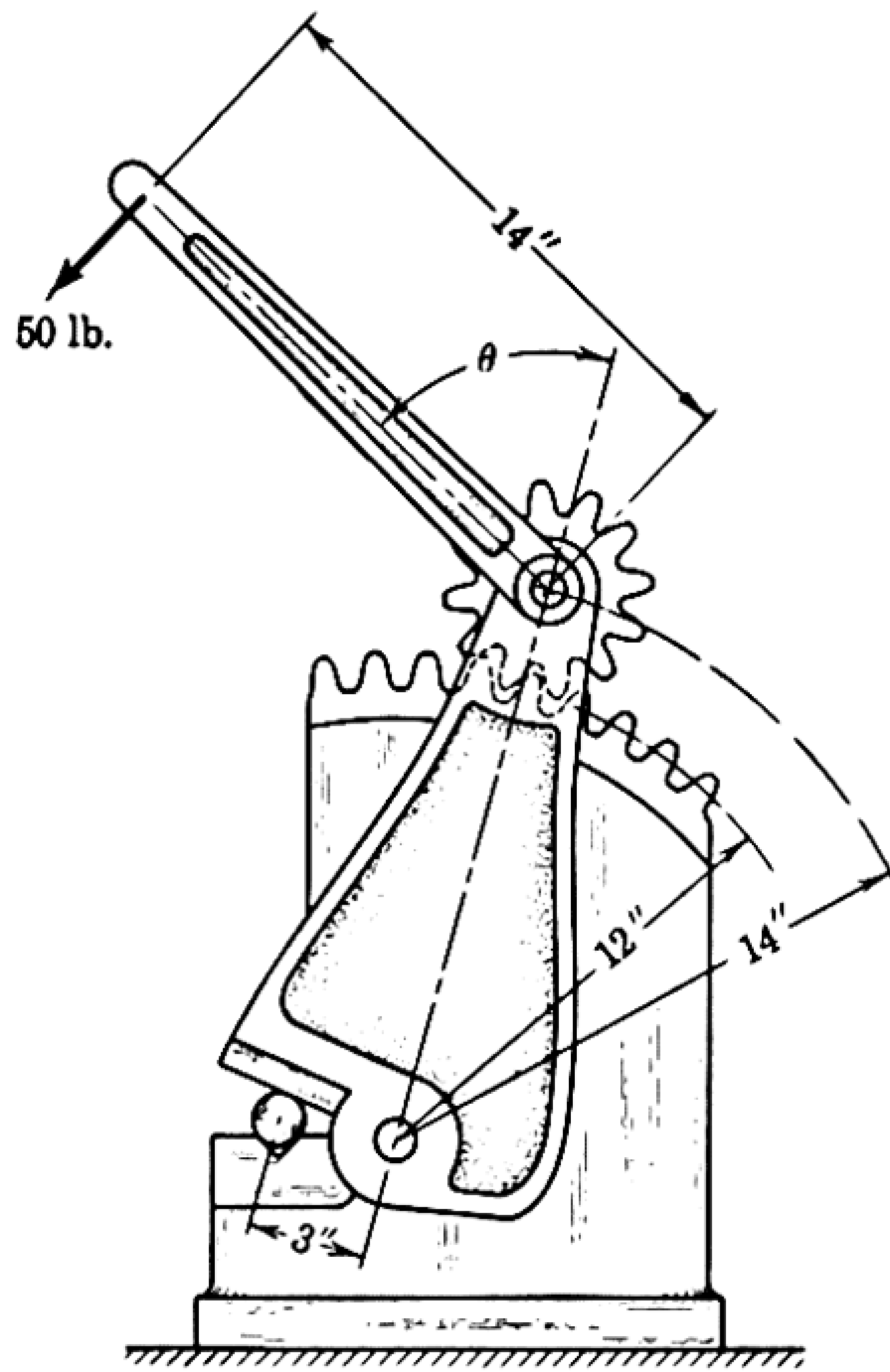
PROB. 319

knuckle assembly AC is 60 lb. and is assumed to act through G. In the equilibrium position shown each of the springs E is compressed 3 in. It is assumed that there is no shock absorber at B, and therefore no moment is applied to AB. Determine the compression P in the link AB and the constant k of the spring E.

Ans. $P = 450$ lb., $k = 409$ lb./in.

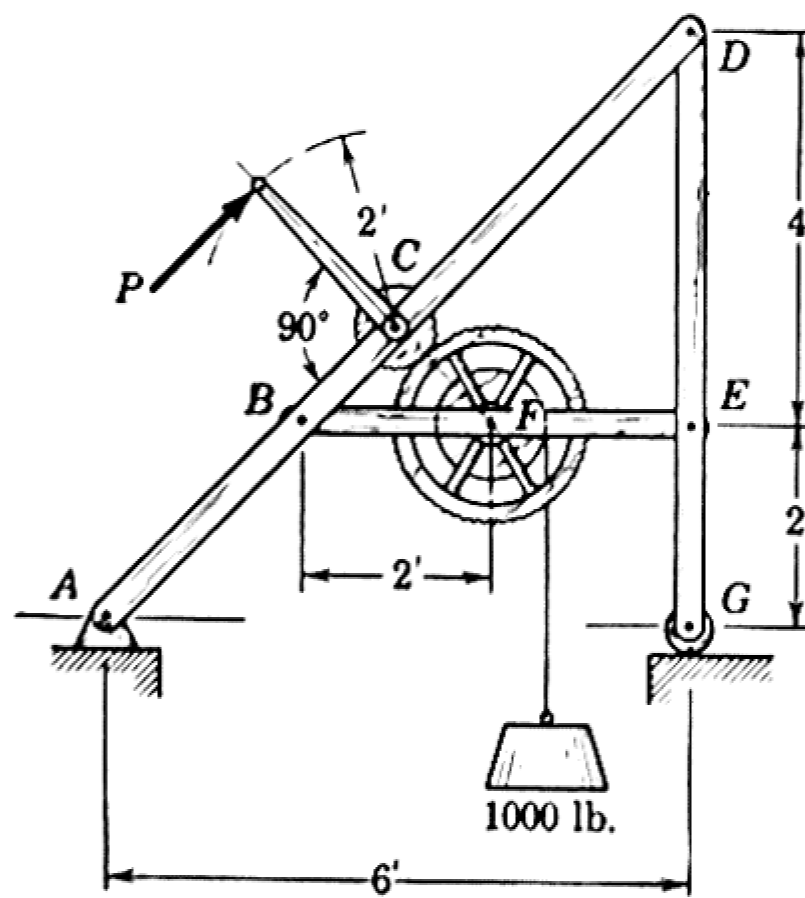
320. Determine the shearing force Q applied to the bar if a 50 lb. force is applied to the handle for $\theta = 30$ deg. For a given applied force what value of θ gives the greatest shear?

Ans. $Q = 1835$ lb., $\theta = 0$ for max. Q



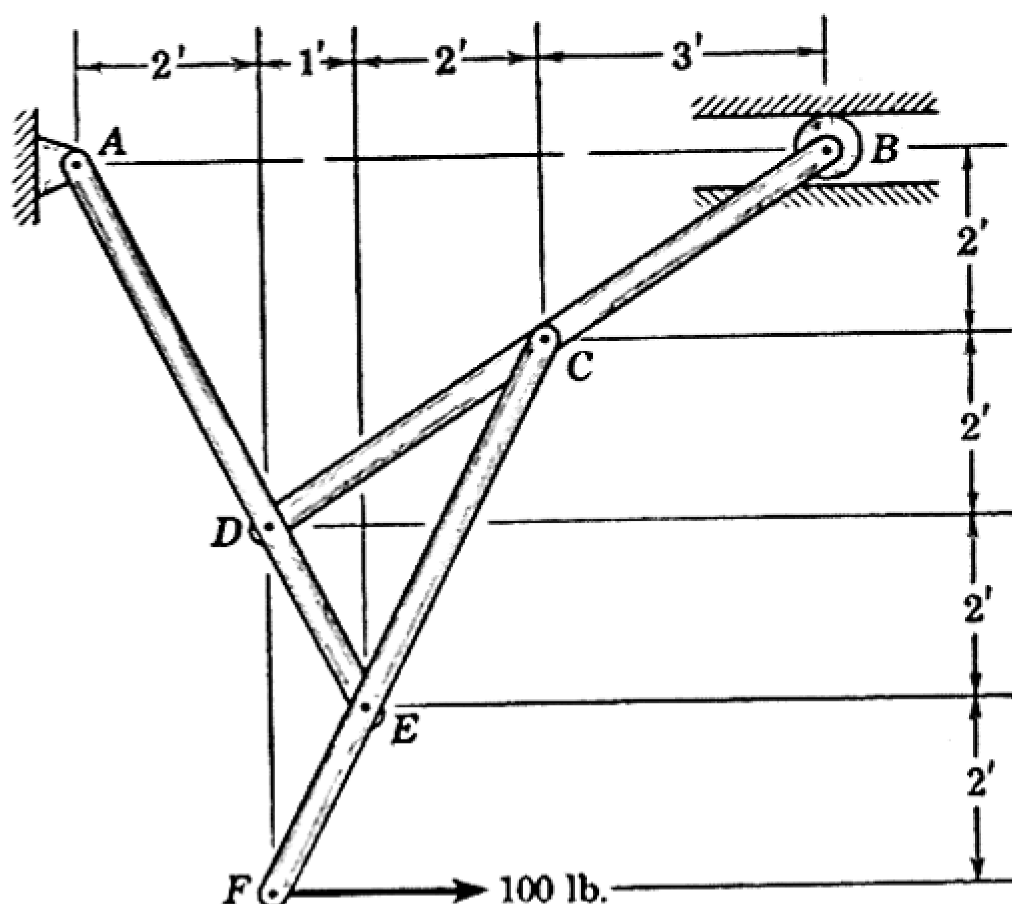
PROB. 320

321. Determine the reaction on the bolt at B for the loaded A-frame hoist with the operating handle held in the position shown. The diameters of the pinion at C, the gear at F, and the attached drum at F are 8.48 in., 25.46 in., and 12.73 in., respectively. Assume that contact between the gears is tangent to their mating circles. *Ans.* $B = 764$ lb.



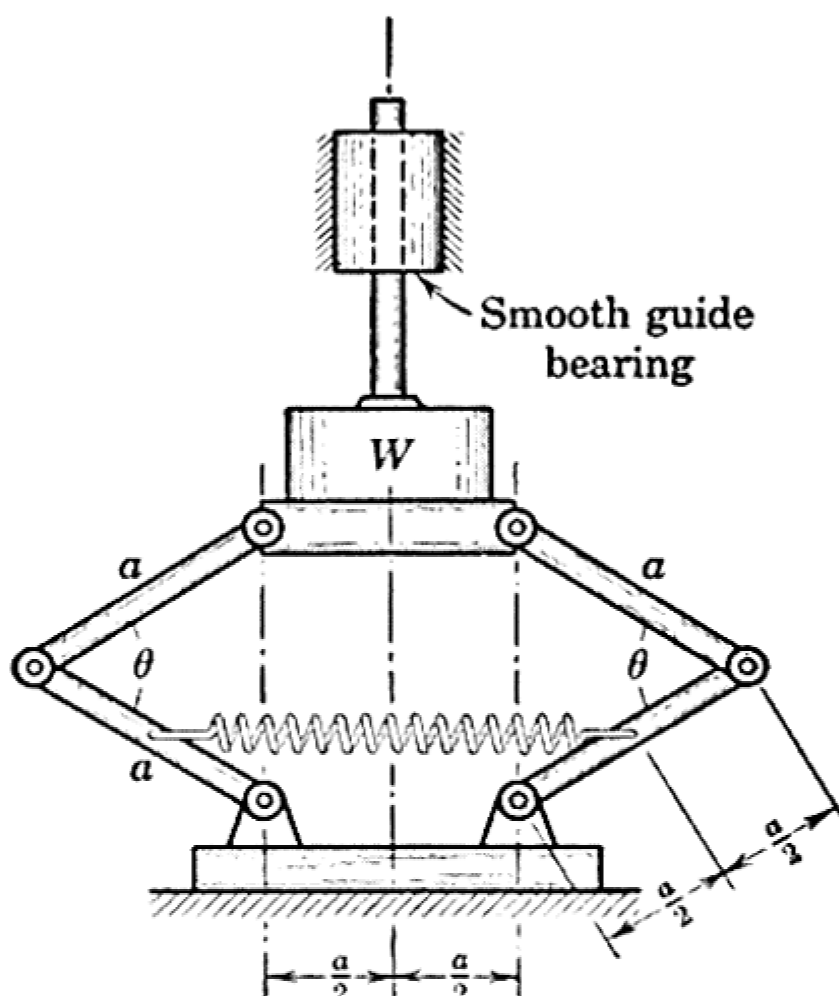
PROB. 321

322. Determine the total pin reaction at D. *Ans.* $D = 168$ lb.



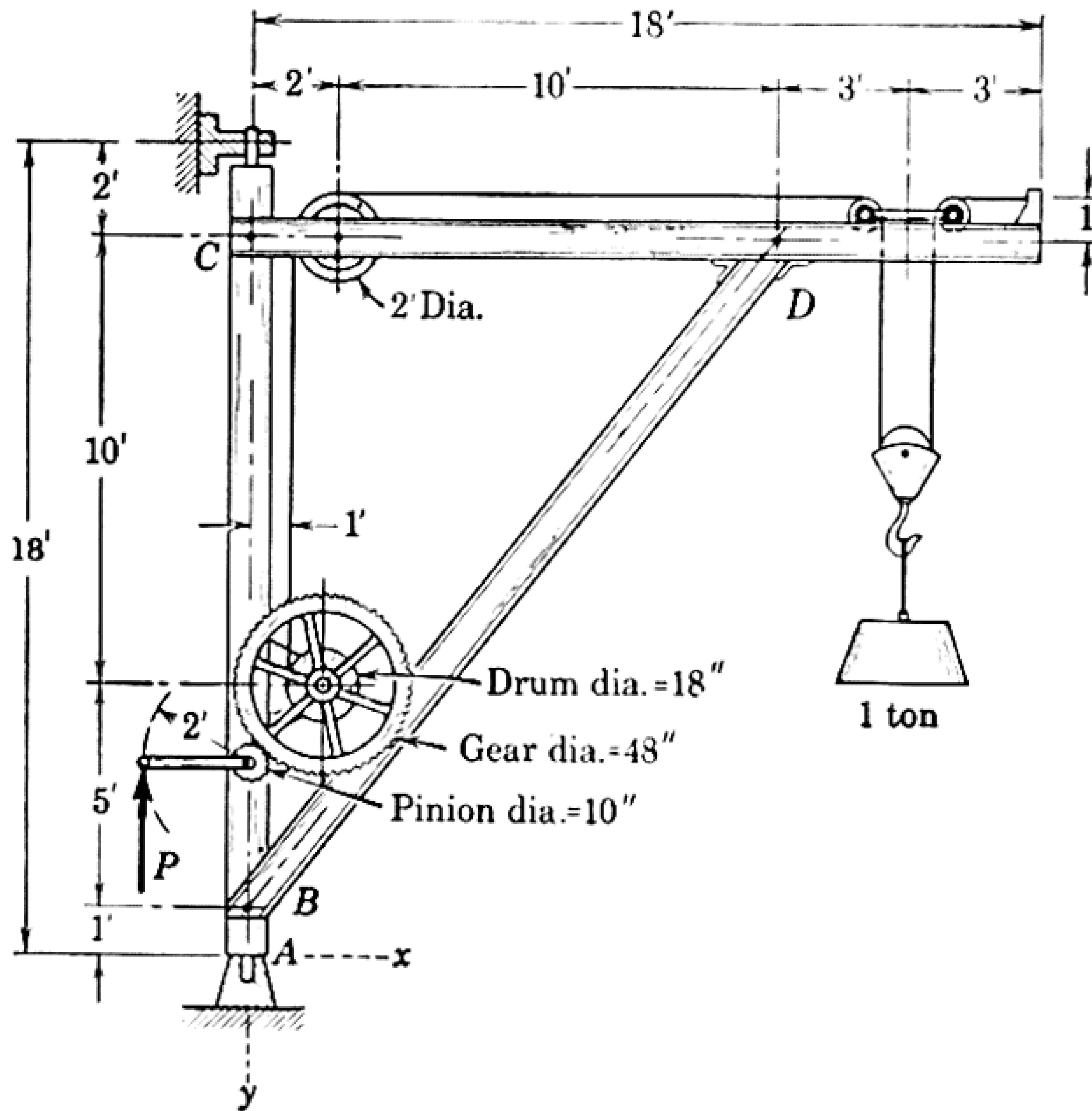
PROB. 322

323. If the force in the spring is zero when θ is 180 deg. and the constant of the spring is k, determine the angle θ for equilibrium under the action of the weight W. (Exclude $\theta = 180$ deg.) *Ans.* $\theta = 2 \sin^{-1} \dfrac{2W}{ka}$



PROB. 323

* **324.** In the small jib crane each of the three structural members is uniform and weighs 400 lb. The connections at B, C, and D may be assumed to be pin joints. The gear drum and pinion are mounted in bearings fastened to the mast. The upper mast bearing is capable of exerting horizontal support only. For the



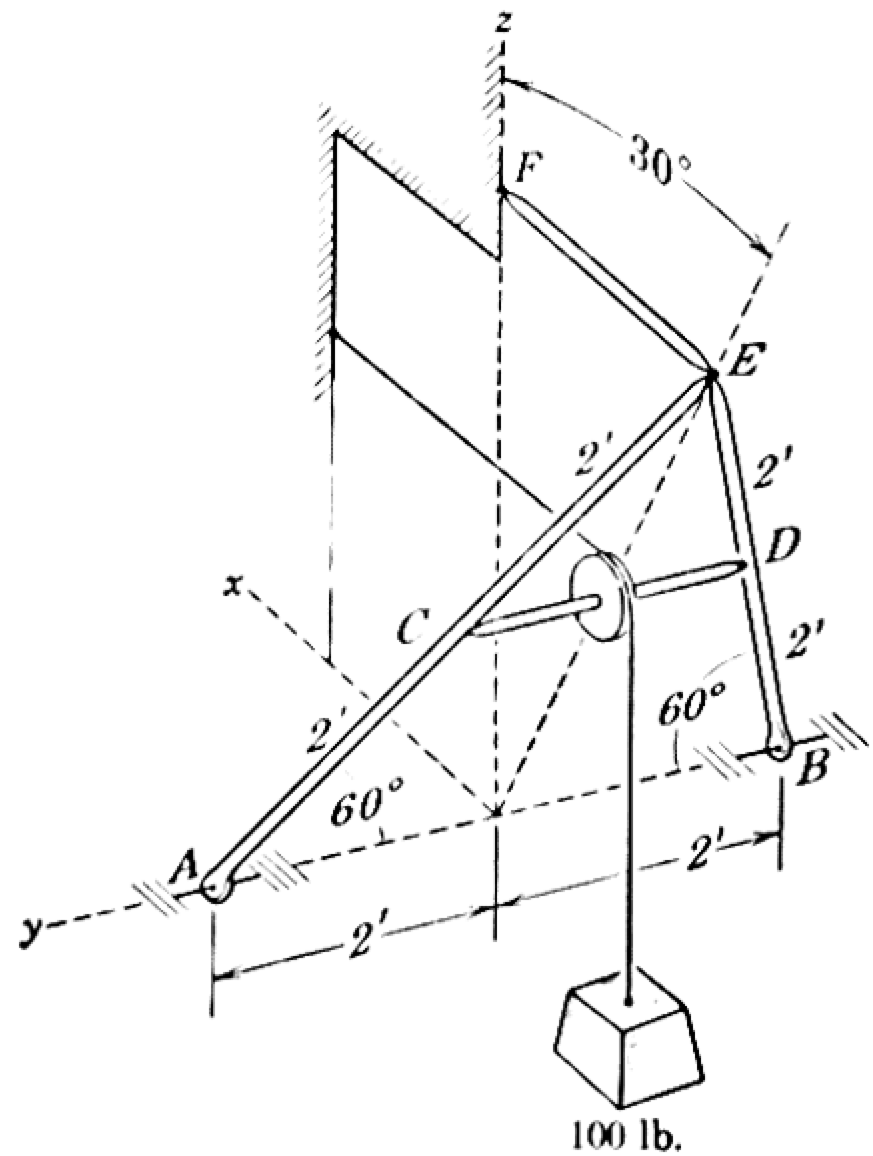
PROB. 324

position shown determine the force P on the crank handle to raise the load and calculate the x- and y-components of the force acting on the connection at B.

Ans. $P = 78.1$ lb., $B_x = 2470$ lb., $B_y = 3280$ lb.

*** 325.** Determine the total reaction at C if the connections at C, D, E, and F act as ball and socket joints. The bearings at A and B provide no support in the direction of AB.

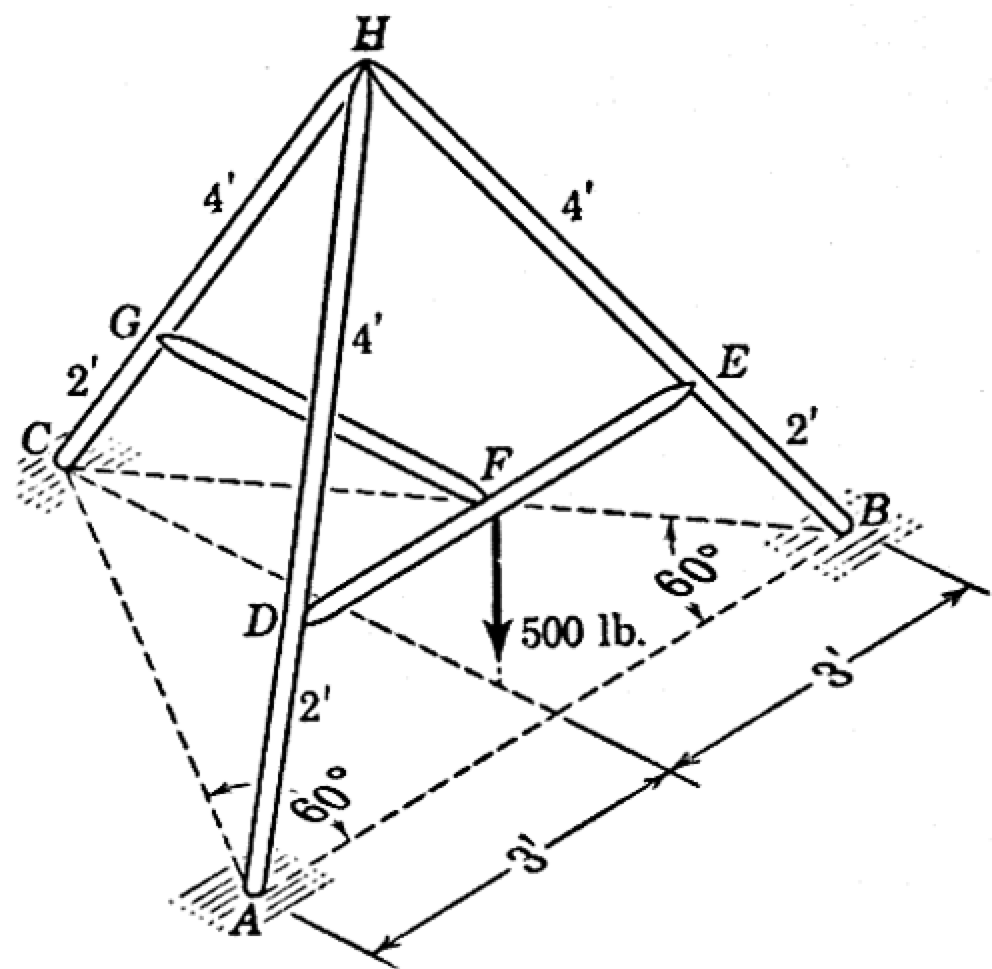
Ans. $C = 78.2$ lb.



PROB. 325

* **326.** The frame shown rests on a smooth horizontal surface, and the connections at D, E, F, G, and H act as ball and socket joints. Determine the total force acting on the connection at D due to the 500 lb. load applied at F.

Ans. $D = 257$ lb.



PROB. 326

CHAPTER V

Distributed Forces

33. Introduction. In the preceding chapters all forces have been assumed to be concentrated and have been represented by single vector arrows at the points of application or along the unique lines of action of the forces. Actually, no "concentrated" forces exist, since every real force applied to a body is distributed over some finite area or volume. In the case of force applied to a surface it is clear that, when the dimensions of the area upon which the force is distributed are negligible compared with the other dimensions of the body, the concept of a concentrated force raises no question. On the other hand force may be applied over a surface area of considerable size, and the variation in the distribution of the force acting over the area must be accounted for. There is also force which is distributed over the volume of a body, such as the earth's gravitational force or electrical and magnetic forces.

A distributed force is measured at any point by its intensity. Thus force distributed over an area is known as *pressure* or *stress* and is measured as force per unit area on which it acts (lb./in.^{2}). The term *pressure* is usually used to denote the intensity of distributed force due to the action of fluids, whereas the term *stress* is more generally used to denote internal distributed force in solids. Force distributed over the volume of a body is known as *body force* and is measured as force per unit of volume (lb./in.^{3}).

34. Center of Gravity. The most common distributed force is the force of attraction of the earth. This body force is distributed over all parts of every object in the earth's field of influence. The resultant of this distribution of body force is known as the *weight* of the body, and it is necessary to determine its magnitude and position for bodies whose weights are appreciable.

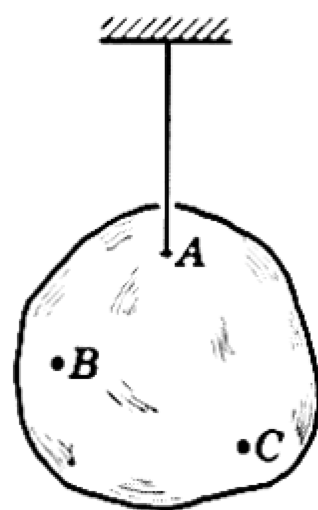


Fig. 45

Consider a three-dimensional body of any size, shape, and weight. If it is suspended, as shown in Fig. 45, by a cord from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant of the gravity or body forces acting on all the particles.

This resultant is clearly collinear with the cord, and it will be assumed that its position will be marked, say, by drilling a hole of negligible size along its line of action. The experiment is repeated by suspension at other points such as B and C, and in each case the line of action of the resultant is marked. For all practical purposes these lines of action will be concurrent at a point which is known as the *center of gravity* or *center of mass* of the body. An exact analysis, however, would take into account the fact that the directions of the gravity forces for the various particles of the body differ slightly because of the fact that they converge toward the center of attraction of the earth. Also, since the particles are at different distances from the earth, the intensity of the earth's force field is not exactly constant over the body. These considerations lead to the conclusion that the lines of action of the gravity force resultants in the experiments just described will not quite be concurrent, and therefore no unique center of gravity exists in the exact sense. This condition is of no practical importance as long as we deal with bodies whose dimensions are small compared with those of the earth. We therefore assume a uniform and parallel field of force due to the earth's gravitational attraction, and this condition results in the concept of a unique center of gravity.

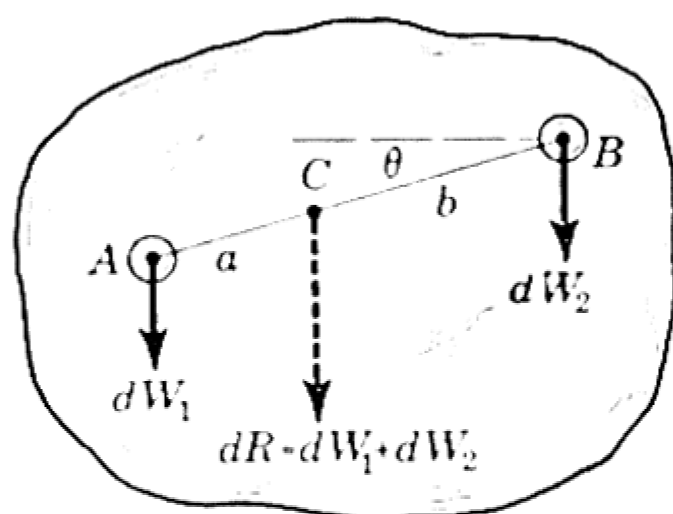


FIG. 46

The position of the center of gravity as a unique point may be determined by experiment, but its existence must not be inferred from this result alone. It may be proved in the following way. Consider any two particles of weight dW_1 and dW_2 in a body as shown in Fig. 46. The line AB joining these particles makes an angle θ with the horizontal plane, and the particles are separated a distance $a + b$. The resultant dR of dW_1 and dW_2 is their sum and can be located by Varignon's theorem. Taking moments about B gives

$$(dW_1 + dW_2)b \cos \theta = dW_1(a + b) \cos \theta,$$

or

$$b = \frac{dW_1}{dW_1 + dW_2} (a + b).$$

The point C on line AB through which the resultant passes is now located, and the result is independent of θ. Hence regardless of the orientation of the body the resultant of dW_1 and dW_2 will always pass through point C. Combining dR with the weight of a third particle will

define another unique point through which the new resultant will always pass. This process is continued until all particles are accounted for and a point is obtained whose coordinates are independent of the orientation of the body. This point is the center of gravity of the body.

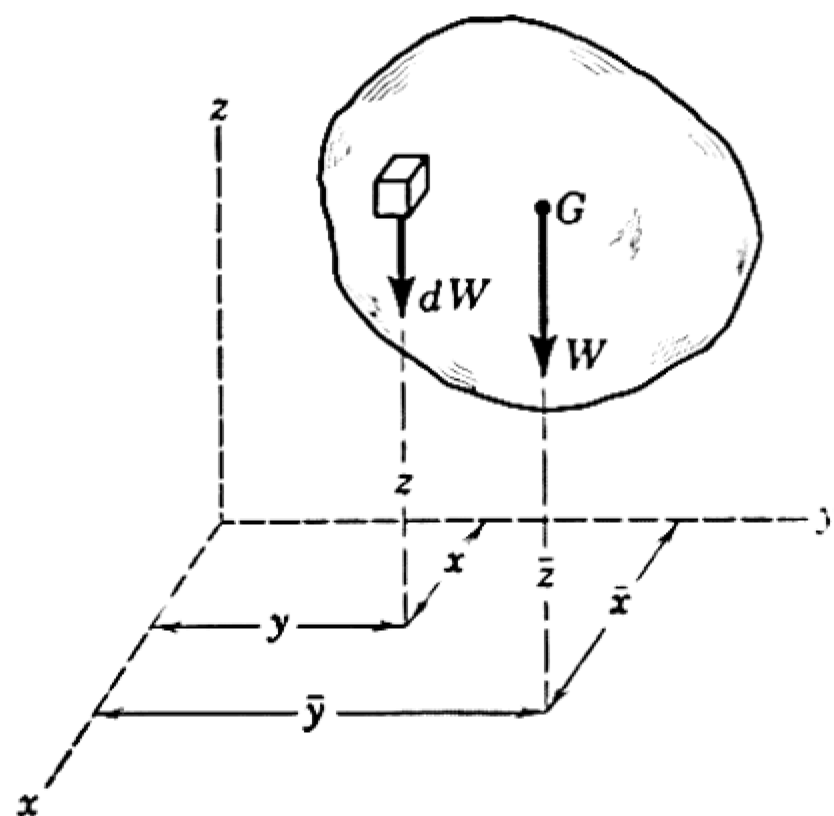


Fig. 47

To locate mathematically the center of gravity G of any body, Fig. 47, an equation may be written which states, by Varignon's theorem, that the moment about any axis of the resultant W of the gravity forces equals the sum of the moments about the same axis due to the gravity forces dW acting on each particle. If the center of gravity is designated by G, its coordinates by $\bar{x}$, $\bar{y}$, and $\bar{z}$, and the total weight by W, the moment principle gives

$$\bar{x} = \frac{\int x\,dW}{W},$$

$$\bar{y} = \frac{\int y\,dW}{W}, \tag{17}$$

$$\bar{z} = \frac{\int z\,dW}{W}.$$

The numerator of each expression represents the *sum of the moments*, and the product of W and the corresponding coordinate of G represents the *moment of the sum*. The third equation is obtained by revolving the body and reference frame 90 deg. about a horizontal axis so that the z-axis is horizontal.

In most problems the calculations may be simplified by an intelligent choice of reference axes. In general the axes should be placed so as to simplify the equations of the boundaries as much as possible. Thus polar coordinates will usually simplify any problem having circular boundaries. Another important clue may be taken from considerations of symmetry.

Whenever there exists a line or plane of symmetry, a coordinate axis or plane should be chosen to coincide with this line or plane. The center of gravity will always lie on such a line or plane since the moments due to symmetrically located elements will always cancel, and the body can be considered as being composed of pairs of these elements. Thus the axis of a right circular cone is such a line of symmetry, and the central vertical plane in the fore-and-aft direction of a ship's hull is such a plane of symmetry.

The proper choice of the differential element for the integration process is important. Whenever possible a first-order differential element of volume should be selected so that only one integration will be required to cover the entire figure. Regardless of the element used, the coordinate x, y, or z which is multiplied by dW in the numerator of Eqs. (17) is the coordinate to the *center of gravity of the element.* When a first-order element is not convenient, a second-order quantity should be selected in preference to a third-order element. As an example of these remarks consider the calculation of the center of gravity of a right circular cone. An element $dx\, dy\, dz$ could be chosen and a triple integration carried out. The process would be relatively laborious. On the other hand if a first-order element defined by a coaxial circular slice of differential thickness were selected, only one simple integration would be required in moving the element from vertex to base. Other considerations in the selection of the element and integration limits are best shown by actual examples, several of which follow the next article.

In the event the density μ is not constant but can be expressed as a function of the coordinates, it will be necessary to account for this variation in the calculation of both the numerators and denominators of Eqs. (17).

35. Centroids of Lines, Areas, and Volumes. Whenever the density μ of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Eqs. (17) and will therefore cancel. The expressions then define a purely geometrical property of the body, since any reference to its physical properties is absent. The term *centroid* is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, the term *center of gravity* or *center of mass* is used. If the density is uniform, the positions of the centroid and center of gravity are identical; whereas if the density varies, these two points will, in general, not coincide.

In the case of a slender rod or wire of length L and weight μ per unit length the body approaches a line, and $dW = \mu\, dL$. Therefore if μ is constant Eqs. (17) become

$$\bar{x} = \frac{\int x\,dL}{L},$$

$$\bar{y} = \frac{\int y\,dL}{L}, \tag{18}$$

$$\bar{z} = \frac{\int z\,dL}{L}.$$

When the body has a small thickness and approaches a surface of area A whose weight per unit area is μ, then $dW = \mu\,dA$, and again if μ is constant, Eqs. (17) give

$$\bar{x} = \frac{\int x\,dA}{A},$$

$$\bar{y} = \frac{\int y\,dA}{A}, \tag{19}$$

$$\bar{z} = \frac{\int z\,dA}{A}.$$

If the surface is curved, as with a shell, all three coordinates will be involved. If it is a plane surface, only the two coordinates of that plane will be involved.

In the case of a body of volume V and density μ the element is $dW = \mu\,dV$. The density μ will cancel if it is constant, and

$$\bar{x} = \frac{\int x\,dV}{V},$$

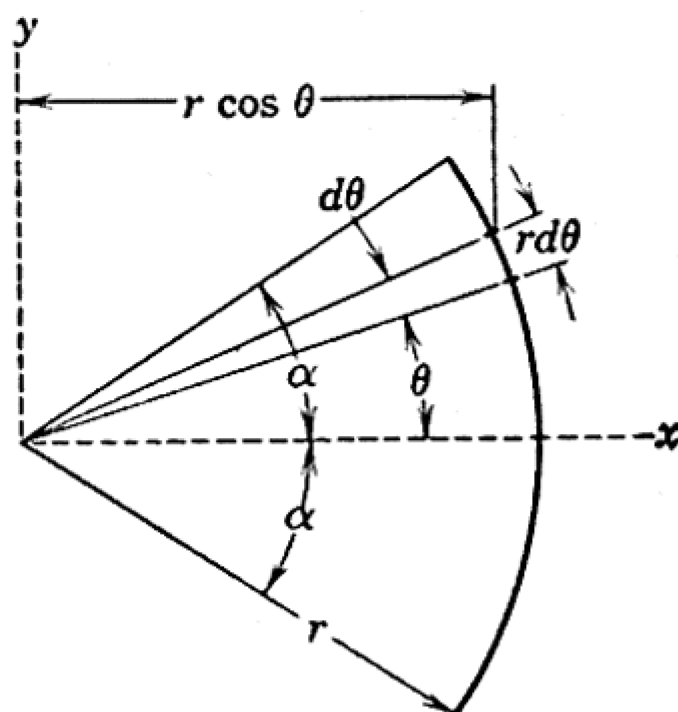
$$\bar{y} = \frac{\int y\,dV}{V}, \tag{20}$$

$$\bar{z} = \frac{\int z\,dV}{V}.$$

A summary of the centroidal coordinates for some of the commonly used shapes is given in Table B4, Appendix B.

SAMPLE PROBLEMS

327. *Centroid of a Circular Arc.* Locate the centroid of a line segment in the form of a circular arc as shown in the figure.



PROB. 327

Solution: Choosing the x-axis as an axis of symmetry makes $\bar{y} = 0$. A differential element of arc has a length $dL = r\,d\theta$, and the x-coordinate of the element is $r \cos \theta$. Applying the first of Eqs. (18) gives

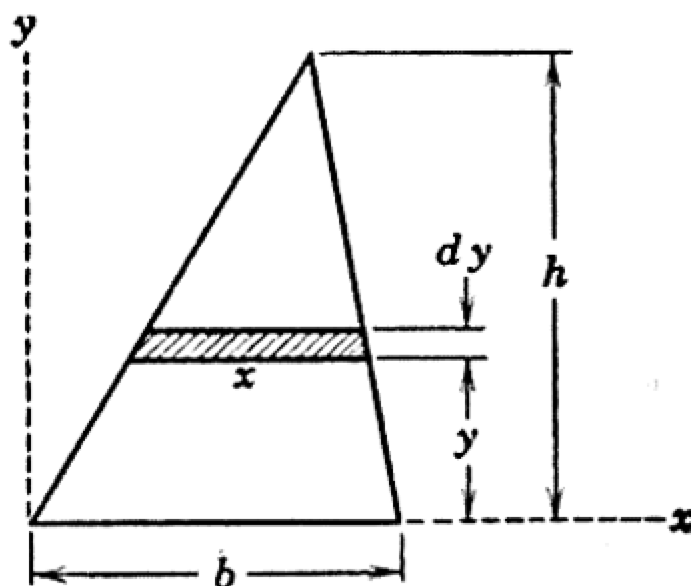
$$\left[L\bar{x} = \int x\,dL\right] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos \theta) r\,d\theta,$$

$$2\alpha r\bar{x} = 2r^2 \sin \alpha,$$

$$\bar{x} = \frac{r \sin \alpha}{\alpha}. \qquad \textit{Ans.}$$

For $2\alpha = \pi/2$, $\bar{x} = 0.900r$; for a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$.

328. *Centroid of a Triangular Area.* Locate the centroid of the area of a triangle of base b and altitude h.



PROB. 328

Solution: The x-axis is taken to coincide with the base. A differential strip of area $x\,dy$ is chosen. By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. (19) gives

$$[A\bar{y} = \int y\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6},$$

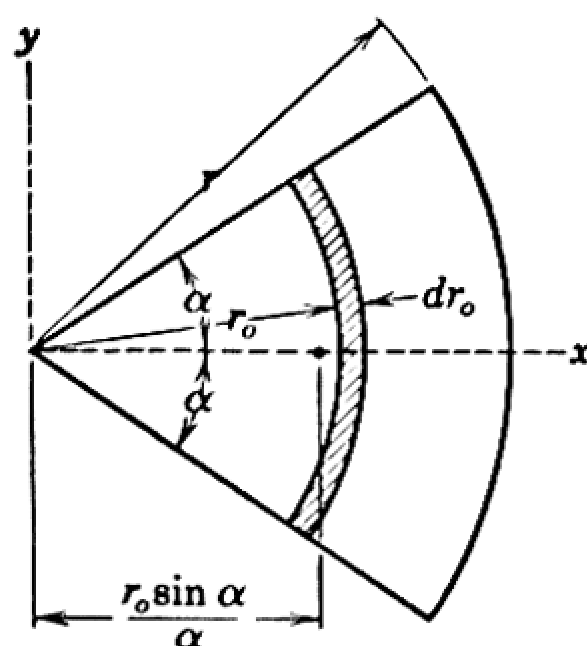
and

$$\bar{y} = \frac{h}{3}. \qquad \textit{Ans.}$$

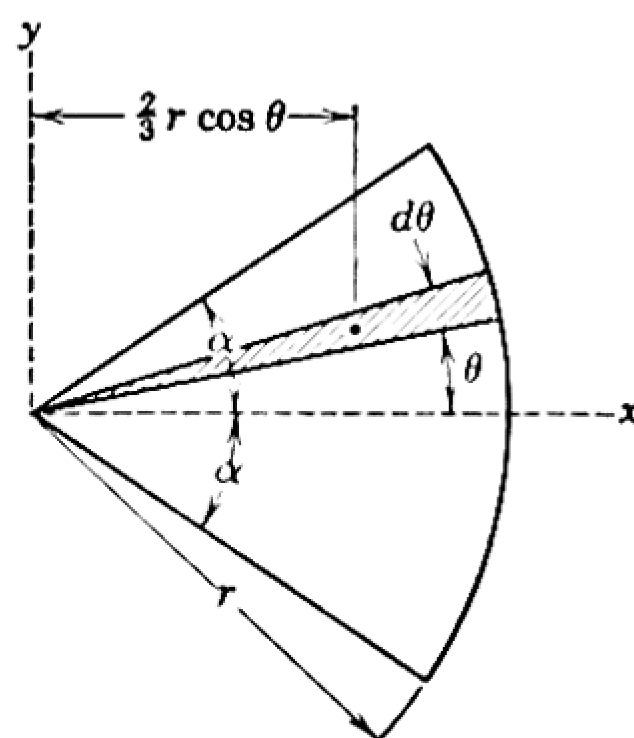
This same result holds with respect to either of the other two sides of the triangle considered as a new base with corresponding new altitude. Thus it can be said that the centroid lies at the intersection of the medians, since the distance of this point from any side is one third the altitude of the triangle with that side considered as base.

It should be noted that the differential strip is a trapezoid, and its area becomes $x\,dy$ only in the limit for which dy approaches zero. In setting up the expression for the elemental area, then, the small triangular end sections are omitted since they involve the higher-order term $dx\,dy$.

329. *Centroid of the Area of a Circular Sector.* Locate the centroid of the area of a circular sector with respect to its vertex.



Solution I



Solution II

PROB. 329

Solution I: The x-axis is chosen as an axis of symmetry, and $\bar{y}$ is therefore automatically zero. The area may be covered by moving an element in the form of a segment of the circular ring, shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0. In the first of Eqs. (19) the coordinate x is the coordinate to the *centroid* of the element dA. From Prob. 327 this distance is $\dfrac{r_0 \sin \alpha}{\alpha}$, where r_0 replaces r. Thus

the first of Eqs. (19) gives

$$\left[A\bar{x} = \int x\, dA\right] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0),$$

$$r^2\alpha\bar{x} = \tfrac{2}{3} r^3 \sin\alpha,$$

$$\bar{x} = \tfrac{2}{3}\frac{r\sin\alpha}{\alpha}. \qquad \textit{Ans.}$$

For a semicircular area $2\alpha = \pi$ and $\bar{x} = 4r/3\pi$.

Solution II: The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\, d\theta)$ where higher-order terms are neglected. Again the x-coordinate of dA is measured to the centroid of the element, and from Prob. 328 this coordinate is seen to be $2r/3$ multiplied by $\cos\theta$. Applying the first of Eqs. (19) gives

$$\left[A\bar{x} = \int x\, dA\right] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\, d\theta),$$

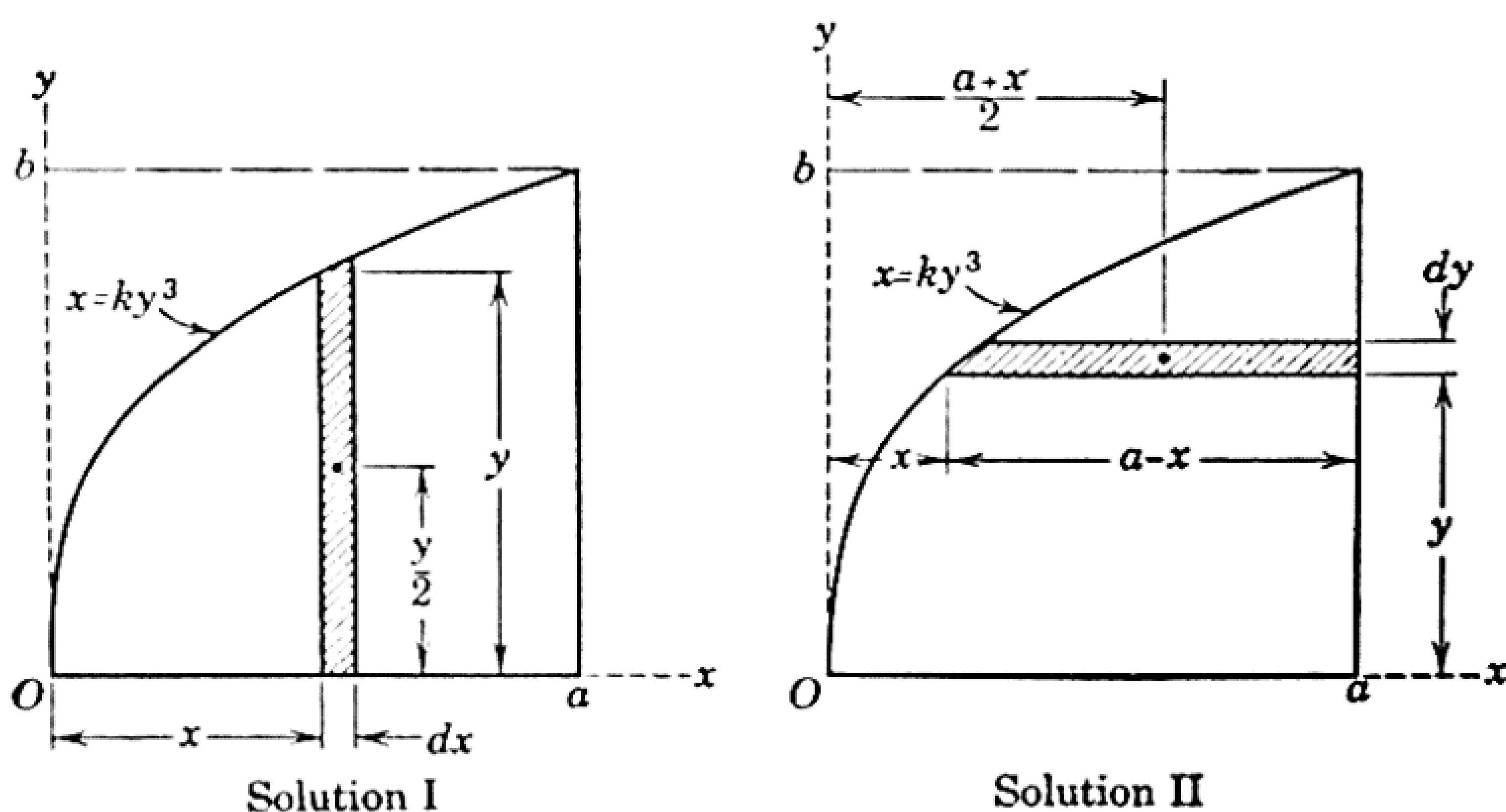
$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3\sin\alpha,$$

and as before

$$\bar{x} = \tfrac{2}{3}\frac{r\sin\alpha}{\alpha}. \qquad \textit{Ans.}$$

It should be noted that, if a second-order element $r\, dr\, d\theta$ is chosen, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand integration with respect to r_0 initially would give the triangular element with which *Solution II* began.

330. Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.



PROB. 330

Solution I: A vertical element of area $dA = y\, dx$ is chosen as shown in the left part of the figure. The x-coordinate of the centroid is found from the first

of Eqs. (19). Thus

$$[A\bar{x} = \int x\,dA] \qquad x\int_0^a y\,dx = \int_0^a xy\,dx.$$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating gives

$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7}, \qquad \bar{x} = \tfrac{4}{7}a. \qquad \textit{Ans.}$$

In solving for $\bar{y}$ by the second of Eqs. (19) the coordinate y is *not* the coordinate to the curve $x = ky^3$ but is the y-distance to the centroid of the element dA. Thus the value $y/2$, which locates the centroid of the rectangular element, must be used. The moment principle becomes

$$[A\bar{y} = \int y\,dA] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx.$$

Substituting $y = (x/k)^{1/3}$ and integrating gives

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10}, \qquad \bar{y} = \tfrac{2}{5}b. \qquad \textit{Ans.}$$

Solution II: The horizontal element of area shown in the right-hand part of the figure may be employed in place of the vertical element. In calculating $\int x\,dA$ the x-coordinate of the centroid of the element must be used. This distance is $x + \frac{a-x}{2} = \frac{a+x}{2}$. Thus

$$[A\bar{x} = \int x\,dA] \qquad \bar{x}\int_0^b (a - x)\,dy = \int_0^b \left(\frac{a+x}{2}\right)(a - x)\,dy.$$

The value of $\bar{y}$ is found from

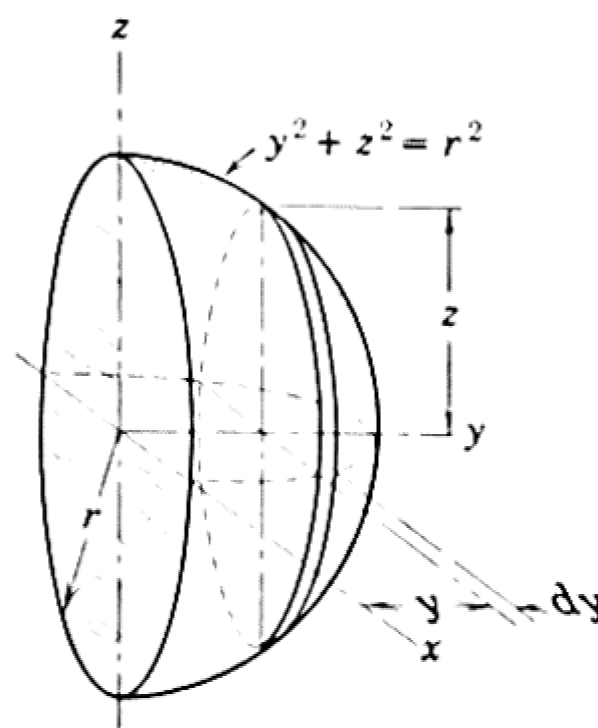
$$[A\bar{y} = \int y\,dA] \qquad \bar{y}\int_0^b (a - x)\,dy = \int_0^b y(a - x)\,dy.$$

The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.

331. *Hemispherical Volume.* Locate the centroid of the volume of a hemisphere of radius r with respect to its base.

Solution: With the axes chosen as shown in the figure $\bar{x} = \bar{z} = 0$ by symmetry. The simplest volume element is a circular slice of thickness dy parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$, the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes

$$dV = \pi(r^2 - y^2)\,dy.$$



PROB. 331

The second of Eqs. (20) requires

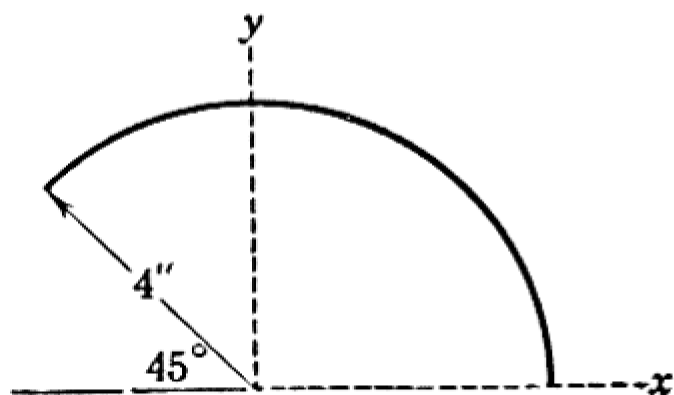
$[V\bar{y} = \int y\,dV]$ $$\bar{y}\int_0^r \pi(r^2 - y^2)\,dy = \int_0^r y\pi(r^2 - y^2)\,dy.$$

Integrating gives

$$\tfrac{2}{3}\pi r^3\bar{y} = \tfrac{1}{4}\pi r^4, \qquad \bar{y} = \tfrac{3}{8}r.$$ *Ans.*

PROBLEMS

332. Locate the center of gravity of the wire bent into the circular shape shown. *Ans.* $\bar{x} = 1.20$ in., $\bar{y} = 2.90$ in.

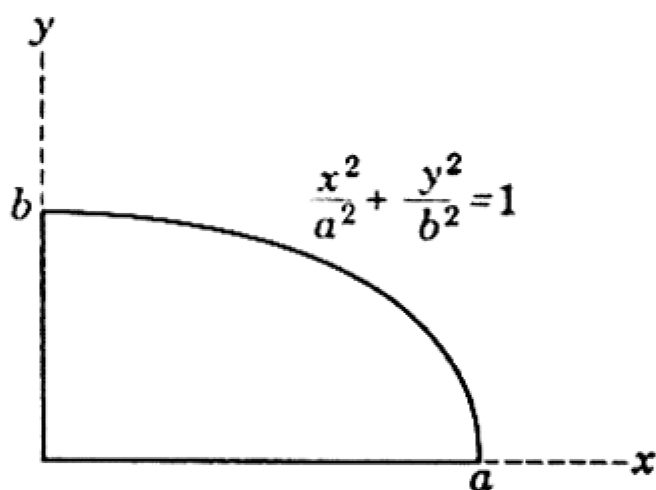


PROB. 332

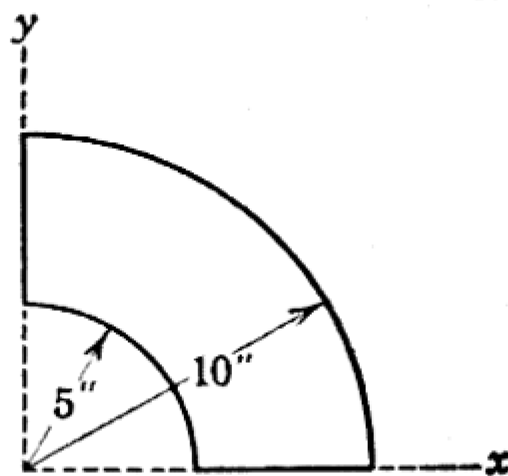
333. Locate the centroid of the area bounded by the parabola $x = y^2$, the x-axis, and the line $x = 1$.

334. Find the distance $\bar{z}$ from the vertex of a right circular cone to the centroid of its volume. *Ans.* $\bar{z} = \frac{3}{4}h$

335. Locate the centroid of the elliptical area shown. *Ans.* $\bar{x} = \dfrac{4a}{3\pi}$, $\bar{y} = \dfrac{4b}{3\pi}$



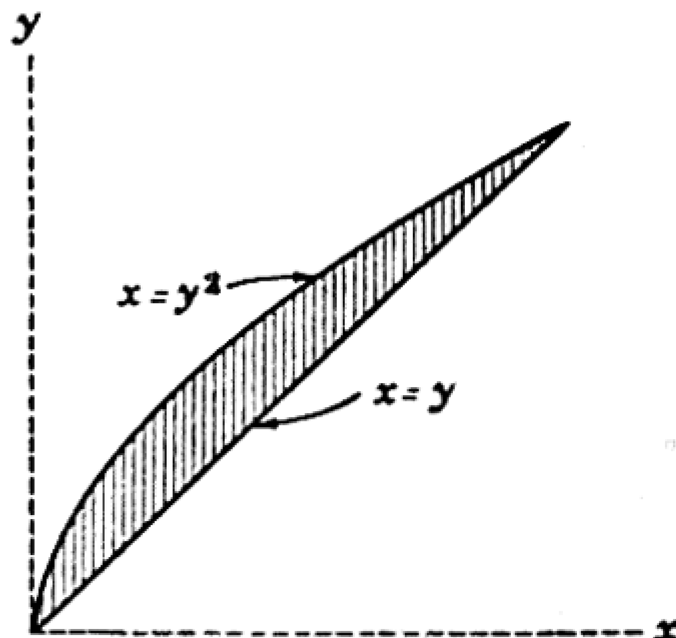
PROB. 335



PROB. 336

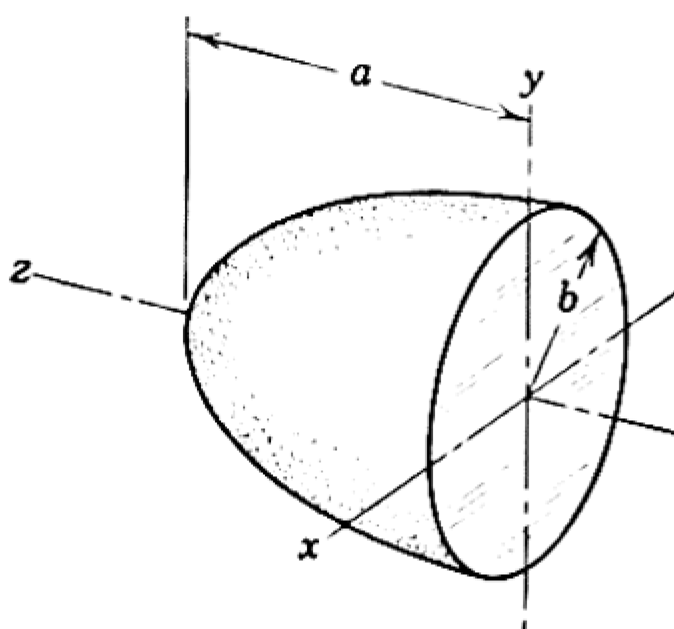
336. A piece of sheet metal is cut into the shape shown. Locate the center of gravity.

337. Determine $\bar{x}$ for the shaded area shown.



PROB. 337

338. A parabola is revolved about the z-axis to obtain the paraboloid shown. Locate the centroid of its volume with respect to the base $z = 0$.



PROB. 338

339. The first quadrant of the ellipse $z^2/a^2 + y^2/b^2 = 1$ is revolved about the z-axis to obtain a solid ellipsoid similar in appearance to the paraboloid of Prob. 338. Locate the centroid.

340. Determine the distance $\bar{h}$ from the centroid of the lateral area of a cone of altitude h to the base of the cone.

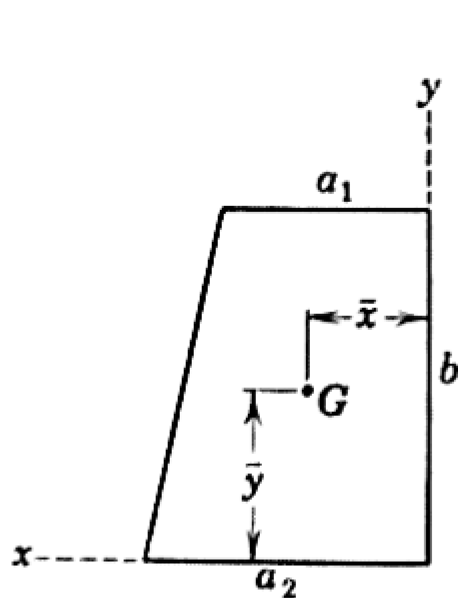
Ans. $\bar{h} = \dfrac{h}{3}$

341. Determine the distance $\bar{z}$ from the base of any pyramid of altitude h to the centroid of its volume.

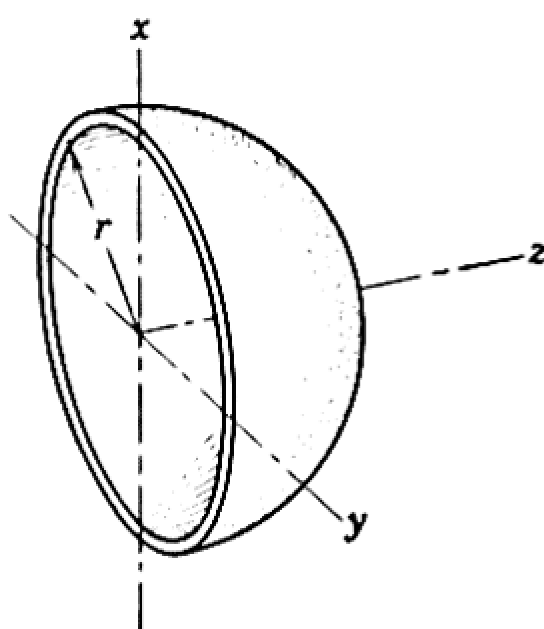
Ans. $\bar{z} = \dfrac{h}{4}$

342. Locate the centroid of the area under the curve $y = a \sin \dfrac{\pi x}{l}$ between $x = 0$ and $x = l$.

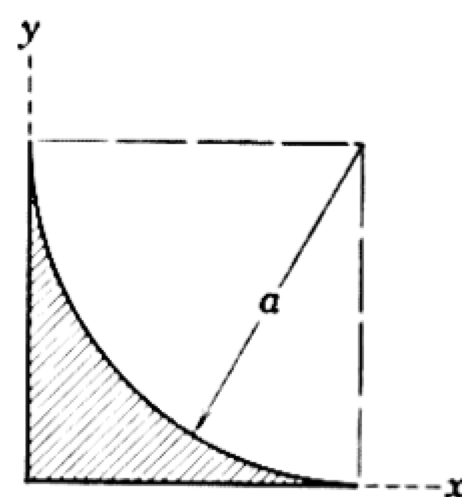
343. Find $\bar{x}$ and $\bar{y}$ for the trapezoidal area shown.



PROB. 343



PROB. 344



PROB. 345

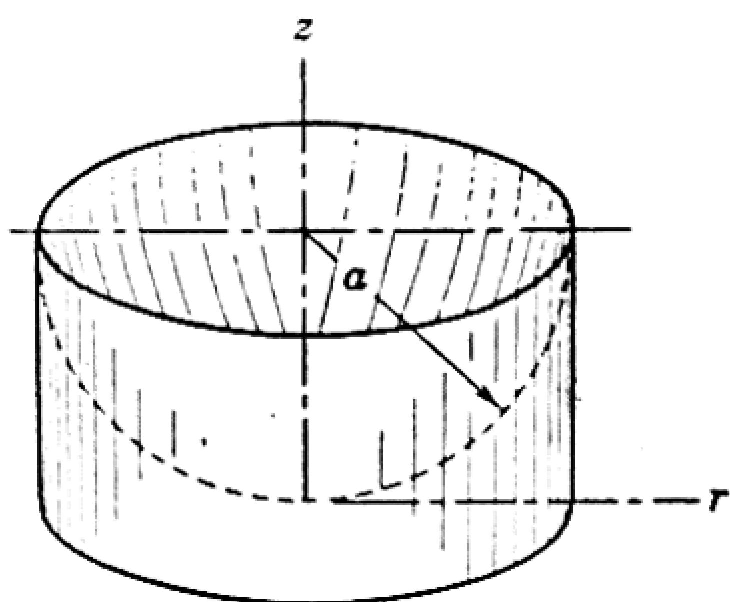
344. Locate the center of gravity of the homogeneous hemispherical shell of radius r and negligible wall thickness.

Ans. $\bar{z} = \dfrac{r}{2}$

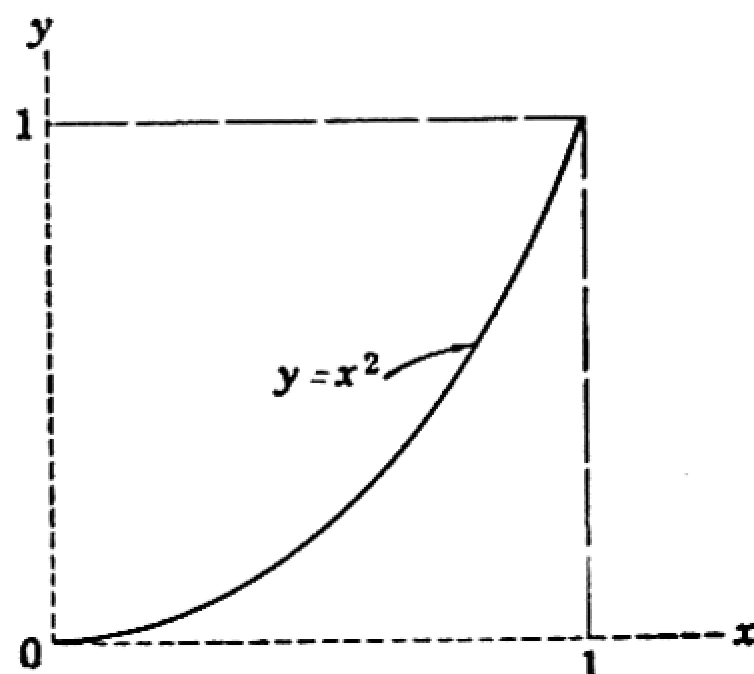
345. Locate the centroid of the area shown in the figure by direct integration.

Ans. $\bar{x} = \dfrac{10 - 3\pi}{4 - \pi} \dfrac{a}{3}$

346. Locate by direct integration the center of gravity of the homogeneous bowl made by cutting out a hemispherical cavity of radius a from a right circular cylinder of the same radius and height. *Ans.* $\bar{z} = \frac{a}{4}$



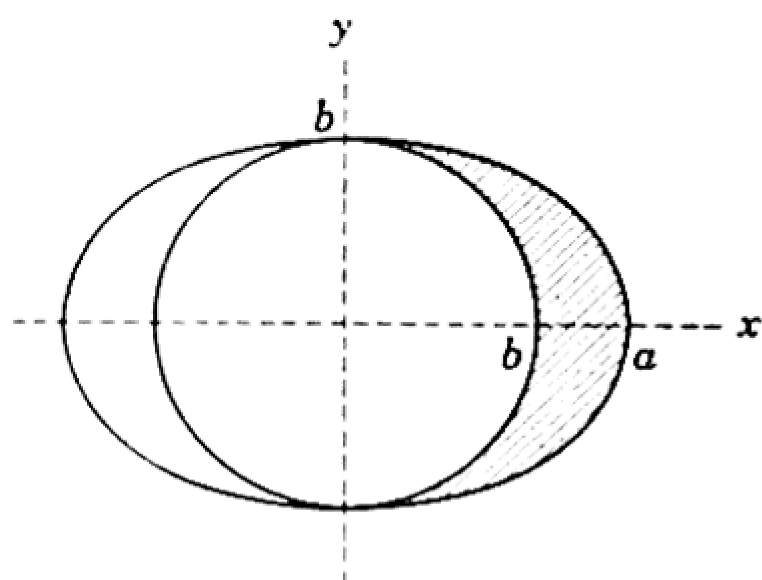
Prob. 346



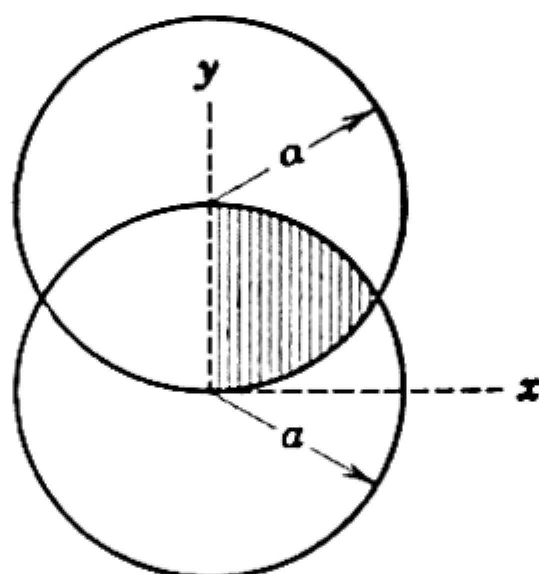
Prob. 347

347. Locate the centroid of the parabolic arc extending from $x = 0$ to $x = 1$.

348. Locate the centroid of the shaded area between the ellipse and the circle. *Ans.* $\bar{x} = \frac{4(a + b)}{3\pi}$



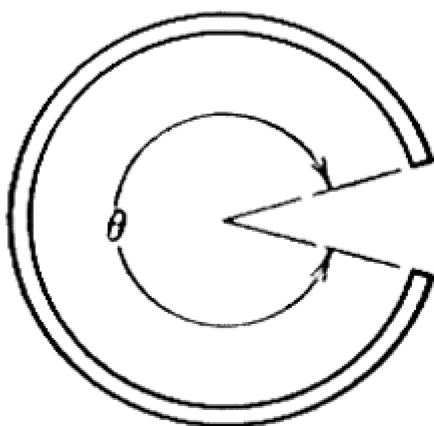
Prob. 348



Prob. 349

349. Locate the centroid of the shaded area shown.

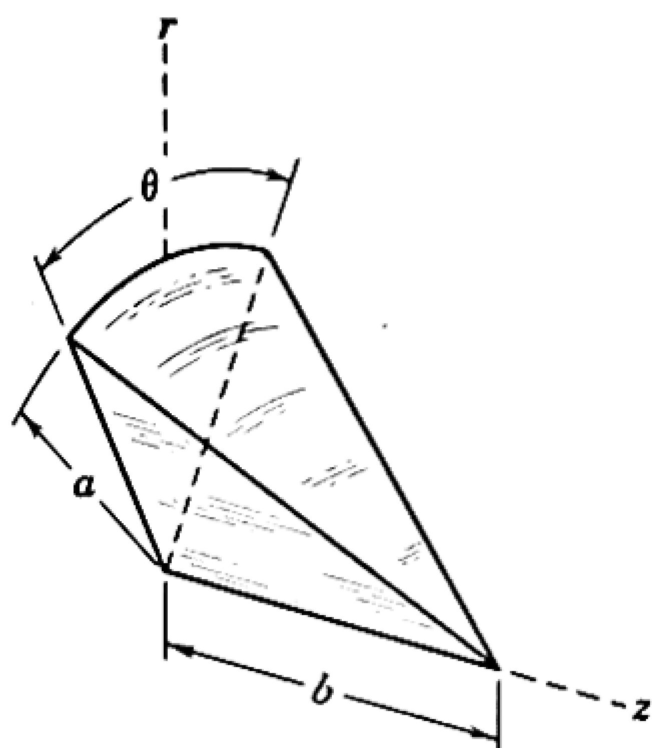
350. A slender rod of small cross section is 10 in. long when initially straight. It is to be bent into a circular arc with a gap between the ends as shown. Determine the angle θ subtended such that the center of gravity is ¼ in. from the center of the arc. *Ans.* $\frac{\sin(\theta/2)}{\theta^2} = \frac{1}{80}$, $\theta = 315.5°$ (graphical solution)



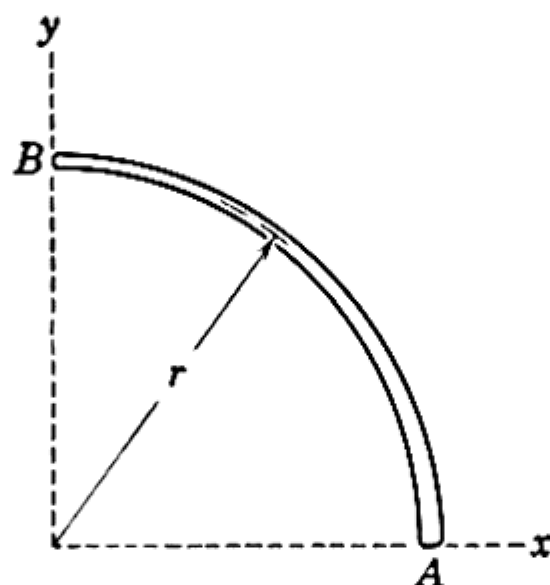
Prob. 350

351. Locate the centroid of the volume generated by revolving the right triangle of altitude a about its base b through an angle θ.

Ans. $\bar{r} = \dfrac{a \sin(\theta/2)}{\theta}, \; \bar{z} = \dfrac{b}{4}$



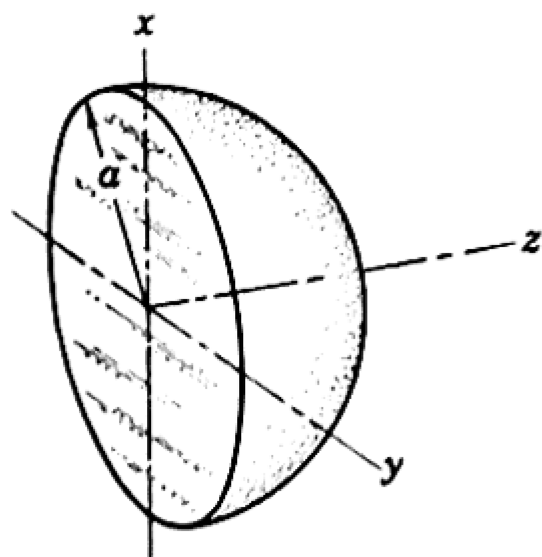
Prob. 351



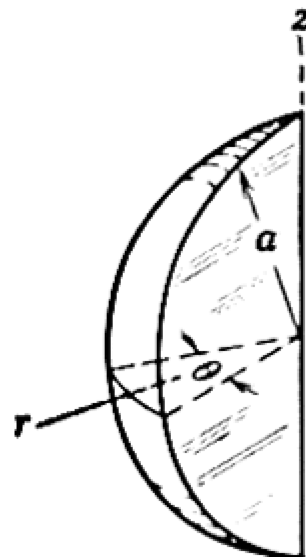
Prob. 352

* **352.** A rod, whose cross-sectional area varies linearly along its length, is bent into the arc of a quarter of a circle as indicated. If the diameter of the rod at A is twice that at B, determine the position of the center of gravity. The diameter is negligible compared with r. *Ans.* $\bar{x} = 0.741r$, $\bar{y} = 0.532r$

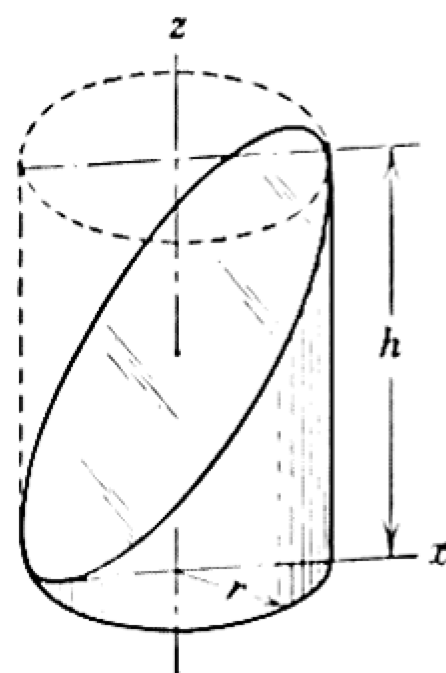
* **353.** The density of a sphere decreases uniformly with radial distance from μ_0 at the center to one half this value at the surface $r = a$. If the sphere is cut on a diametral plane, determine the distance $\bar{z}$ from this base plane to the center of gravity of either hemisphere. (*Hint:* Use the results of Prob. 344 in selecting the element of volume.)



Prob. 353



Prob. 354



Prob. 355

* **354.** Determine the distance $\bar{r}$ to the center of gravity of the solid spherical wedge shown.

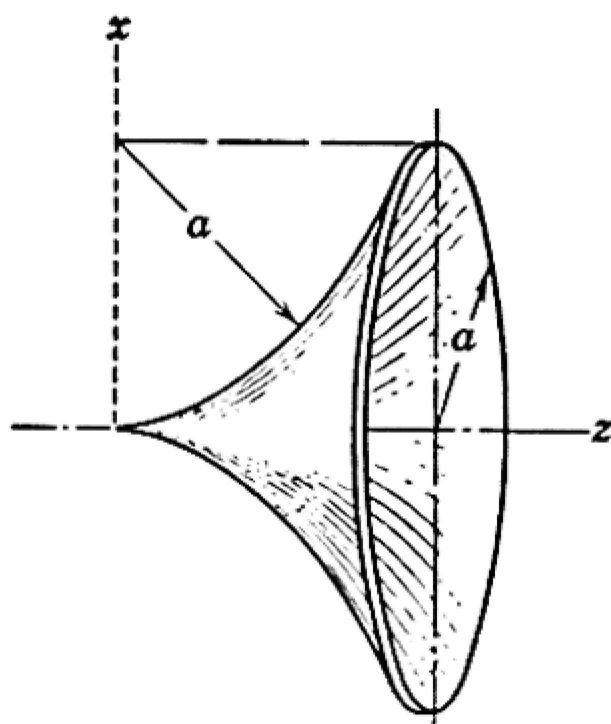
Ans. $\bar{r} = \dfrac{3}{8} \dfrac{\pi a \sin(\theta/2)}{\theta}$

* **355.** A cylindrical body of radius r and height h is divided in half by a diagonal slice as shown. Locate the center of gravity of the part indicated.

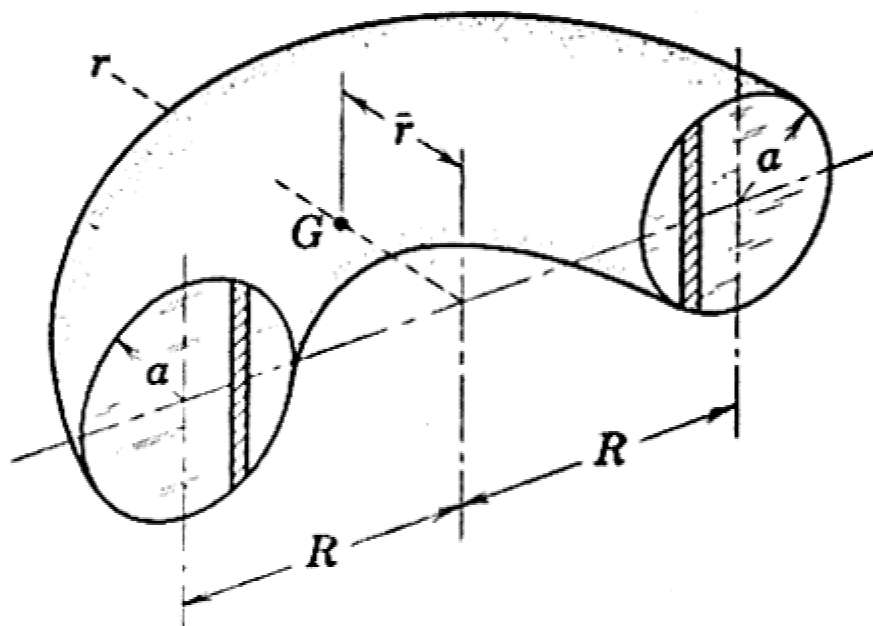
Ans. $\bar{x} = \frac{1}{4}r$, $\bar{z} = \frac{5}{16}h$

* **356.** Locate the center of gravity of the bell-shaped shell of uniform but negligible thickness.

Ans. $\bar{z} = \dfrac{a}{\pi - 2}$



PROB. 356



PROB. 357

* **357.** Locate the center of gravity G of the steel half-ring. (*Hint:* Choose an element of volume in the form of a cylindrical shell whose intersection with the plane of the ends is shown in section.)

Ans. $\bar{r} = \dfrac{a^2 + 4R^2}{2\pi R}$

36. Composite Bodies and Figures. When a body or figure can be conveniently divided into several parts of simple shape, the principle of Varignon may be used if each part is treated as a finite element of the whole. Thus for a body whose several parts weigh W_1, W_2, W_3, $\cdots$, and whose separate coordinates of the centers of gravity of these parts in, say, the x-direction are $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\cdots$, the moment principle gives

$$(W_1 + W_2 + W_3 + \cdots)\bar{X} = W_1\bar{x}_1 + W_2\bar{x}_2 + W_3\bar{x}_3 + \cdots,$$

where $\bar{X}$ is the x-coordinate of the center of gravity of the whole. Similar relations hold for the other two coordinate directions. These sums may be expressed in condensed form and written as

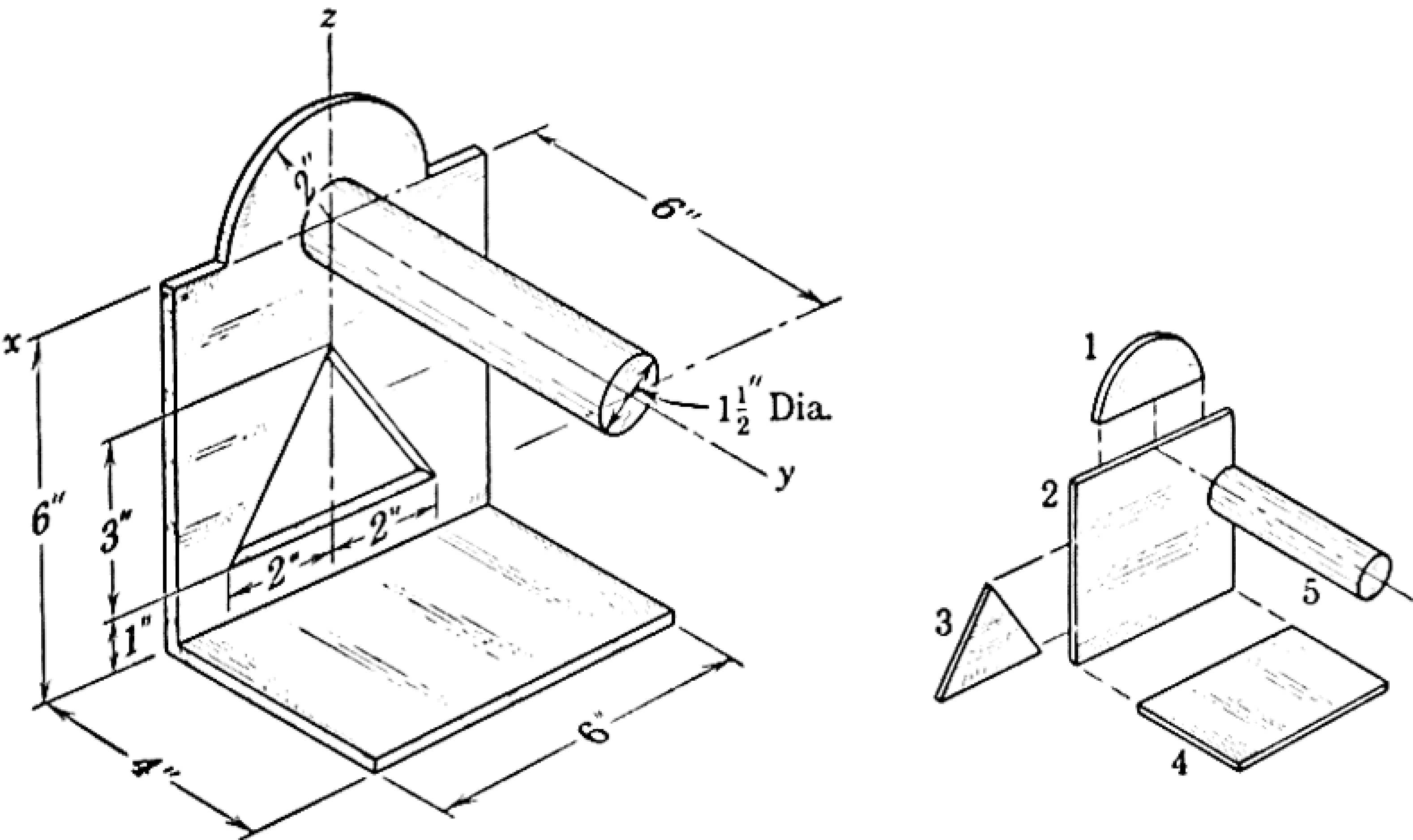
$$\bar{X} = \frac{\Sigma W\bar{x}}{\Sigma W},\qquad \bar{Y} = \frac{\Sigma W\bar{y}}{\Sigma W},\qquad \bar{Z} = \frac{\Sigma W\bar{z}}{\Sigma W}. \tag{21}$$

Analogous relations hold for composite lines, areas, and volumes, where the W's are replaced by L's, A's, and V's, respectively. It should be pointed out that, if a hole or cavity is considered as one of the component

parts of a composite body or figure, the corresponding weight or area represented by the cavity or hole is considered a negative quantity.

SAMPLE PROBLEM

358. Locate the center of gravity of the bracket and shaft combination. The vertical face is made from sheet metal which weighs 5 lb./ft.^{2}, the material of the horizontal base weighs 8 lb./ft.^{2}, and the steel shaft has a weight density of 0.283 lb./in.^{3}



PROB. 358

Solution: The composite body may be considered to be composed of the five elements shown in the right-hand part of the illustration. The triangular part will be taken as a negative area. For the reference axes indicated it is clear by symmetry that the x-component of the center of gravity is zero.

The weight W of each part and the coordinates $\bar{y}$ and $\bar{z}$ of their respective centers of gravity are determined. The terms involved in applying Eqs. (21) are best handled in the form of a table as follows:

Part	W, lb.	$\bar{y}$, in.	$\bar{z}$, in.	$W\bar{y}$, lb. in.	$W\bar{z}$, lb. in.
1	0.218	0	0.849	0	0.185
2	1.25	0	−3.00	0	−3.75
3	−0.208	0	−4.00	0	0.832
4	1.33	2.00	−6.00	2.66	−7.98
5	3.00	3.00	0	9.00	0
Totals	5.59			11.66	−10.71

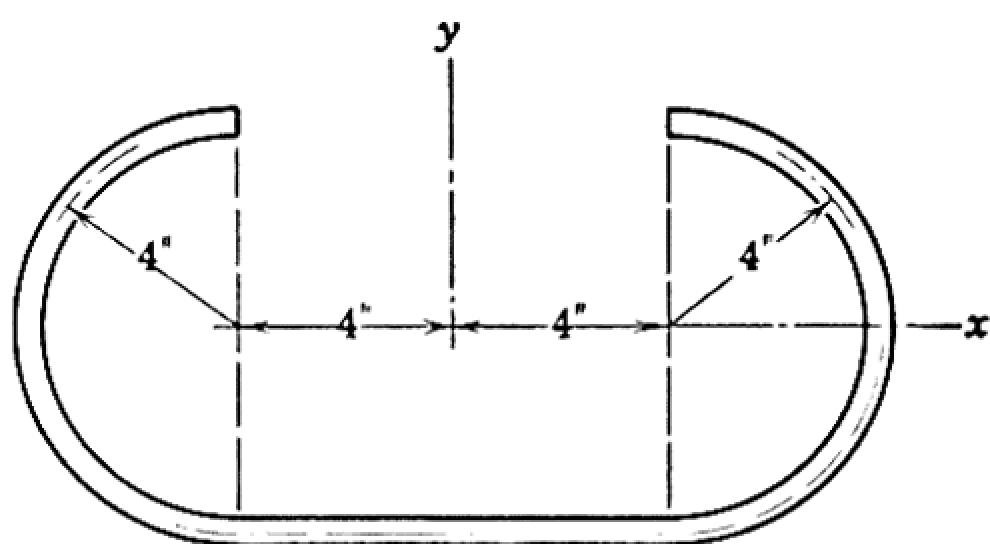
Equations (21) are now applied and the results are

$$\left[\bar{Y} = \frac{\Sigma W\bar{y}}{\Sigma W}\right] \qquad \bar{Y} = \frac{11.66}{5.59} = 2.08 \text{ in.},$$

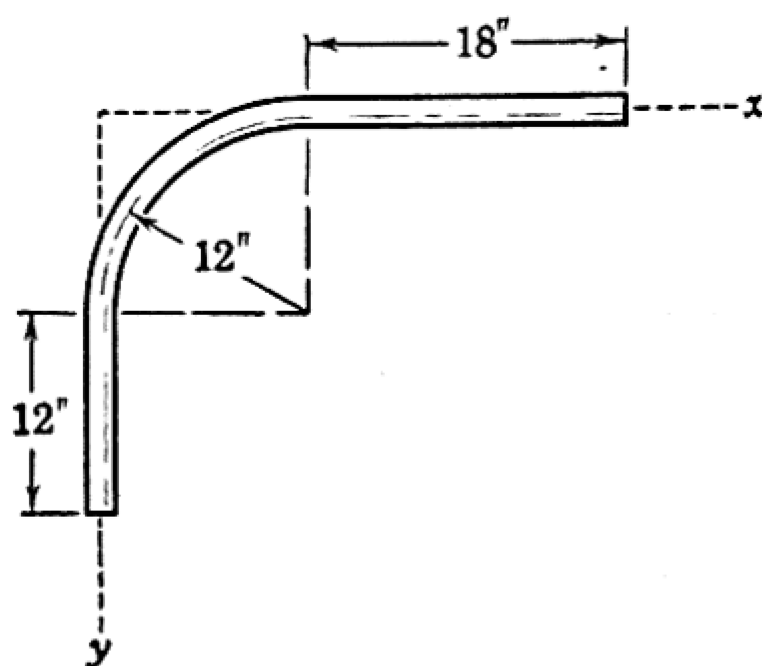
$$\left[\bar{Z} = \frac{\Sigma W\bar{z}}{\Sigma W}\right] \qquad \bar{Z} = \frac{-10.71}{5.59} = -1.92 \text{ in.}$$

PROBLEMS

359. Locate the center of gravity of the slender rod bent into the shape shown.
Ans. $\bar{Y} = -0.966$ in.



Prob. 359



Prob. 360

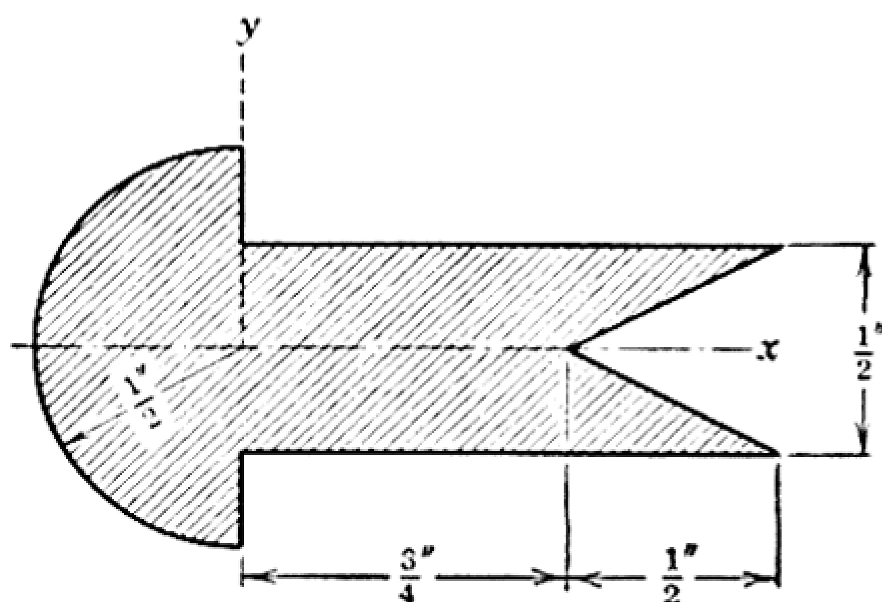
360. Locate the center of gravity of the rod shown. The diameter of the rod is small compared with the other dimensions.

361. Solve Prob. 348 by the method of component parts.

362. Solve Prob. 345 by the method of component parts.

363. Solve Prob. 346 by the method of component parts.

364. Locate the centroid of the shaded area.

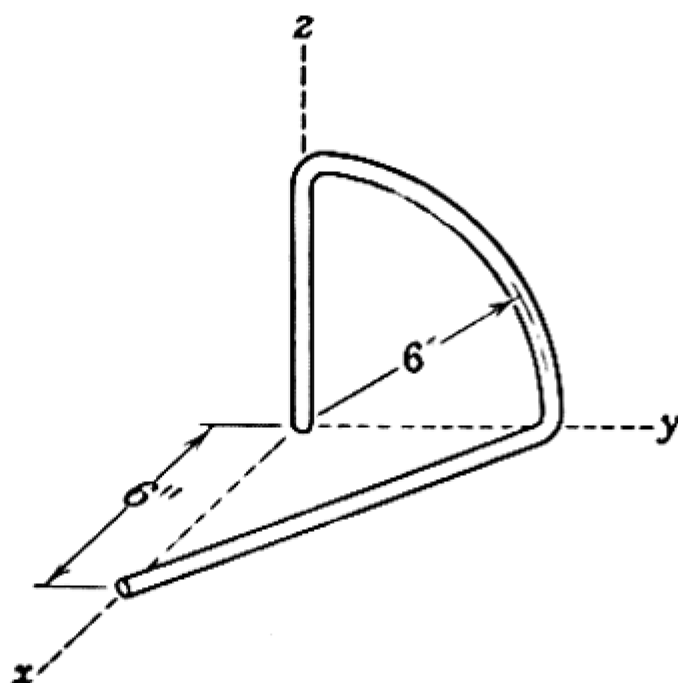


Prob. 364

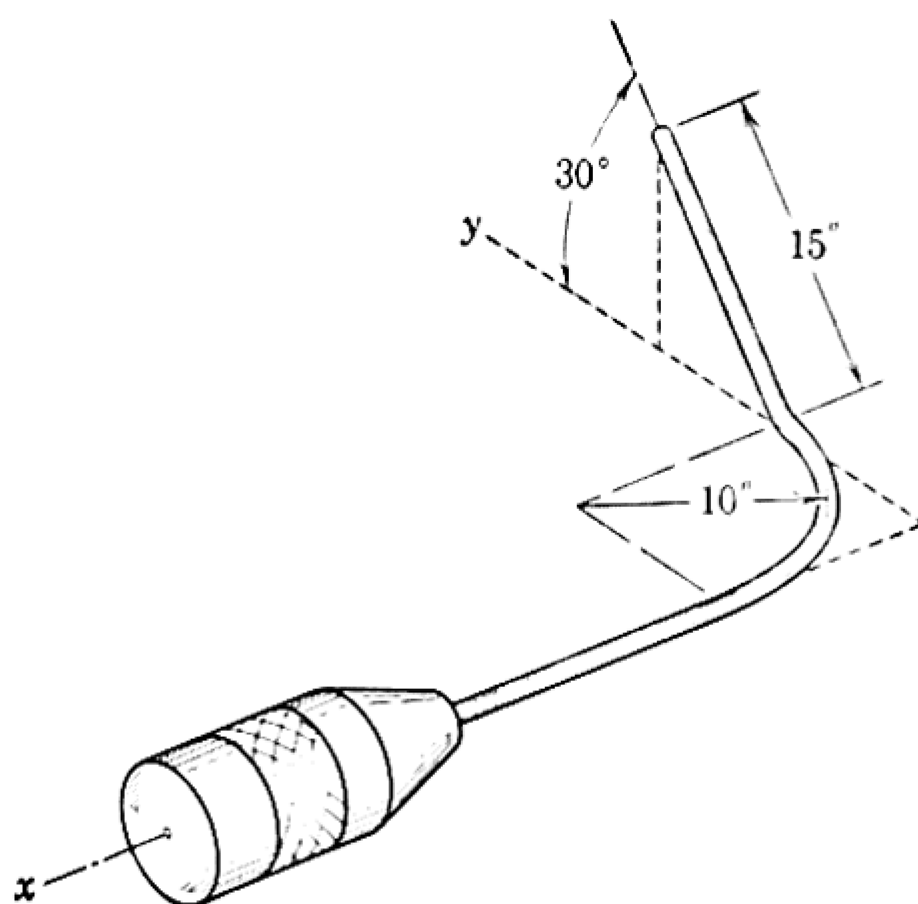
365. Locate the centroid of the volume obtained by revolving the shaded area in Prob. 364 about the x-axis.

366. Locate the center of gravity of the bent wire shown.

Ans. $\bar{X} = 1.065$ in., $\bar{Y} = 2.57$ in., $\bar{Z} = 2.26$ in.



PROB. 366

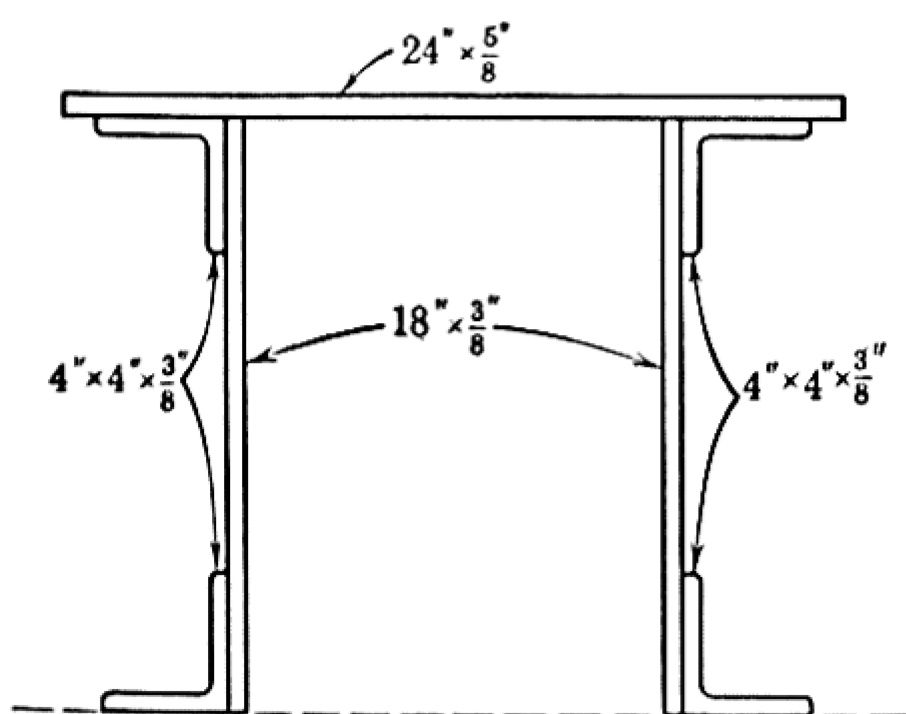


PROB. 367

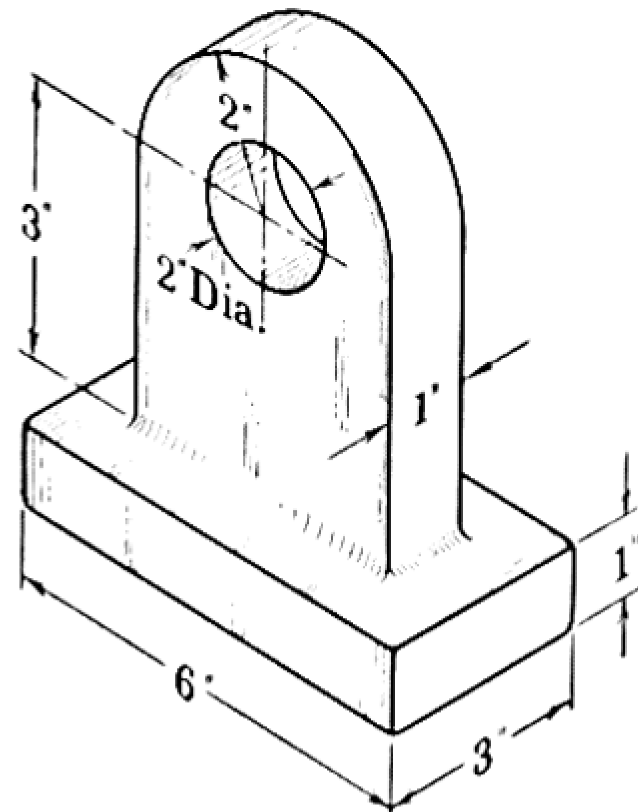
367. The bent rod shown weighs 2 lb. per foot of length. Find the moment about the x-axis which the chuck exerts on the rod to hold it in place.

Ans. $M_x = 50.8$ lb. in.

368. Locate the distance $\bar{H}$ from the top of the cover plate to the centroid of the built-up structural section shown.



PROB. 368

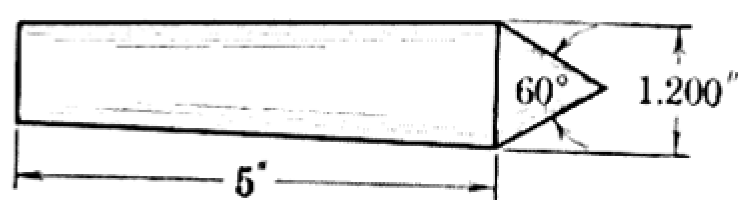


PROB. 369

369. Determine the distance $\bar{H}$ from the bottom of the base to the center of gravity of the bracket casting.

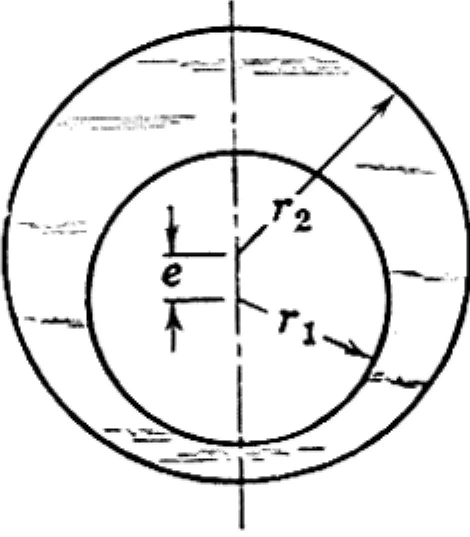
370. Locate the distance $\bar{Z}$ from the point of the lathe center to the center of gravity. The shank has a taper of 0.05193 in. (change in diameter) per inch of axial length.

Ans. $\bar{Z} = 3.14$ in.

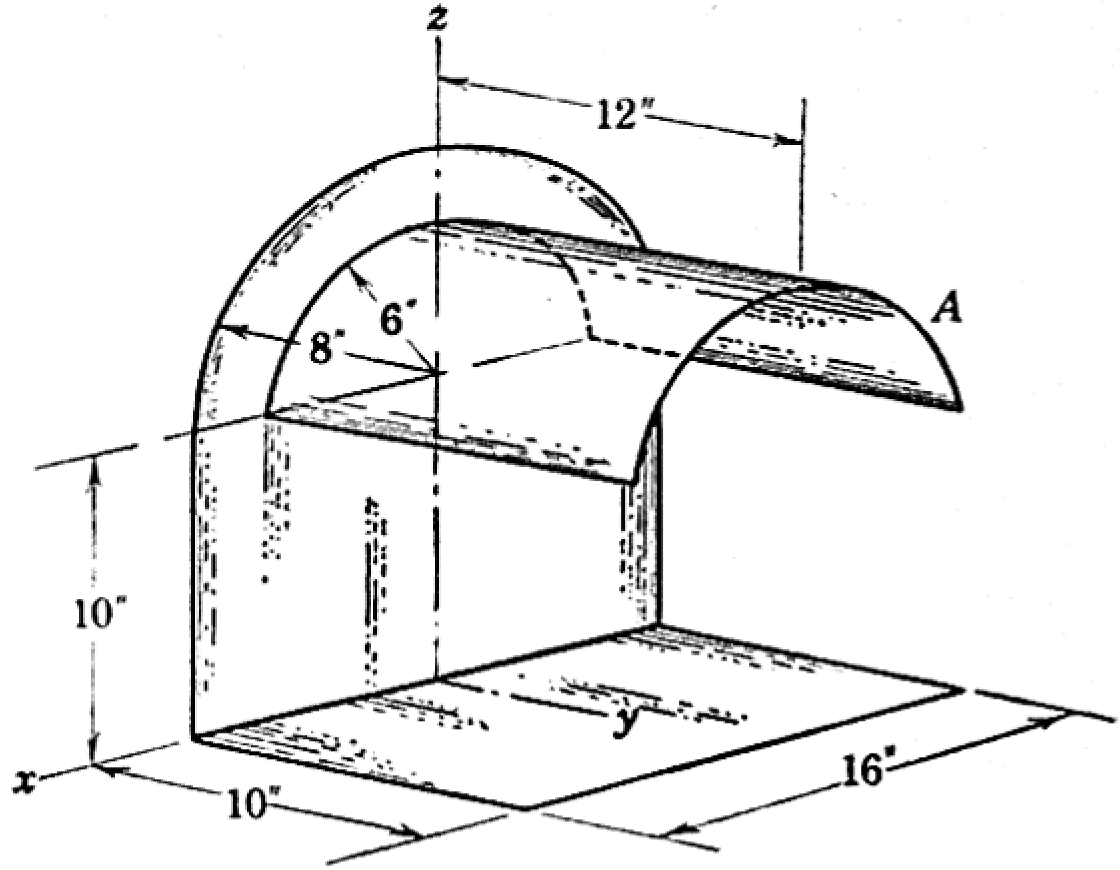


PROB. 370

371. Find the radius r_1 which will locate the centroid of the net area a distance equal to e above the center of the outer circle.



PROB. 371

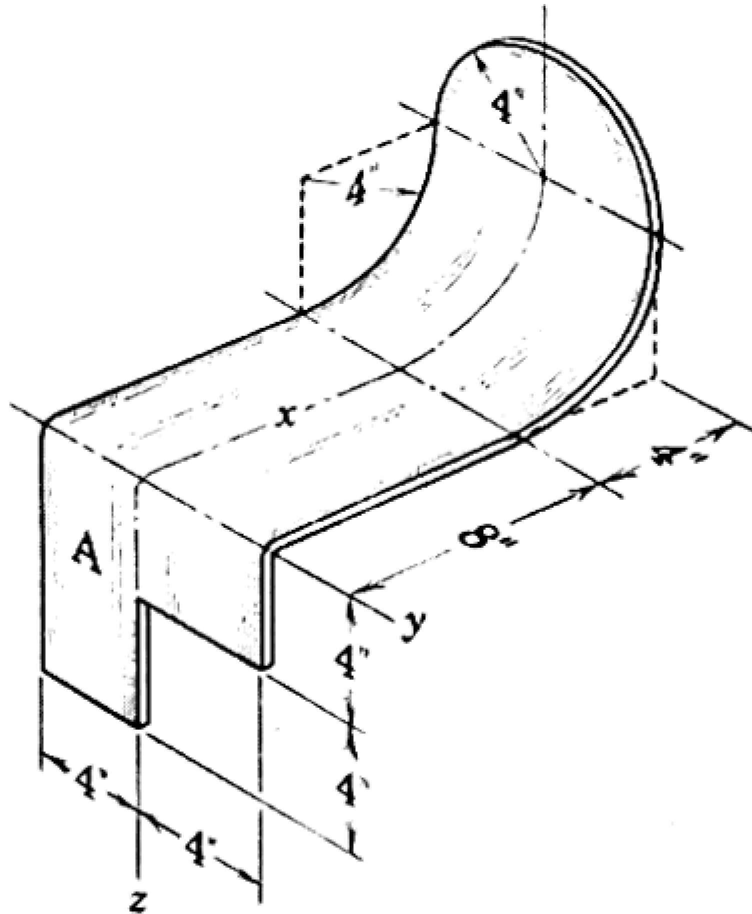


PROB. 372

372. Locate the center of gravity of the sheet-metal part shown. The material of part A weighs 3 lb./ft.^{2}, and the remainder weighs 2 lb./ft.^{2}

373. Locate the center of gravity of the sheet-metal form shown. The end A is made from material weighing 5 lb./ft.^{2}, and the remainder is made from material weighing 3 lb./ft.^{2}

Ans. $\bar{X} = 4.96$ in., $\bar{Y} = -0.243$ in., $\bar{Z} = 0.229$ in.



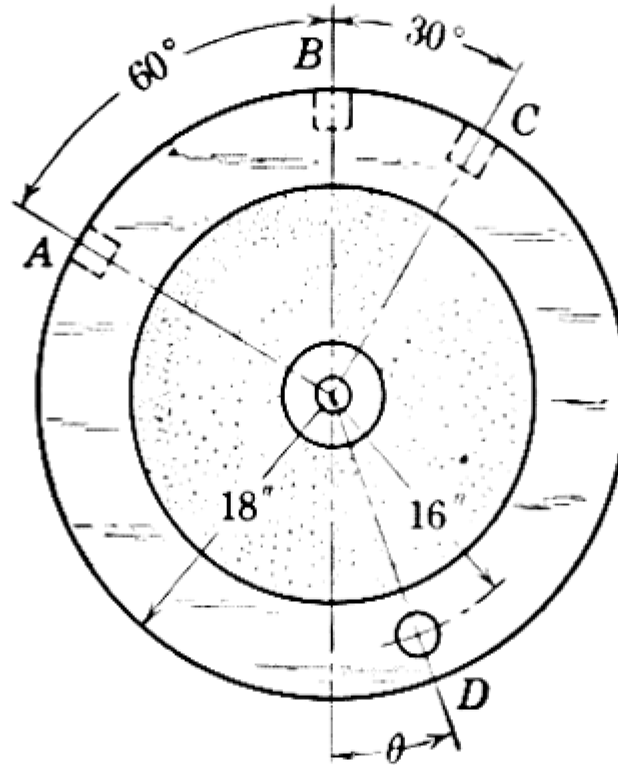
PROB. 373

374. A cylindrical tank with vertical axis and open top has a radius of 2 ft. and a height of 6 ft. The tank is filled with water and the axis is tilted 30 deg. from the vertical. Find the distance $\bar{h}$ from the level of the water to the center of gravity of the remaining water. (*Hint:* Use the results of Prob. 355.)

Ans. $\bar{h} = 2.13$ ft.

375. A flywheel with three radial holes A, B, and C in its rim is to be balanced by drilling one hole D completely through the rim. If the original three holes are ½ in. in diameter and 1 in. deep and the rim is 2 in. wide, determine the necessary diameter d of the balancing hole and the angle θ.

Ans. $d = 0.572$ in., $\theta = 8° 47'$



PROB. 375

37. Graphical Approximations. Frequently in practice the boundaries of an area or volume are not expressible in terms of the simple geometrical shapes or in shapes which can be represented mathematically. For such cases it is necessary to resort to a method of approximations.

Consider the problem of locating the centroid of the irregular area shown in Fig. 48. The area may be divided into strips of width Δx and variable height h. The area A of each strip, such as the one crosshatched, is multiplied by the coordinates x and y to its *centroid* to obtain the moments of the element of area. The sum of the moments for all strips divided by the total area of the strips will give the corresponding centroidal component. A tabulation of the results, similar to that used in Sample Prob. 358, will permit an orderly evaluation of the total area ΣA, the sums ΣAx and ΣAy, and the results

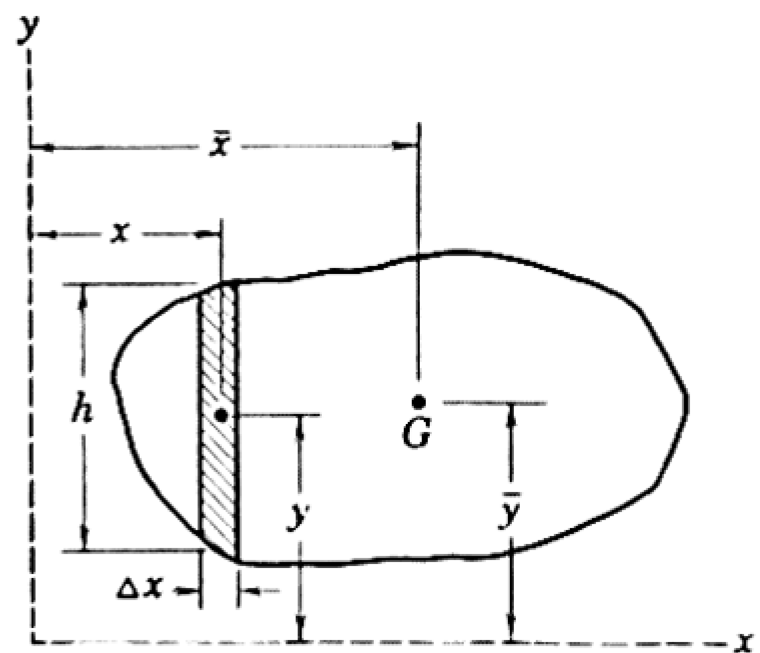


FIG. 48

$$\bar{x} = \frac{\Sigma Ax}{\Sigma A}, \qquad \bar{y} = \frac{\Sigma Ay}{\Sigma A}.$$

The accuracy of the approximation will be increased by decreasing the widths of the strips used. In all cases the average height of the strip

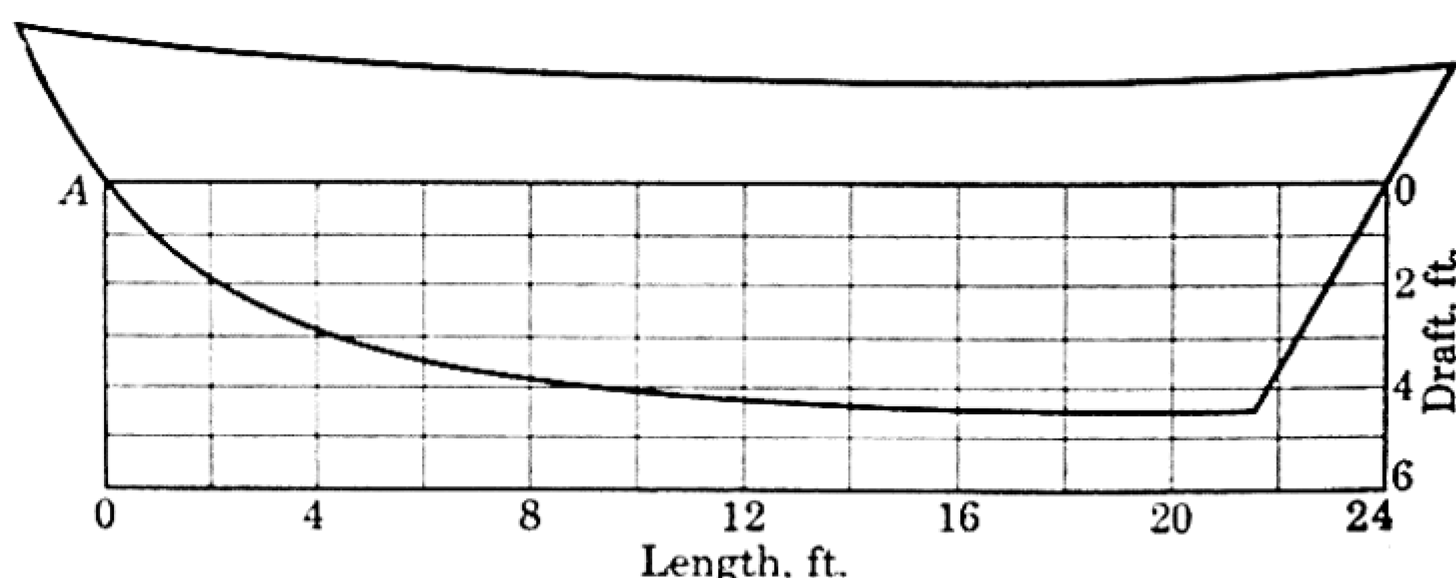
should be estimated in approximating the areas. Although it is usually of advantage to use elements of constant width, it is not necessary to do so. In fact elements of any size and shape which approximate the given area to satisfactory accuracy may be used.

In locating the centroid of an irregular volume the problem may be reduced to one of determining the centroid of an area. It is necessary only to plot a curve representing the magnitude of the areas of cross sections of the volume normal to a desired axis. The position along the axis of the centroid of the figure defined by this curve of areas will be the corresponding position of the volume centroid.

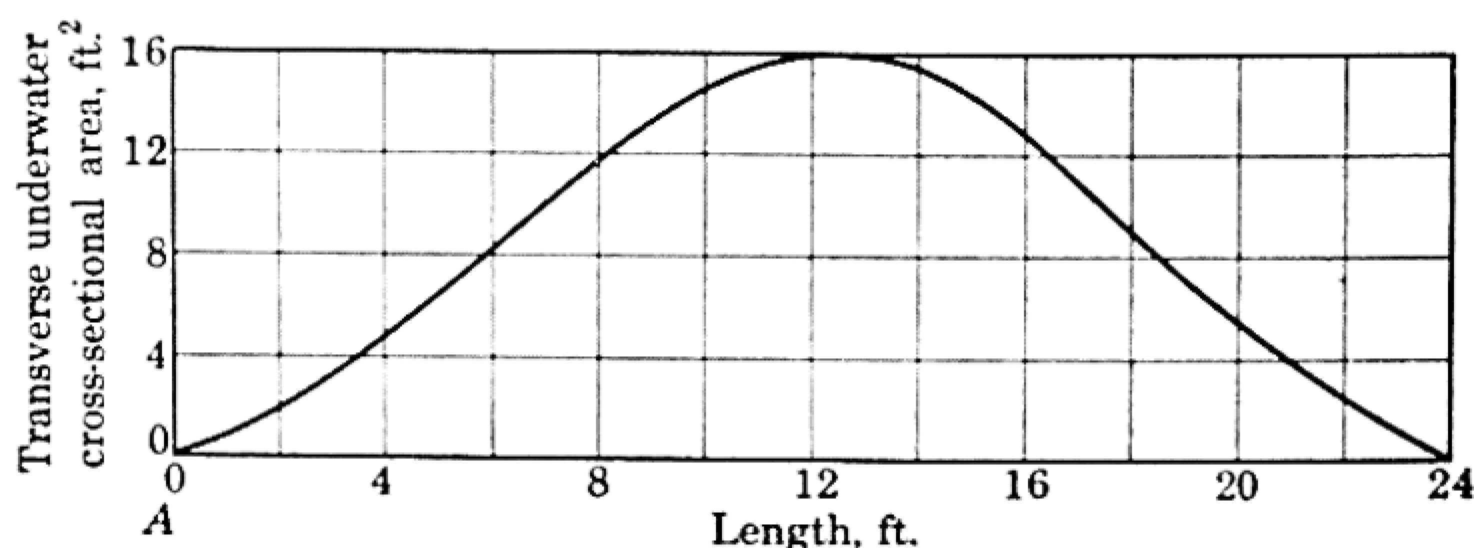
PROBLEMS

376. The cross-sectional area of a concrete culvert is determined every 5 ft. along its length. These measurements in square feet are 4.1, 5.7, 7.3, 8.2, 7.5, 5.7, 4.7, where the first and last numbers represent the cross-sectional areas of the two ends. Approximate the distance from the first end (area 4.1 ft.^{2}) to the center of gravity of the culvert.

377. The center of lateral resistance for a ship's hull is the centroid of the projection of the underwater portion of the hull on the vertical central plane. Find the distance $\bar{x}$ of this point aft of point A for the scale drawing of the sailboat hull shown.



PROB. 377



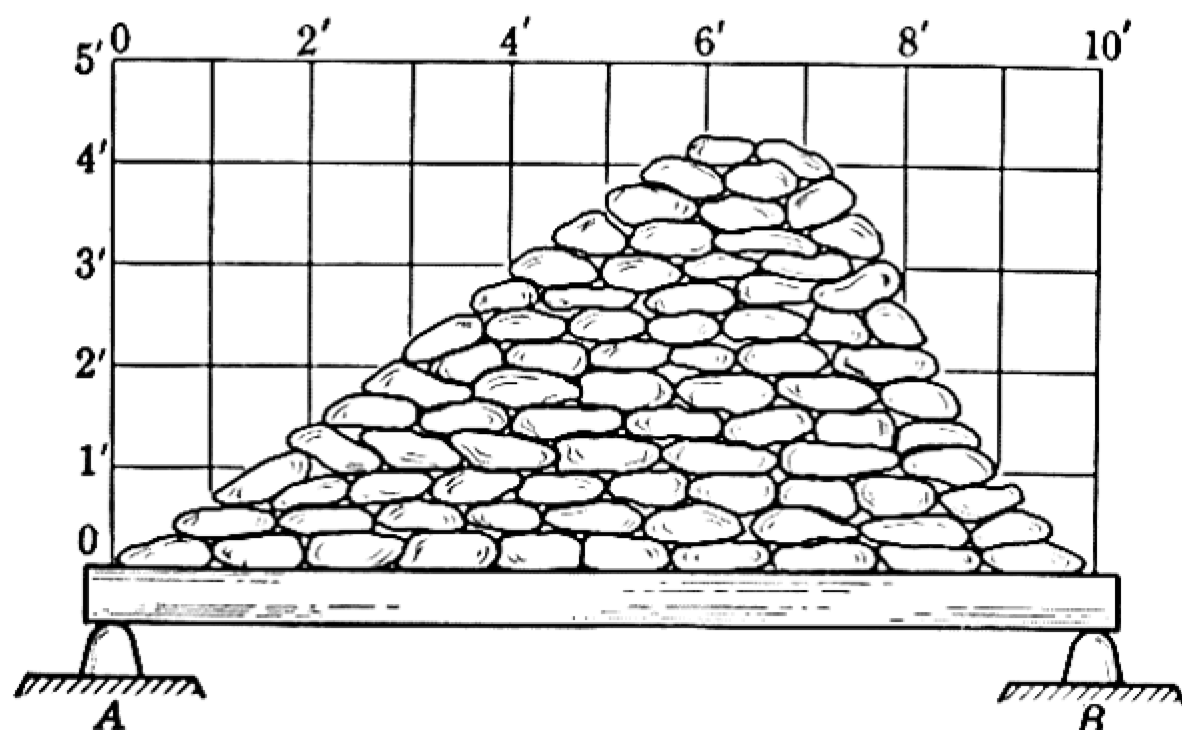
PROB. 378

378. The center of buoyancy of a ship's hull is the centroid of the displaced volume. Find the distance aft of point A to the center of buoyancy for the sail-

boat hull of Prob. 377 if the longitudinal distribution of the transverse underwater cross-sectional areas (curve of areas) is as plotted here.

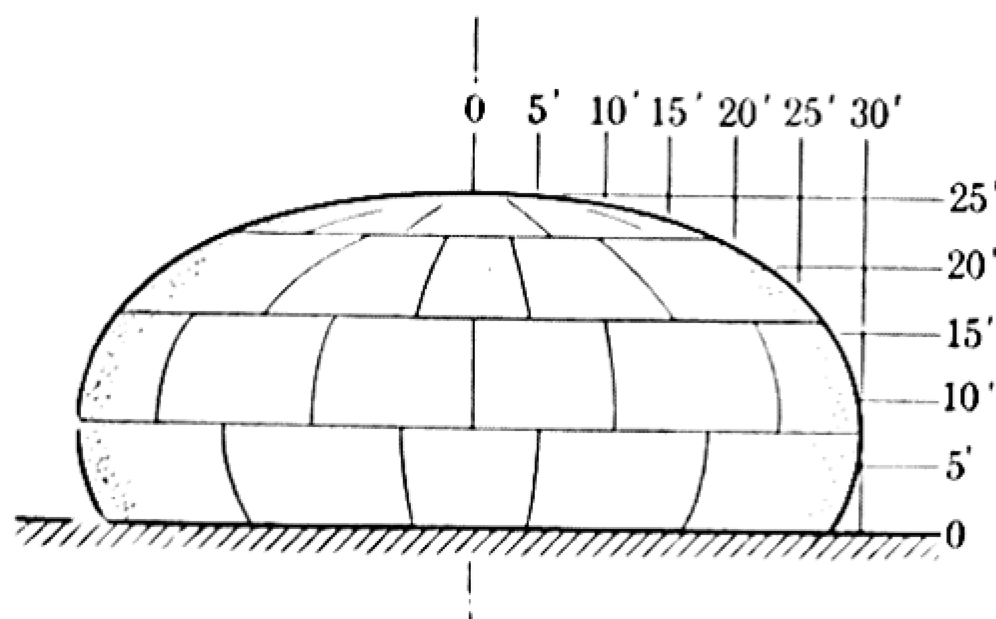
379. Locate the horizontal position of the centroid of the area shown in Prob. 378 by graphical means in place of a tabular computation. (*Hint:* Consider the area as a sheet-metal figure and represent the weight of each strip by a vertical vector through the centroid of each strip. The centroid of the area will lie on the resultant of the parallel force vectors which is obtained by a funicular polygon.)

380. A beam supports 82 sandbags weighing 100 lb. each and stacked as shown. Approximate the reactions at A and B due to the weight of the bags.



Prob. 380

381. The storage tank in the form of a surface of revolution is to be sprayed with two coats of paint. The surface area of the tank is unknown, but a scale drawing of the tank, reproduced here, is available. Determine the number of whole gallons of paint which must be purchased if each gallon will cover 500 ft.^{2} with one coat. *Ans.* 24 gal.



Prob. 381

38. Theorems of Pappus.* A very simple method exists for calculating the surface area generated by revolving a plane curve about a non-

* Attributed to Pappus of Alexandria, a Greek geometer, who lived in the third century A.D. The theorems often bear the name of Guldinus (Paul Guldin, 1577–1643), who claimed original authorship, although the works of Pappus were apparently known to him.

intersecting axis in the plane of the curve. In Fig. 49 the line segment in the x–y plane of length L generates a surface when revolved about the x-axis. An element of this surface is the ring generated by dL. The area of this ring is

$$dA = 2\pi y\,dL,$$

and the total area is then

$$A = 2\pi \int y\,dL.$$

But since $\bar{y}L = \int y\,dL$ the area becomes

$$A = 2\pi\bar{y}L, \tag{22}$$

where $\bar{y}$ is the y-coordinate of the centroid for the line of length L. Thus

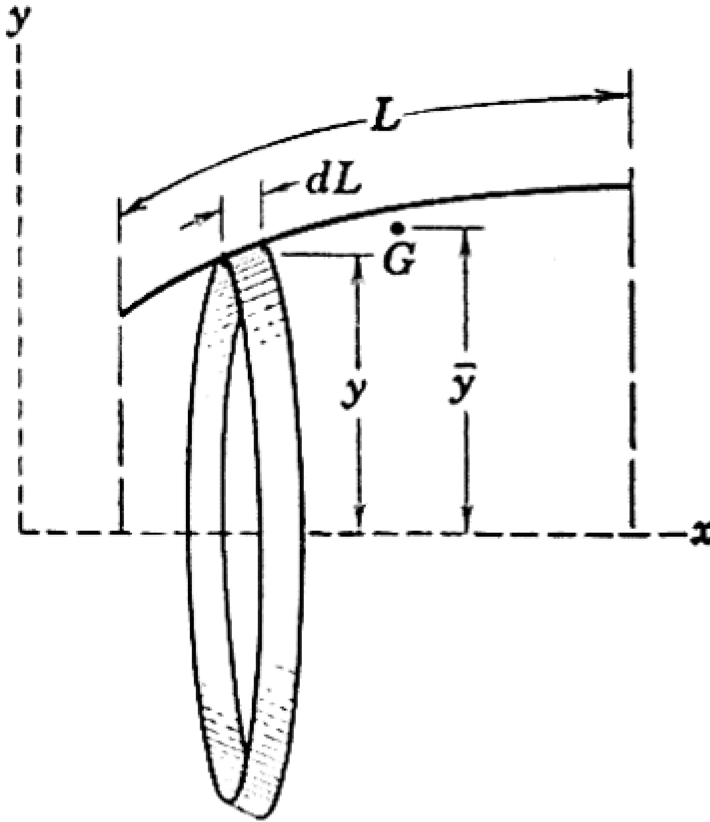


FIG. 49

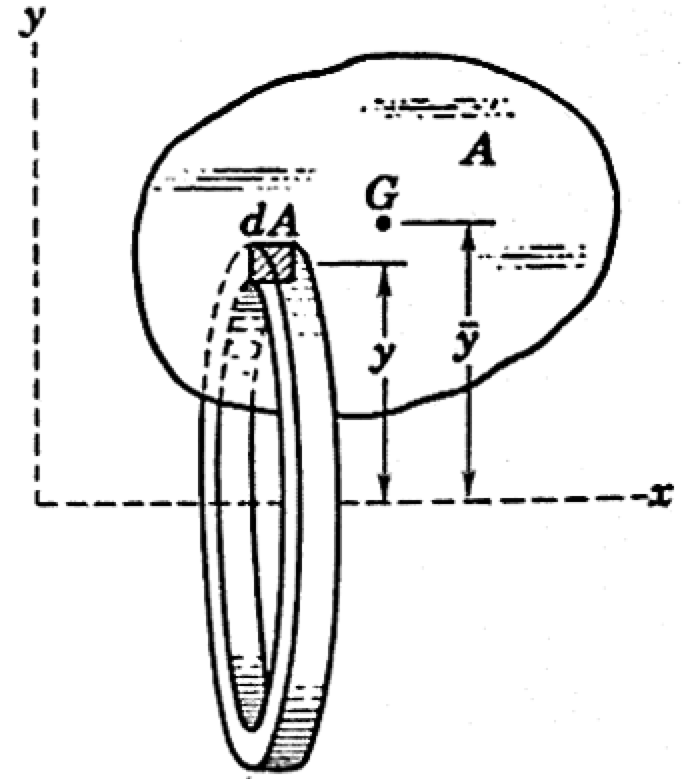


FIG. 50

the generated area is the same as the lateral area of a right circular cylinder of length L and radius $\bar{y}$.

In the case of a volume generated by revolving an area about a nonintersecting line in its plane an equally simple relation exists for finding the volume. An element of the volume generated by revolving the area A about the x-axis, Fig. 50, is the elemental ring of cross section dA and radius y. The volume of the element is $dV = 2\pi y\,dA$, and the total volume is

$$V = 2\pi \int y\,dA.$$

But since $\bar{y}A = \int y\,dA$ the volume becomes

$$V = 2\pi\bar{y}A, \tag{23}$$

where $\bar{y}$ is the y-coordinate of the centroid of the revolved area A. Thus

the generated volume is obtained by multiplying the generating area by the circumference of the circular path described by its centroid.

The two theorems of Pappus, expressed by Eqs. (22) and (23), not only are useful in determining areas and volumes of generation, but they are also employed to find the centroids of plane curves and plane areas when the corresponding areas and volumes due to revolution of these figures about a nonintersecting axis are known. Dividing the area or volume by 2π times the corresponding line segment length or plane area will give the distance from the centroid to the axis of revolution.

In the event that a line or an area is revolved through an angle θ less than 2π, the generated surface or volume may be found by replacing 2π by θ in Eqs. (22) and (23). Thus

$$A = \theta \bar{y} L \quad \text{and} \quad V = \theta \bar{y} A,$$

where θ is expressed in radians.

PROBLEMS

382. Determine the volume of a sphere from the semicircular area used to generate the sphere.

383. Determine the volume of a right circular cone from the right triangle used to generate the cone.

384. Use the notation of the half-torus of Prob. 357 and determine the volume V and the surface area A of a complete torus.

385. Find the volume of the spherical wedge of Prob. 354 by the method of this article.

386. Find the volume of the conical wedge of Prob. 351 by the method of this article.

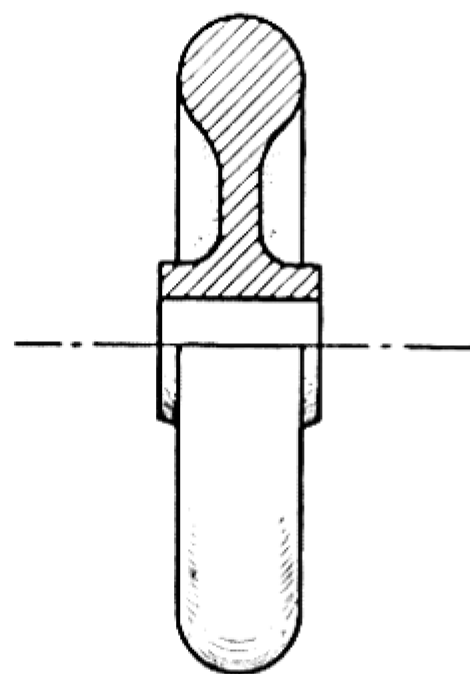
387. Obtain the answer to Prob. 327 for the semicircular arc by the method of this article.

388. Determine the volume of the ellipsoid obtained by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the y-axis.

389. Find the volume obtained by revolving the shaded area in Prob. 345 about the x-axis.

$$Ans. \quad V = \frac{(10 - 3\pi)\pi a^3}{6}$$

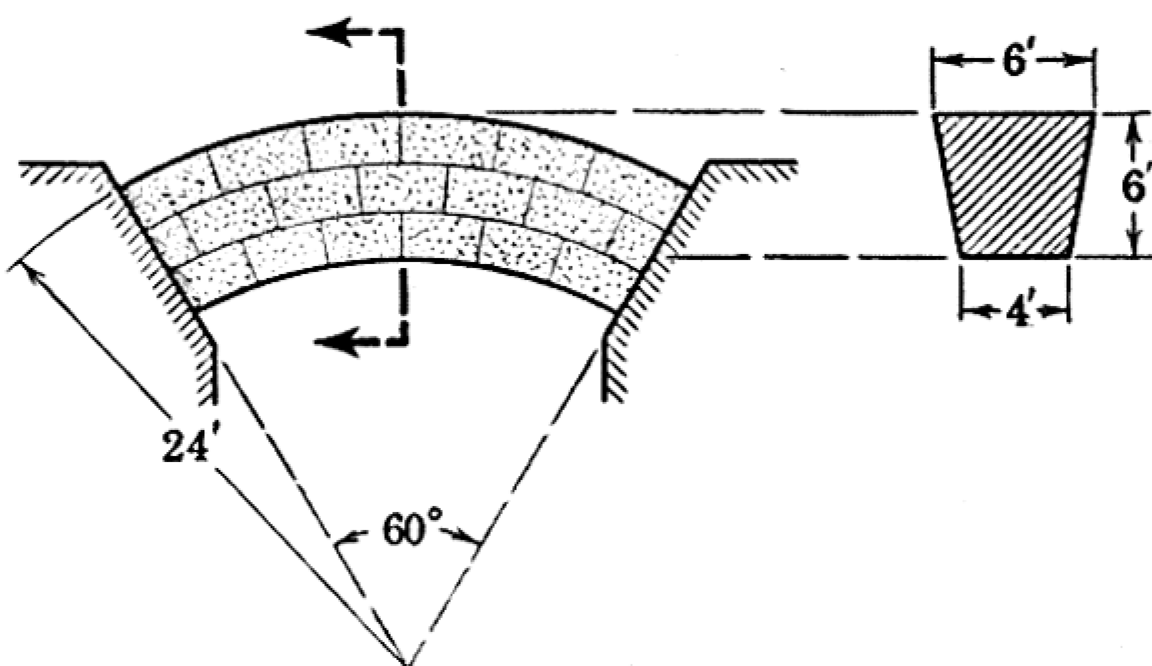
390. A hand-control wheel made of steel has the proportions shown in the sectional view. The shaded section has an area of 18.4 in.^{2}, and the wheel weighs 195 lb. Find the distance $\bar{r}$ from the axis of the wheel to the centroid of the shaded section. Steel weighs 0.283 lb./in.^{3}



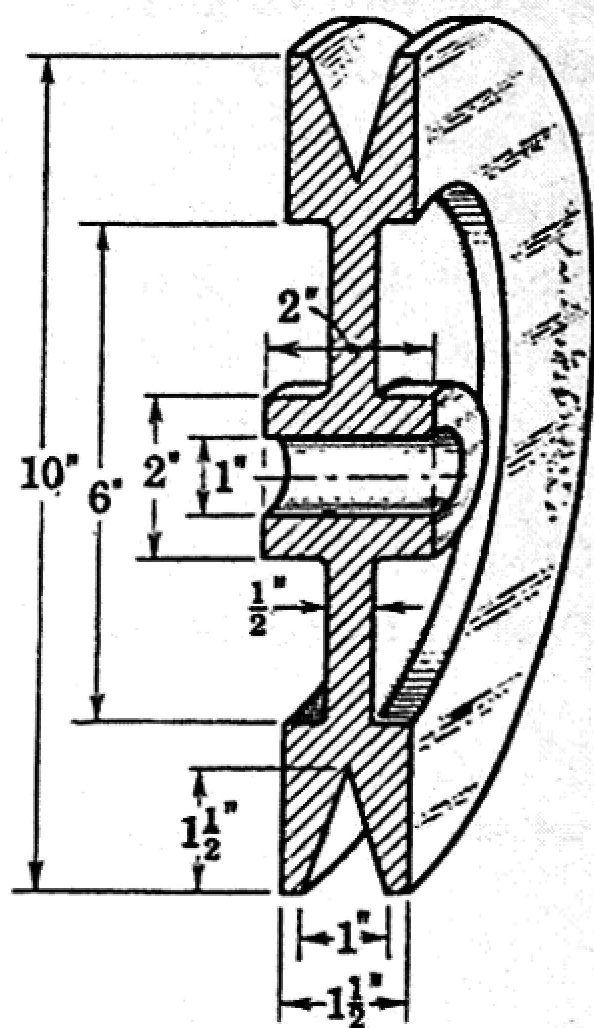
PROB. 390

391. In order to provide sufficient support for the stone masonry arch it is necessary to know the weight W of the arch. Find W if the stone weighs 165 lb./ft.^{3}

Ans. $W = 54.9$ tons



PROB. 391



PROB. 392

392. The cross-sectional view of a steel V-belt pulley has the dimensions shown. Determine the weight W of the entire pulley. The density of steel is 0.283 lb./in.^{3}

39. Flexible Cables. In the design of suspension bridges, transmission lines, and messenger cables for heavy trolley or telephone lines, it is necessary to know the relations between the tension, span, sag, and length of the cables. These are obtained by examining the cable as a body in equilibrium. In the study of flexible cables it is assumed that any resistance offered to bending is negligible. This assumption means that the force in the cable is always in the direction of the cable.

There are two extreme cases to be investigated. In the first the weight of the cable is negligible compared with the load it supports, such as in most suspension bridges, and if the load is uniformly distributed along the horizontal, the cable is found to assume the shape of a parabola. In the second case the cable hangs under the action of its own weight only, as in a transmission line. For this condition the cable is found to take the shape of the catenary.

Case I. Parabola. Figure 51 shows a suspension bridge of span L and sag h. Let that portion of the uniform road bed supported by each cable weigh w pounds per unit of length. The equilibrium of a finite portion of the cable is represented by the free-body diagram in the figure.

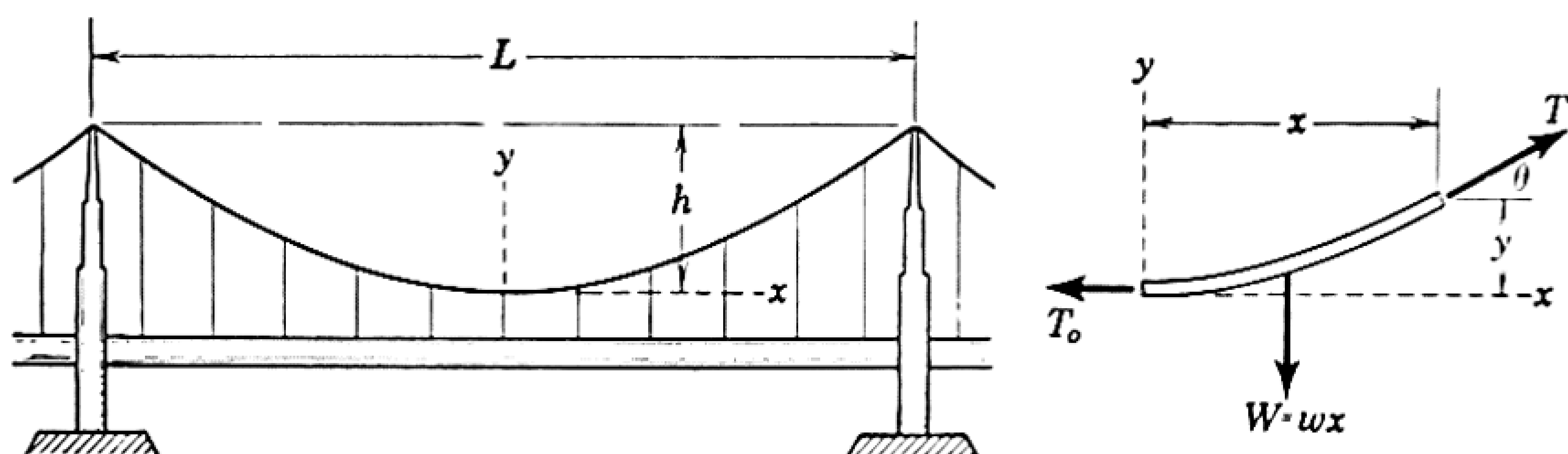


FIG. 51

The cable segment is in equilibrium under the action of a horizontal cable tension T_0 at the origin, the tension T at a point whose coordinates are x and y, and the force $W = wx$ which is the total vertical load supported by this portion of the cable. Equilibrium is specified by

$[\Sigma F_y = 0]$ $\qquad T \sin \theta = wx,$

$[\Sigma F_x = 0]$ $\qquad T \cos \theta = T_0.$

Combining gives $\tan \theta = dy/dx = wx/T_0$. Thus the differential equation of the curve for this cable is the simple equation $dy = (wx/T_0)dx$. Integration from the origin to any point x, y on the cable gives

$$\int_0^y dy = \frac{w}{T_0}\int_0^x x\,dx \qquad \text{or} \qquad y = \frac{wx^2}{2T_0}. \tag{24}$$

Equation (24) describes a parabola and gives the shape of the cable in terms of the constants w and T_0. Substitution of the boundary conditions $x = L/2$ and $y = h$ into Eq. (24) gives

$$T_0 = \frac{wL^2}{8h} \qquad \text{and} \qquad y = \frac{4hx^2}{L^2}.$$

Eliminating the angle θ from the equations of equilibrium gives $T^2 = T_0{}^2 + w^2x^2$. Thus the cable tension is

$$T = w\sqrt{x^2 + \frac{L^4}{64h^2}}. \tag{25}$$

The maximum tension occurs when $x = L/2$ and is

$$T_{\text{max.}} = \frac{wL}{2}\sqrt{1 + \frac{L^2}{16h^2}}.$$

The length S of the complete cable is obtained from the differential relation $dS = \sqrt{(dx)^2 + (dy)^2}$. Thus

$$\frac{S}{2} = \int_0^{L/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{L/2} \sqrt{1 + \left(\frac{wx}{T_0}\right)^2}\, dx.$$

For convenience in computation this expression is changed to a convergent series and then integrated term by term. From the expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

the integral may be written as

$$S = 2\int_0^{L/2} \left[1 + \frac{w^2x^2}{2T_0^2} - \frac{w^4x^4}{8T_0^4} + \cdots\right] dx$$

$$= L\left(1 + \frac{w^2L^2}{24T_0^2} - \frac{w^4L^4}{640T_0^4} + \cdots\right).$$

Substitution of $w/T_0 = 8h/L^2$ yields

$$S = L\left(1 + \frac{8}{3}\left[\frac{h}{L}\right]^2 - \frac{32}{5}\left[\frac{h}{L}\right]^4 + \cdots\right). \tag{26}$$

When the properties of this series are examined, it is found that it converges for all values of $h/L \leqq 1/4$. In most cases h is much smaller than $L/4$ so that the three terms of Eq. (26) give a sufficiently accurate approximation.

Case II. Catenary. Consider now a uniform cable, Fig. 52, suspended at two points in the same horizontal plane and hanging under the action

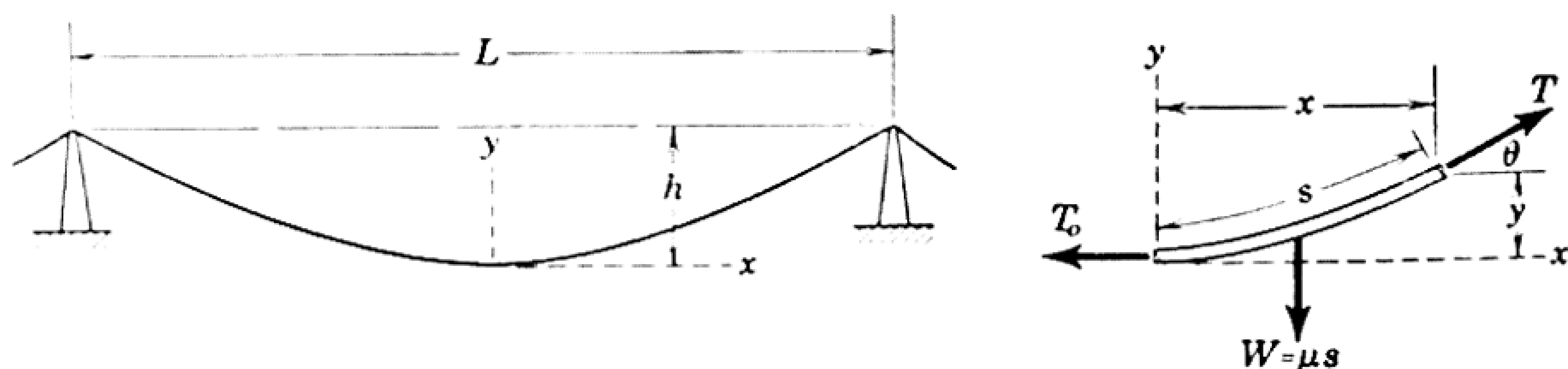


Fig. 52

of its own weight only. The free-body diagram of a finite portion of the cable of length s is shown in the right-hand part of the figure. This free-body diagram differs from that in Fig. 51 in that the total vertical force supported is equal to the weight of the section of cable of length s in place

of the uniform horizontal load. If the cable weighs μ pounds per foot of length, this load is μs.

Equilibrium is specified by

$$[\Sigma F_y = 0] \qquad T \sin\theta = \mu s,$$

$$[\Sigma F_x = 0] \qquad T \cos\theta = T_0.$$

Combining gives

$$\frac{dy}{dx} = \frac{\mu}{T_0} s.$$

Since $s = f(x,y)$, it is necessary to change this equation to one containing only the two variables. Differentiating gives

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx}.$$

Substituting the identity $(ds)^2 = (dx)^2 + (dy)^2$ yields

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{27}$$

Equation (27) is the differential equation of the curve (catenary) assumed by the cable. Solution of this equation is facilitated by the substitution $p = dy/dx$, which gives

$$\frac{dp}{\sqrt{1 + p^2}} = \frac{\mu}{T_0} dx.$$

Integrating this equation produces

$$\log\left(p + \sqrt{1 + p^2}\right) = \frac{\mu}{T_0} x + C.$$

The constant C is zero since $dy/dx = p = 0$ when $x = 0$. Substituting $p = dy/dx = (\mu/T_0)s$, changing to exponential form, and clearing the equation of the radical give

$$s = \frac{T_0}{\mu}\frac{e^{\mu x/T_0} - e^{-\mu x/T_0}}{2}.$$

At this point the hyperbolic functions * are introduced for convenience. Thus the arc length s may be written as

$$s = \frac{T_0}{\mu}\sinh\frac{\mu x}{T_0}. \tag{28}$$

* See Table B3, Appendix B.

Since $\mu s/T_0 = dy/dx$, Eq. (28) may be integrated to obtain

$$y = \frac{T_0}{\mu} \cosh \frac{\mu x}{T_0} + K.$$

The integration constant K is evaluated from the boundary condition $x = 0$ when $y = 0$. This substitution requires that $K = -T_0/\mu$, and hence

$$y = \frac{T_0}{\mu}\left(\cosh \frac{\mu x}{T_0} - 1\right). \tag{29}$$

Equation (29) is the equation of the curve (catenary) assumed by the cable hanging under the action of its weight only.

The tension T in the cable is obtained from the two equilibrium equations for the cable segment by squaring and adding. This operation gives

$$T^2 = \mu^2 s^2 + T_0^2,$$

which, upon combination with Eq. (28), becomes

$$T^2 = T_0^2\left(1 + \sinh^2 \frac{\mu x}{T_0}\right) = T_0^2 \cosh^2 \frac{\mu x}{T_0},$$

or

$$T = T_0 \cosh \frac{\mu x}{T_0}. \tag{30}$$

The tension may also be expressed in terms of y with the aid of Eq. (29), which, when substituted into Eq. (30), gives

$$T = T_0 + \mu y. \tag{31}$$

Equation (31) shows that the increment in cable tension from that at the lowest position depends only on y.

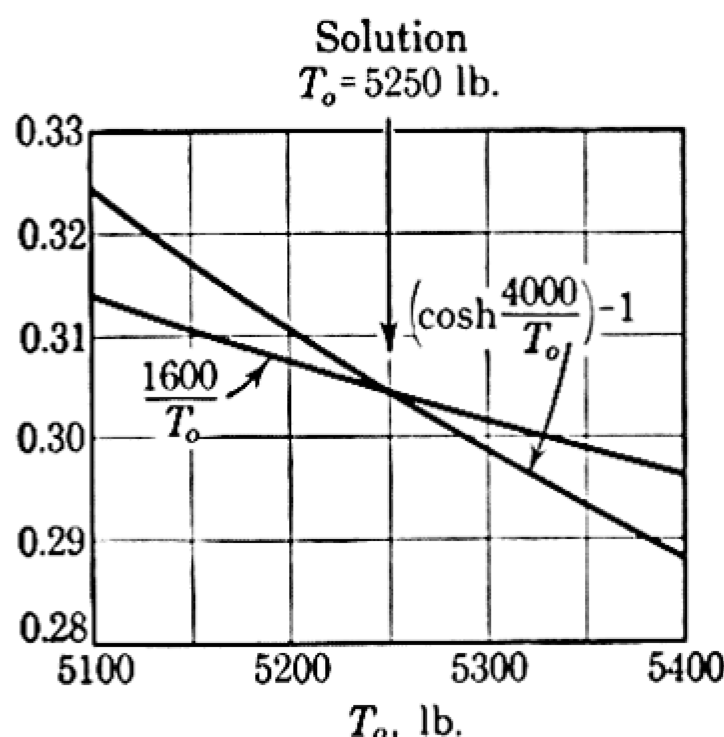
Most problems dealing with the catenary involve solutions of Eqs. (28) through (31) which are best handled graphically. This procedure is illustrated in the sample problem following this article.

The solution of catenary problems where the sag-to-span ratio is small may be approximated by the relations developed for the parabolic cable. A small sag-to-span ratio means a tight cable, and the uniform distribution of weight along the cable is not much different from the same load intensity distributed uniformly along the horizontal.

Many problems dealing with both the catenary and parabolic cable involve suspension points which are not on the same level. In such cases the relations may be applied to each part of the cable on either side of the lowest point.

SAMPLE PROBLEM

393. A cable weighing 8 lb./ft. is suspended between two points on the same level and 1000 ft. apart. If the sag is 200 ft., find the length of the cable and the maximum tension.



PROB. 393

Solution: Equations (28) and (30) for the cable length and tension both involve the minimum tension T_0 which must be found from Eq. (29). Thus, for $x = 500$ ft., $y = 200$ ft., and $\mu = 8$ lb./ft.,

$$200 = \frac{T_0}{8}\left(\cosh\frac{8 \times 500}{T_0} - 1\right) \qquad \text{or} \qquad \frac{1600}{T_0} = \cosh\frac{4000}{T_0} - 1.$$

This equation is most easily solved graphically. The expression on each side of the equals sign is computed and plotted as a function of various values of T_0. The intersection of the two curves establishes the equality and determines the correct value of T_0. This plot is shown in the figure accompanying this problem and yields the solution

$$T_0 = 5250 \text{ lb.}$$

The maximum tension occurs for maximum y and from Eq. (31) is

$$T = 5250 + 8 \times 200 = 6850 \text{ lb.} \qquad \textit{Ans.}$$

From Eq. (28) the total length of the cable becomes

$$2s = 2\,\frac{5250}{8}\sinh\frac{8 \times 500}{5250} = 1100 \text{ ft.} \qquad \textit{Ans.}$$

PROBLEMS

394. A cable with a span of 500 ft. and a sag of 50 ft. supports a uniformly distributed load of 800 lb. per horizontal foot. Determine the length S of the cable and the maximum tension T.

395. Expand Eq. (29) in a power series for cosh $(\mu x/T_0)$ and show that the equation for the parabola, Eq. (24), is obtained by taking only the first two terms in the series. (See Table B3, Appendix B, for the series expansion of the hyperbolic function.)

396. A cable hanging under the action of its own weight is suspended between two points on the same level and 400 ft. apart. If the sag is 100 ft., determine the total length S of the cable. What error is involved if the length is computed from the expression for the parabolic cable?

Ans. $S = 460$ ft., 0.65% error

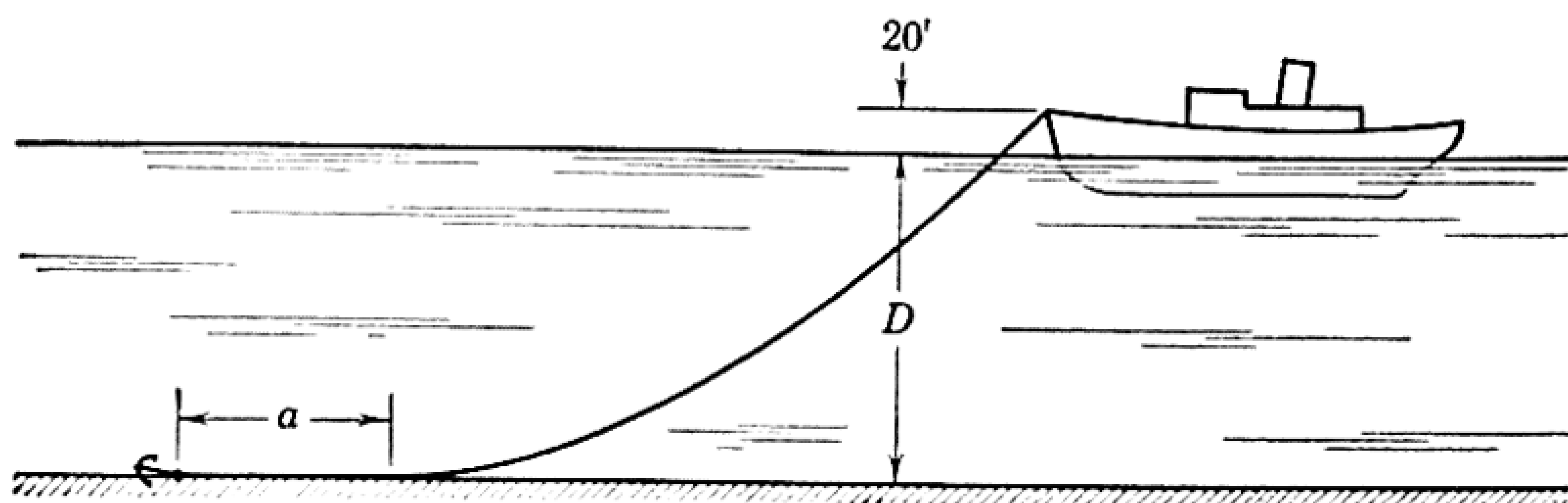
397. A catenary and a parabolic cable are both suspended from two points 500 ft. apart and on the same level. Each cable has a sag of 50 ft. Plot the shape of both curves and determine the difference Δy in their vertical coordinates at a position 125 ft. measured horizontally from the middle of the span.

398. The Golden Gate Bridge has a main span of 4200 ft., a sag of 470 ft., and a total static loading of 21,300 lb. per lineal foot of horizontal measurement. The weight of both cables is included in this figure and is assumed uniformly distributed along the horizontal. Each of the two towers weighs 44,400,000 lb. The angle made by the cables with the vertical at the top of the towers is the same on either side of the towers. Determine the compressive force P exerted by each of the two cables on the top of one tower and the total compressive force C exerted by each tower on its foundation.

Ans. $P = 44.7 \times 10^6$ lb., $C = 133.8 \times 10^6$ lb.

399. A ship is anchored with 200 ft. of chain from the bow to the anchor. The chain weighs 10 lb./ft. and makes an angle of 45 deg. with the horizontal at the bow. In addition the tension in the chain at the bow winch which secures it is 2500 lb. Determine the length a of chain which is lying flat on the bottom and the depth D of the water. Neglect the buoyancy of the chain in the water.

Ans. $a = 23.2$ ft., $D = 53.2$ ft.

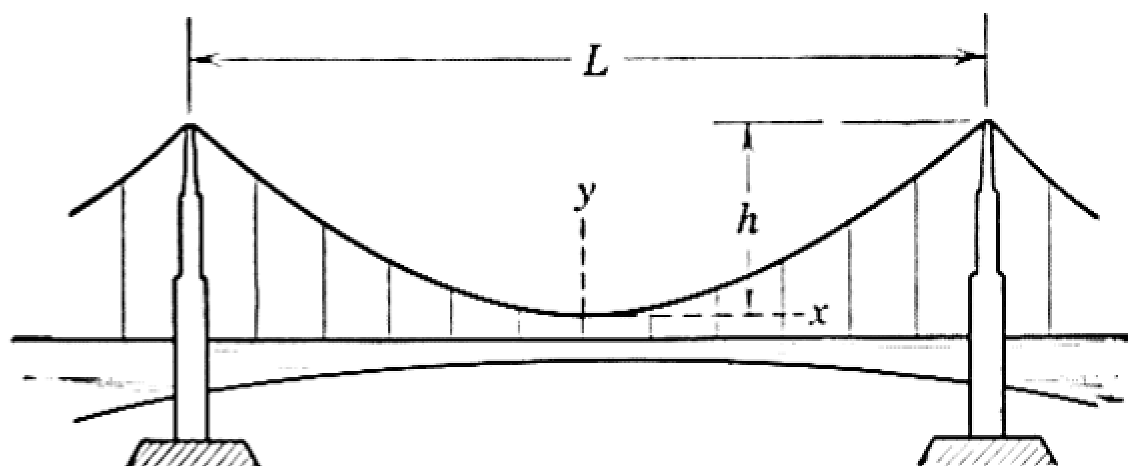


PROB. 399

400. A cable 200 ft. long is suspended between two points on the same level 180 ft. apart. If the cable supports a large load uniformly distributed in the horizontal direction, find the sag h in the cable. [*Hint:* Solve Eq. (26) by successive approximations.]

401. The bridge shown has a road structure whose weight per unit length in the horizontal direction is $w = w_0 + kx^2$. Find the equation of the curve

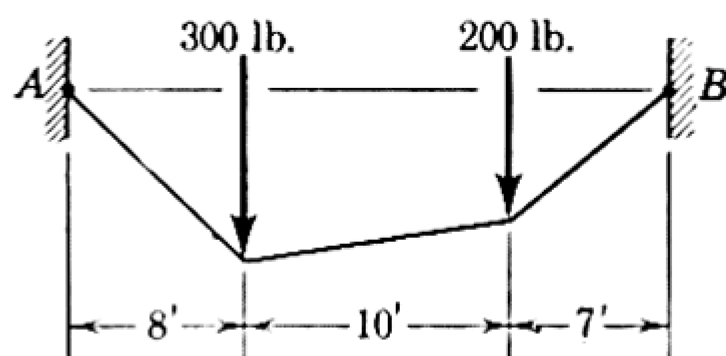
assumed by the cable and the expression for the constant horizontal component T_0 of the cable tension. *Ans.* $y = \frac{w_0x^2}{2T_0} + \frac{kx^4}{12T_0}$, $T_0 = \frac{L^2}{8h}\left(w_0 + k\frac{L^2}{24}\right)$



PROB. 401

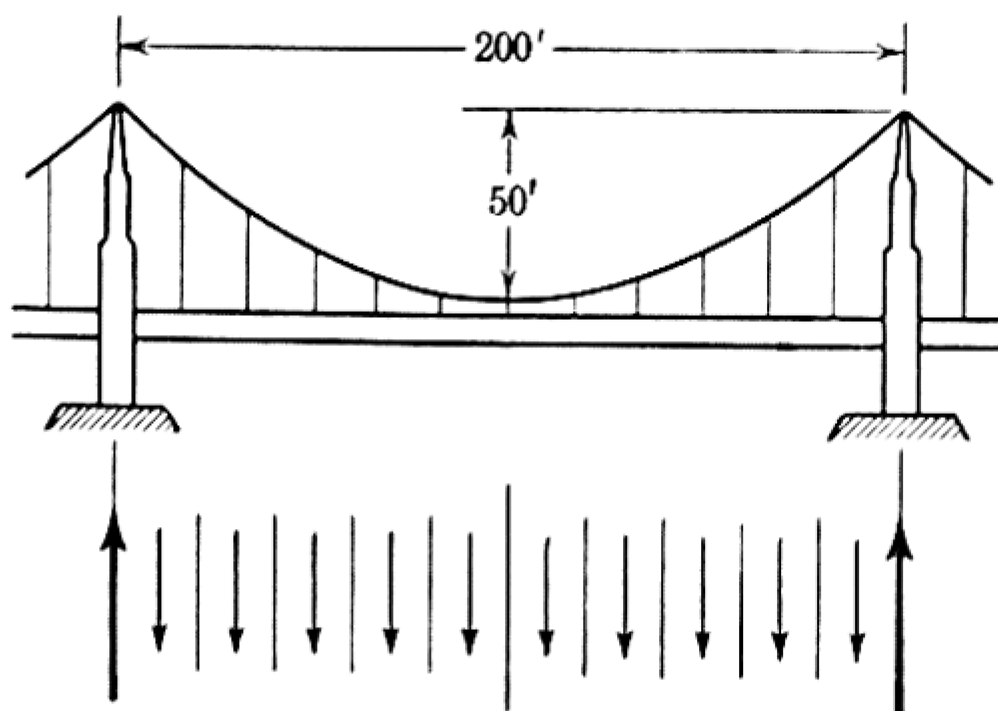
402. A cable supports a uniform horizontal load of 400 lb./ft. and is suspended from two points which differ in elevation by 20 ft. If the supports are separated by 200 ft. measured along the horizontal and the sag of the cable below the upper support is 40 ft., determine the maximum cable tension T and the total length S of cable. *Ans.* $T = 83{,}100$ lb., $S = 212$ ft

403. A light cable 30 ft. long supports the two concentrated loads shown. Determine graphically the shape assumed by the cable and the constant horizontal component T_0 of the cable tension. (*Hint:* Show that a funicular polygon passing through points A and B and representing the equilibrium of the two given loads and the vertical components of the support reactions also represents to some scale the correct shape of the cable. By adjusting the horizontal position of the pole O a funicular polygon representing the proper cable length may be drawn.) *Ans.* $T_0 = 286$ lb.



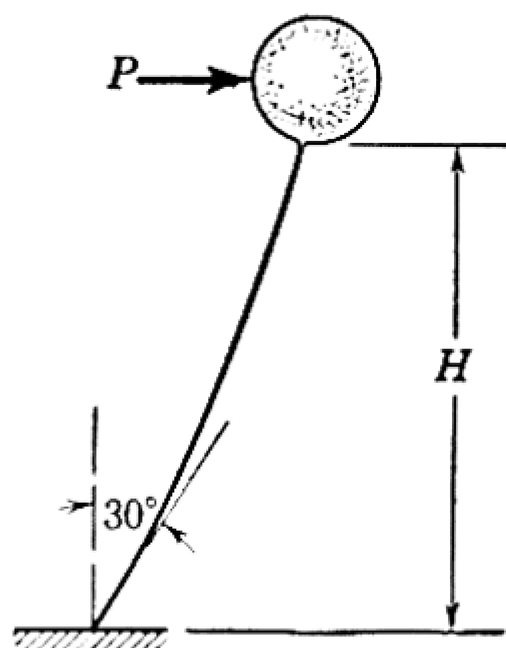
PROB. 403

404. In the suspension bridge shown the total weight of the roadway between towers is 20,000 lb. carried equally by two cables. Without the aid of equations determine graphically the shape of the cable, the constant horizontal component T_0 of cable tension, and the length S of the cable. (*Hint:* Divide the span into, say, 10 equal parts as indicated and form the funicular polygon for the equal vertical loads. By trying various horizontal positions of the pole O a funicular polygon representing the proper sag and shape can be drawn. Read the hint for Prob. 403.)



PROB. 404

* **405.** A balloon exerts an upward force of 80 lb. on its 100 ft. ground cable at the attachment point of the cable on the balloon. A horizontal wind causes a shift to the position shown, and the angle made by the cable with the vertical at the ground is observed to be 30 deg. If the cable weighs 0.5 lb./ft., determine the altitude H and the horizontal wind force P on the balloon.



PROB. 405

Ans. $H = 94.4$ ft., $P = 17.3$ lb.

* **406.** A cable hanging under the action of its own weight is suspended from the same two supports as in Prob. 402 and has the same sag. Without solving, outline a procedure for determining the maximum tension T in the cable and the total length S of the cable, using the exact relations.

40. Fluid Statics. The equilibrium of bodies subjected to the forces of fluid pressure is an important problem in statics. A fluid is any continuous substance which, when at rest, can transmit compressive forces but not tensile or shear forces. A shear force is one tangent to the surface upon which it acts. Thus a fluid at rest can exert only normal forces on a bounding surface. Fluids may be either gaseous or liquid.

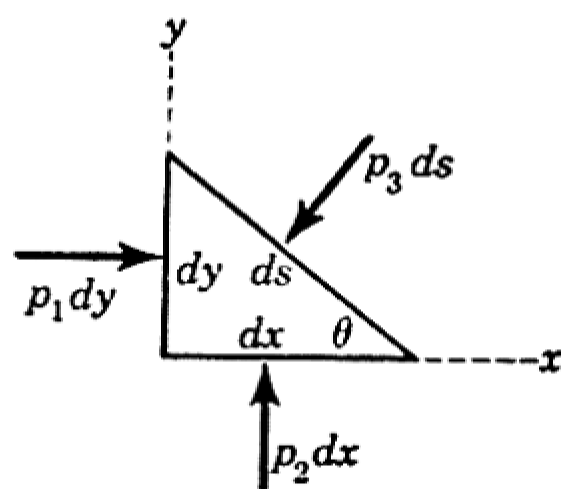


FIG. 53

The pressure in a fluid is the same in all directions (Pascal's law). This fact may be shown by considering the equilibrium of an infinitesimal triangular element of a fluid as shown in Fig. 53. By taking the z-dimension to be unity and the pressures on the three sides to be p_1, p_2, and p_3, the equilibrium of forces in the x- and y-directions may be expressed by the equations

$$p_1\,dy = p_3\,ds \sin\theta \qquad \text{and} \qquad p_2\,dx = p_3\,ds\cos\theta.$$

Since $ds \sin\theta = dy$ and $ds\cos\theta = dx$, these equations require that

$$p_1 = p_2 = p_3 = p.$$

Thus the pressure p is the same in all directions. In this analysis it is unnecessary to account for the weight of the fluid element, since when multiplying the fluid density by the volume of the element, a differential quantity of second order results which may be neglected compared with the first-order pressure force terms.

In all fluids at rest the pressure is a function of the vertical dimension. To determine this function a change in the vertical dimension must be considered and account of the weight of the fluid must be taken. Figure

54 shows a differential element of fluid in the form of a cylinder with vertical axis and cross-sectional area dA. The positive direction of vertical measurement z is taken down. The pressure on the upper face is p, and that on the lower face is p plus the change in p, or $p + dp$. The weight of the element equals its weight density μ multiplied by the volume. The normal forces on the lateral surface have nothing to do with the balance of forces in the vertical direction and are not shown. Equilibrium of the fluid element in the z-direction requires

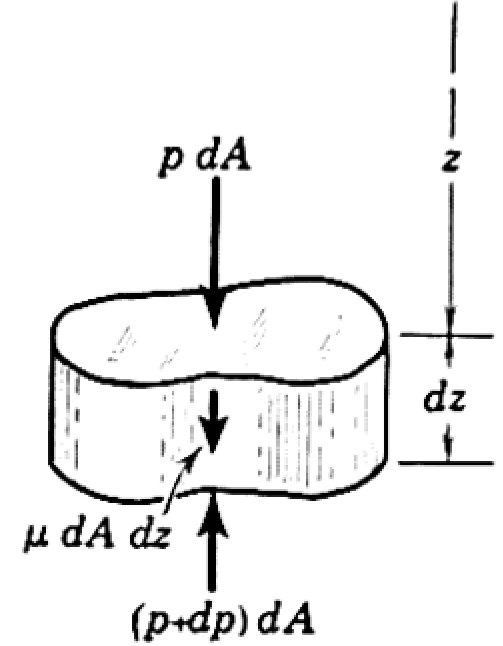


FIG. 54

$$p\,dA + \mu\,dA\,dz - (p + dp)\,dA = 0,$$

$$dp = \mu\,dz. \tag{32}$$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Equation (32) holds for both liquids and gases and is in accord with common knowledge of air and water pressures.

Fluids which are essentially incompressible are called liquids, and it follows that for the most practical purposes the density may be considered constant for every part of the liquid.* With μ a constant Eq. (32) may be integrated as it stands, and the result is

$$p = p_0 + \mu z. \tag{33}$$

The pressure p_0 is the pressure on the surface of the liquid where $z = 0$. If p_0 is due to atmospheric pressure and the measuring instrument records only the increment above atmospheric pressure,† then the measurement gives what is known as "gage pressure" and is $p = \mu z$.

Gases, on the other hand, are compressible, and here the density varies with vertical distance. For most engineering problems this variation is negligible when considering gas pressures on a structure, since the height of the structure usually represents only a small change in altitude. If the gas temperature is constant (isothermal), the gas law $p = \mu K$, where K is a constant, may be used to determine the variation of pressure with altitude. Substitution of this value of μ into Eq. (32) and replacement of the downward measurement z by the upward measurement h, ($dz = -dh$), give $K\,dp = -p\,dh$. Integration from the conditions of pressure p_0 at zero altitude to pressure p at altitude h yields

$$h = K \log \frac{p_0}{p}.$$

* See Table of Densities, Table B1, Appendix B.

† Atmospheric pressure at sea level may be taken to be 14.7 lb./in.^{2}

When the temperature of a gas is not constant with altitude, such as with the earth's atmosphere, the effect of changing temperature must be accounted for.

41. Pressure on Submerged Surfaces. A surface submerged in a liquid, such as a gate valve in a dam or reservoir, is subjected to distributed fluid forces. Consider the fluid pressure on the upper surface of the submerged fixed vane, Fig. 55, which intersects the plane of the figure in the section AB. The dimension perpendicular to the plane of

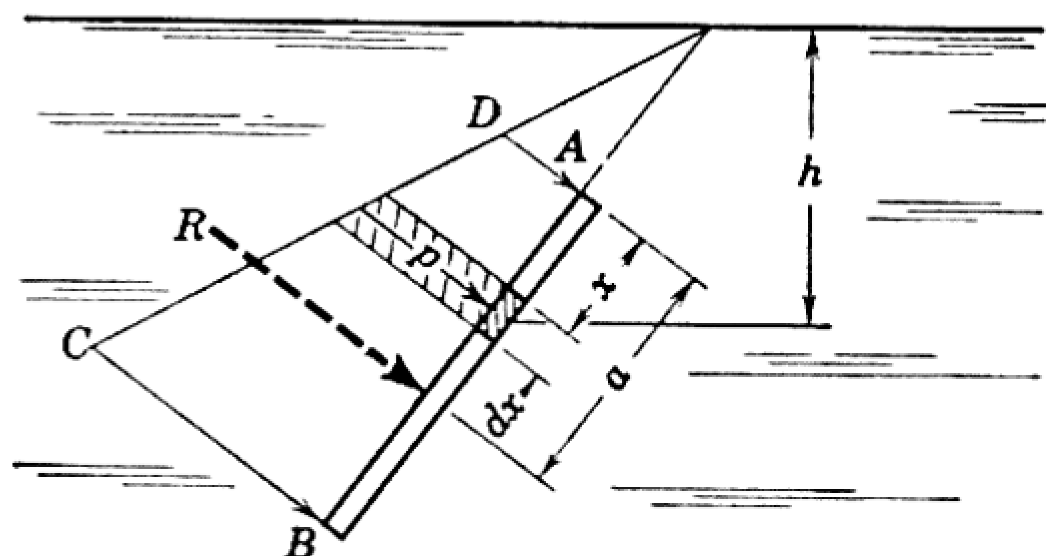


FIG. 55

the paper is taken to be unity. The effect of the atmospheric pressure p_0 may be omitted since it acts over all surfaces and yields a zero resultant. The variation of fluid pressure (gage) is linear according to Eq. (33) and is $p = \mu h$. Thus the pressure at the top of the section has the magnitude DA and that at the bottom of the section has the magnitude CB. The resultant R of the distributed pressure forces on this unit section AB is obtained by integrating the force acting on an element dx over the surface of the vane. Thus

$$R = \int p\,dx = \int dA = A,$$

where dA is the differential area $p\,dx$ and A is the area $ABCD$ under the pressure curve. The units of A are those of pressure times distance (lb./in.^{2})(in.) = (lb./in.) which represents force per unit width of section perpendicular to the paper. For a linear distribution of pressure the force R per unit width of vane is merely the average of the pressure at each end times the area of the vane. The position of R is found from the principle of moments, which requires that

$$Ra = \int px\,dx = \int x\,dA = \bar{x}A.$$

Hence $a = \bar{x}$ and the resultant of the forces always passes through the centroid of the figure defined by the curve of pressure versus distance.

This relation holds irrespective of the manner in which the pressure varies along the surface. For the linear pressure distribution of Fig. 55 the resultant passes through the centroid of the trapezoid $ABCD$.

For a submerged curved surface the resultant R caused by distributed pressure involves more calculation than for a flat surface. In Fig. 56 let the figure ABC be the vertical cross section of a curved vane of unit dimension normal to the paper. The force on an element of length dL

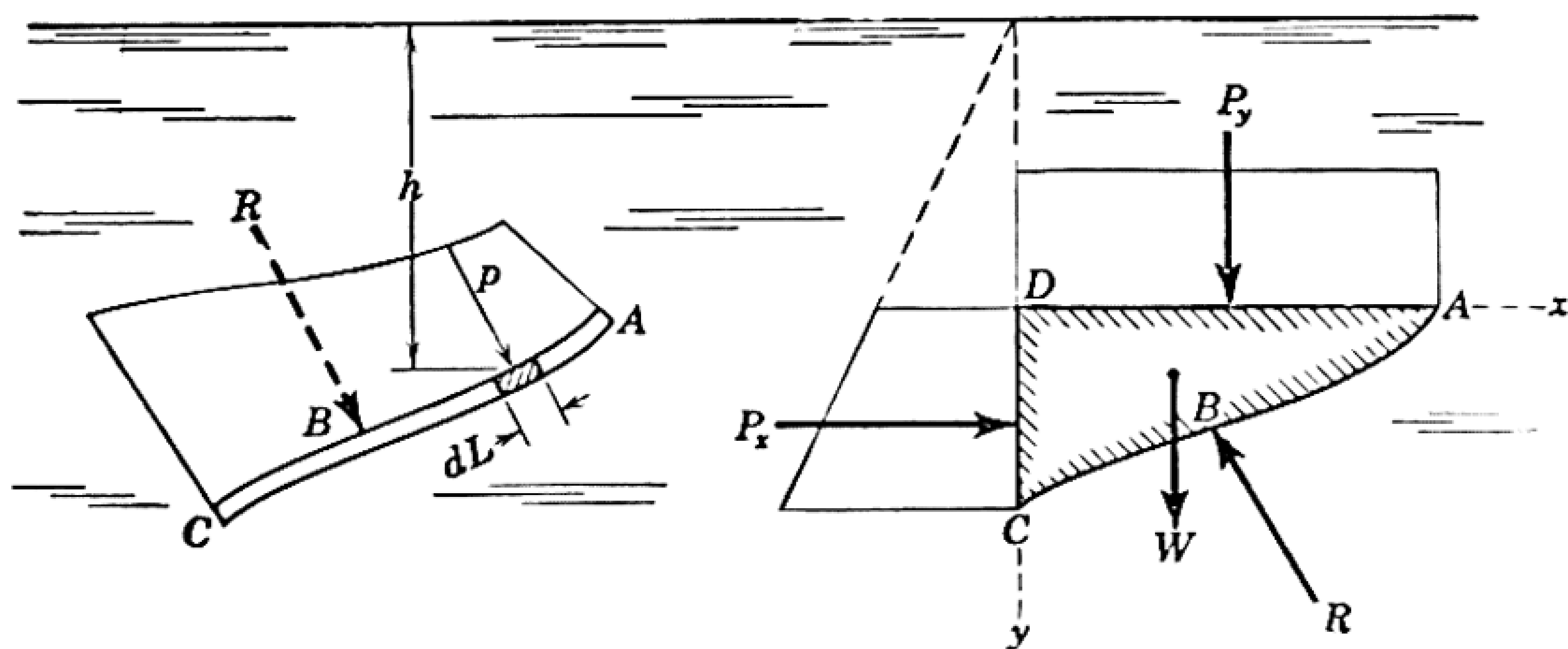


FIG. 56

is $p\,dL$ and is normal to the surface. Since the direction of the force changes over the cross section, it is not possible to integrate $p\,dL$ as it stands. The total horizontal component of R is $\int p_x\,dL$ or $\int p\,dy$, and the vertical component is $\int p_y\,dL$ or $\int p\,dx$. The resultant R is then found from its two components. The position of R can be found by the principle of moments.

A second method for finding R is often much simpler. The equilibrium of the block of liquid $ABCD$ above the vane, shown in Fig. 56, is established. The resultants of the pressures along DA and along DC are P_y and P_x, respectively, and are easily obtained. The weight W of the liquid block is calculated from the area $ABCD$ of the section and is shown acting through the centroid of the section. The equilibrant R is the resultant force exerted on the liquid by the vane and is equal and opposite to the force R exerted on the vane by the liquid.

For a flat plate of *any* shape submerged in a liquid the force on an element dA of its surface is $p\,dA$. Since the pressure is linear with the depth y, this elemental force becomes $\mu y\,dA$. Integration over the surface gives for the total force

$$R = \int p\,dA = \mu \int y\,dA = \mu \bar{y} A.$$

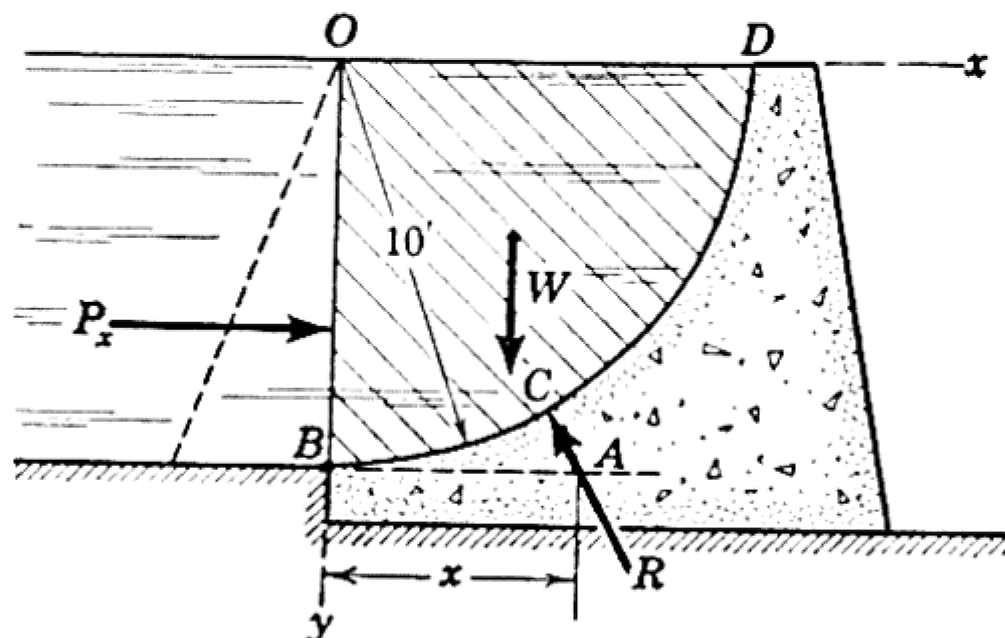
Thus the resultant force may be found by multiplying the area of the plate by the average pressure $\mu\bar{y}$, which is the pressure existing at the centroid of the submerged surface. The resultant force $R = \int p\,dA$ may also be viewed as the volume V of the three-dimensional figure whose base is the plate surface and whose altitude has a magnitude equal to the variable pressure acting over the surface. The resultant R does *not* act at the centroid of the plate surface but passes through the centroid of this described volume as may be seen from the principle of moments. Thus, if $\bar{h}$ is the depth to the centroid of this volume, a moment center at the surface gives

$$R\bar{h} = \int py\,dA = \int y\,dV = \bar{y}V = \bar{y}R.$$

Hence the coordinate $\bar{y}$ to the centroid of the volume equals the depth $\bar{h}$ at which R acts.

SAMPLE PROBLEM

407. Determine completely the resultant force R exerted on a unit section of the cylindrical dam surface by the water. The density of fresh water is 62.4 lb./ft.^{3}



PROB. 407

Solution: The circular block of water *BCDO* is isolated and its free-body diagram is drawn. The force P_x is

$$P_x = \frac{\mu r}{2} r = \frac{62.4 \times 10}{2} \times 10 = 3120 \text{ lb.}$$

The weight W of the water is

$$W = \mu A = 62.4 \times \frac{\pi(10)^2}{4} = 4900 \text{ lb.}$$

Equilibrium of the section of water requires

$[\Sigma F_x = 0]$ $\qquad R_x = P_x = 3120$ lb.,

$[\Sigma F_y = 0]$ $\qquad R_y = W = 4900$ lb.

The resultant force R exerted by the fluid on the dam is equal and opposite to that shown acting on the fluid and is

$$[R = \sqrt{R_x^2 + R_y^2}] \qquad R = 5810 \text{ lb.} \qquad Ans.$$

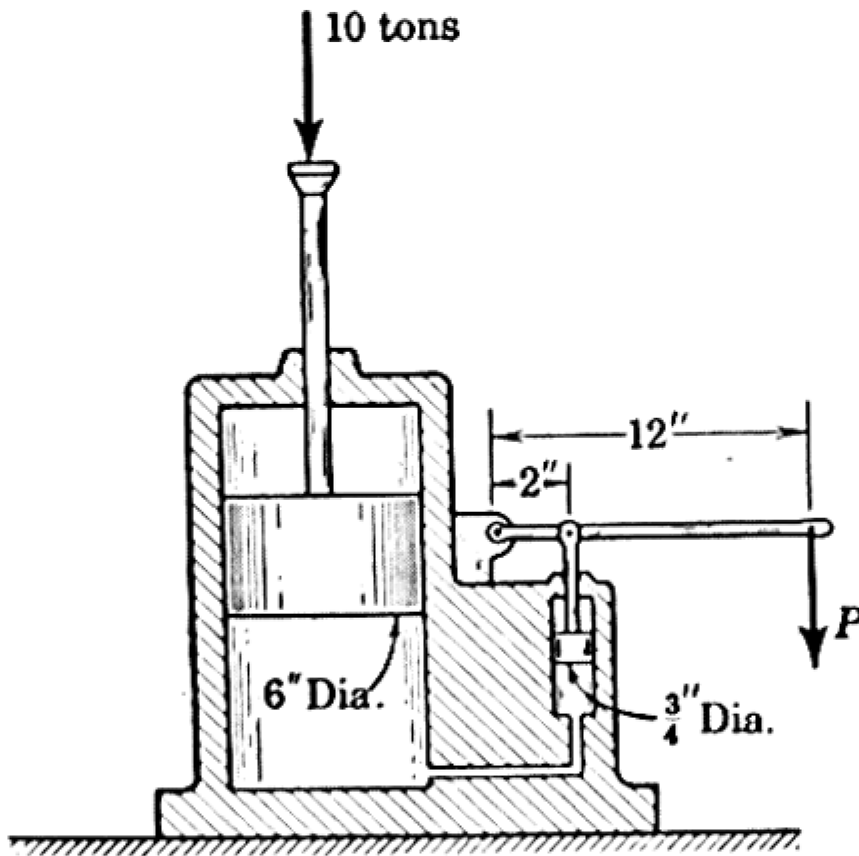
The x-coordinate of the point A through which R passes may be located graphically if desired. Algebraically x is found from the principle of moments. Using B as a moment center gives

$$P_x \frac{r}{3} + W \frac{4r}{3\pi} - R_y x = 0, \qquad x = \frac{3120 \times \frac{10}{3} + 4900 \times \frac{40}{3\pi}}{4900} = 6.37 \text{ ft.} \qquad Ans.$$

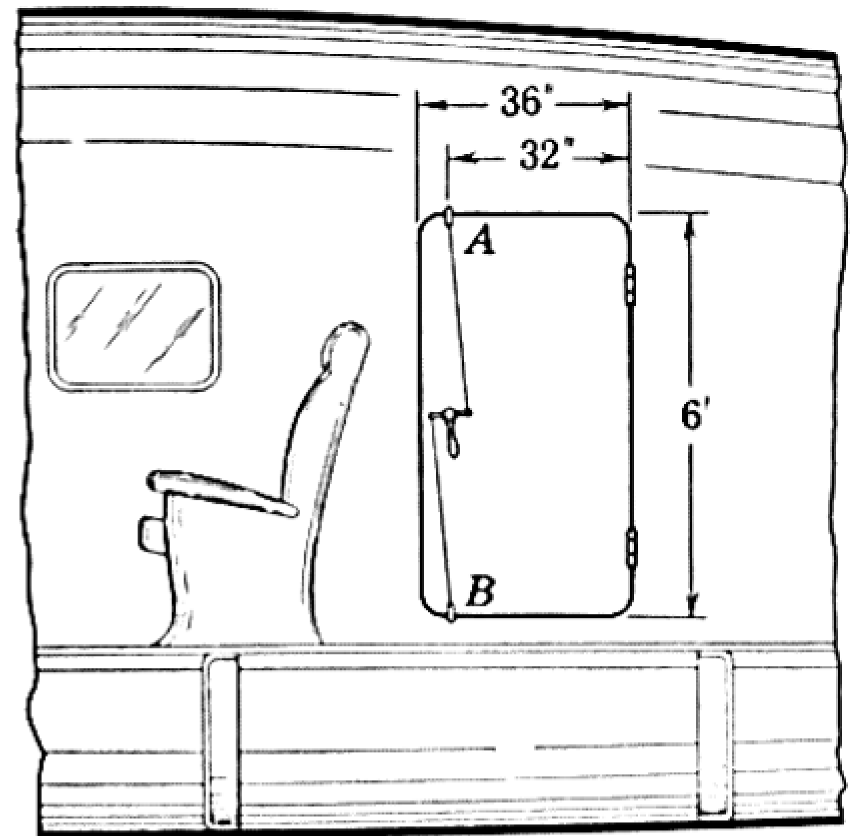
If the point C on the curved surface is desired, it will be necessary to combine the moment relation with the equation of the circle for solution.

PROBLEMS

408. Find the force P on the pump handle of the hydraulic jack in order to raise the 10 ton load.



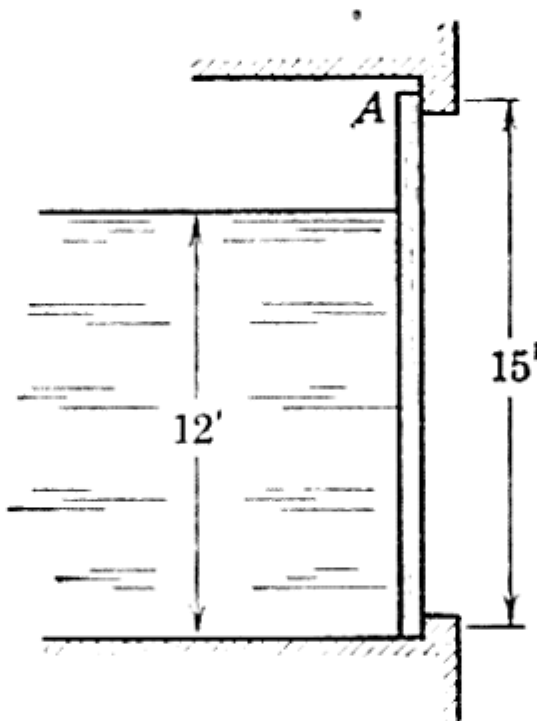
PROB. 408



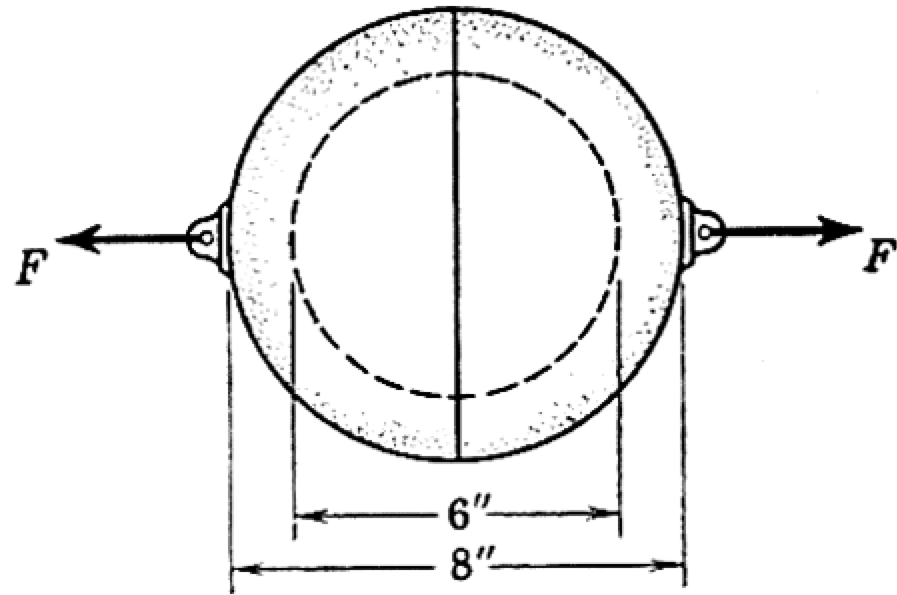
PROB. 409

409. The ratios of atmospheric pressure at 30,000 ft. and 5000 ft. altitudes to the pressure at sea level are 0.297 and 0.832, respectively. Determine the force on each of the latches A and B on the cabin door of the airplane flying at 30,000 ft. with the cabin pressurized to the conditions at 5000 ft. Assume that the door is secured by the two latches and the two hinges only and that the door is located at a position along the body where additional external pressure due to the movement of air is negligible.

410. The width of the opening closed by the gate for the fresh-water reservoir is 10 ft. Find the force F exerted by the top of the gate on the support at A.



Prob. 410



Prob. 411

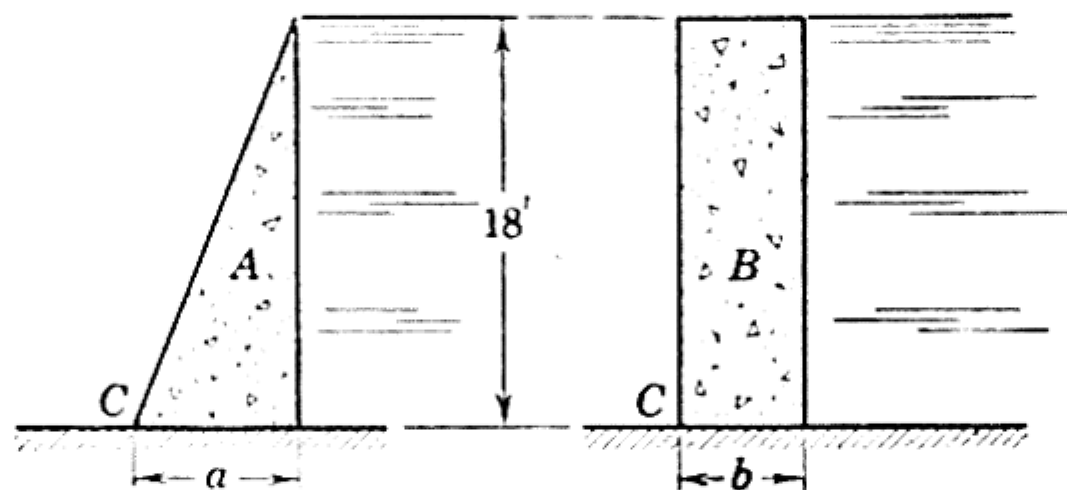
411. The two hemispherical shells are perfectly sealed together over their entire mating surface, and the air inside is partially evacuated to a pressure of 2 lb./in.^{2} absolute. Determine the force F required to pull the shells apart.

Ans. $F = 682$ lb.

412. An instrument for recording water pressures below the surface of the ocean consists of a tube with top end closed and bottom end open. Before submergence the tube is open to the atmosphere at a pressure of 14.7 lb./in.^{2} When the tube is submerged to a depth h, the air is compressed to one fourth of its original volume in the tube. Find h, assuming constant temperature. Salt water weighs 64 lb./ft.^{3}

Ans. $h = 99.2$ ft.

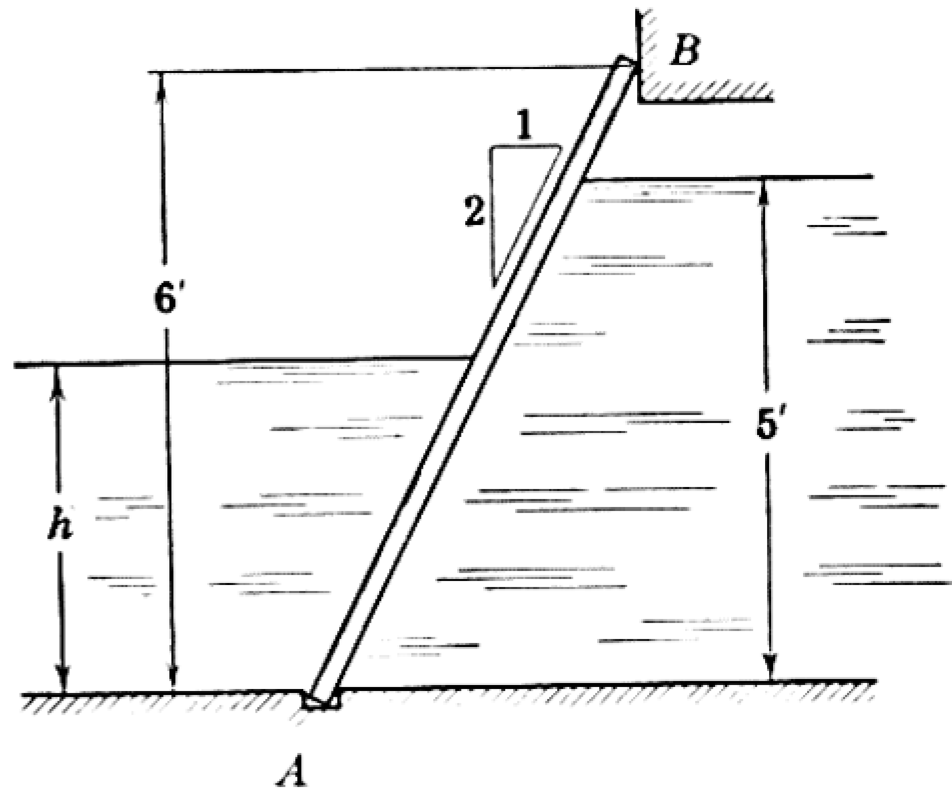
413. The triangular and rectangular sections are being considered for a small concrete dam. From the standpoint of resistance to overturning about C, which section will require less concrete, and how much less per foot of dam length? Concrete weighs 150 lb./ft.^{3}



Prob. 413

414. The slanted gate AB is a uniform plate weighing 2000 lb. which separates two bodies of fresh water in a channel 8 ft. wide. As the water is drained from

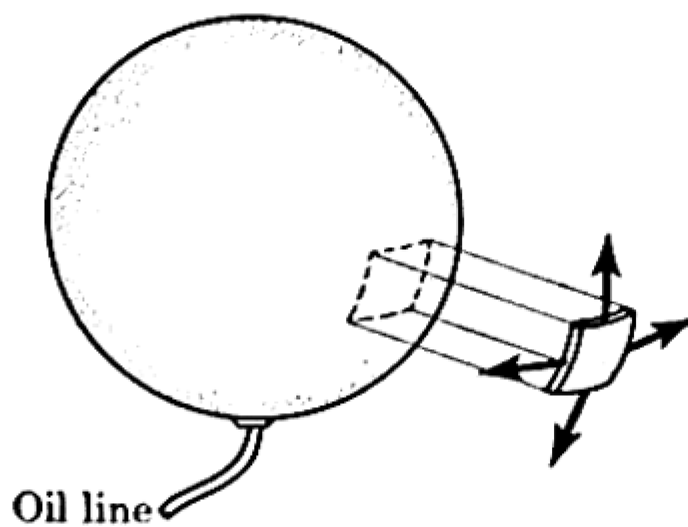
the left side of the gate, the reaction against the support at B is changed. Determine the depth h at which this reaction is zero. Assume that the gate is hinged at its lower end. *Ans.* $h = 4.58$ ft.



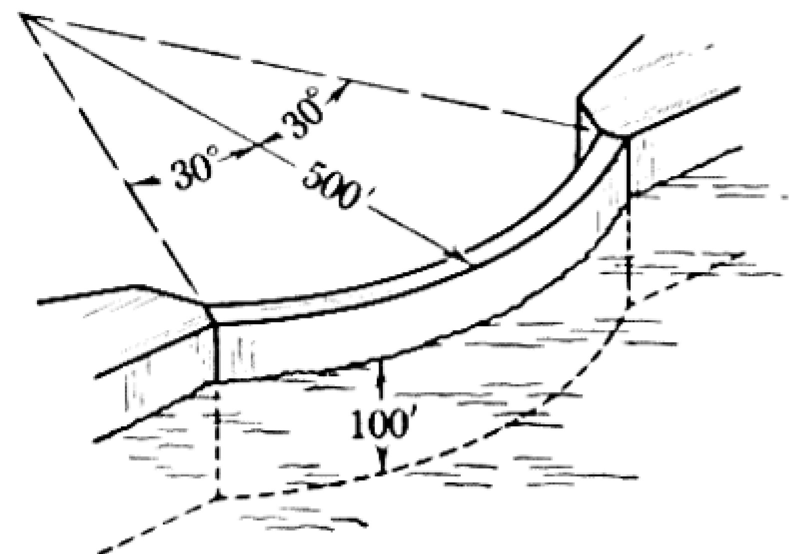
PROB. 414

415. The spherical aluminum shell has an inside diameter of 12 in. and a wall thickness of ¼ in. Oil under pressure is pumped into the shell, causing every element of the shell, such as the one illustrated, to be subjected to equal tensile forces on each side. The force per unit area of shell section is the *stress*. If the aluminum used can withstand a stress of 40,000 lb./in.² before fracture occurs, determine the oil pressure required to burst the shell.

Ans. $p = 3400$ lb./in.² gage



PROB. 415

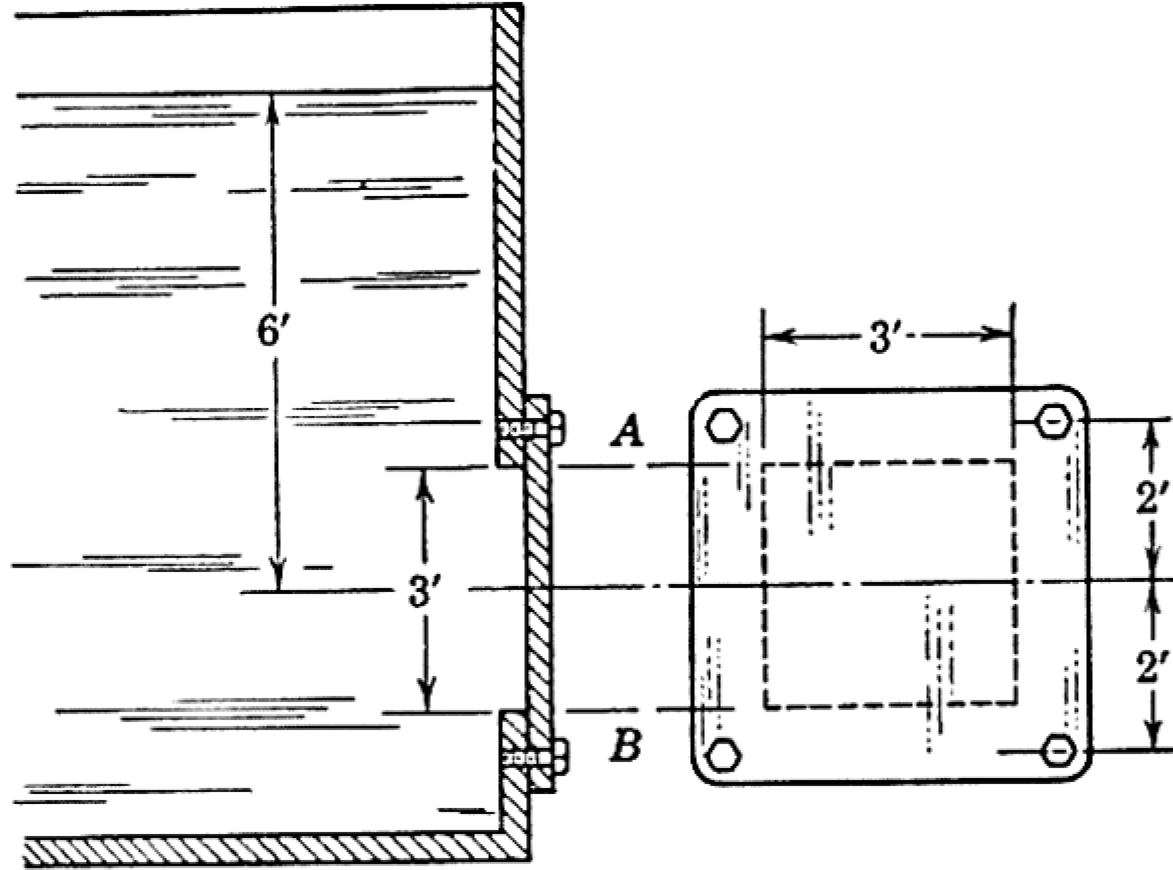


PROB. 416

416. The upstream side of an arched dam has the form of a cylindrical surface of 500 ft. radius and subtends an angle of 60 deg. If the water is 100 ft. deep, determine the total force P exerted by the water on the dam.

417. Find the total force R exerted by a liquid of density μ on a vertical gate which blocks the end of an open channel of semicircular section with radius a, if the liquid fills the channel. *Ans.* $R = \frac{2}{3}\mu a^3$

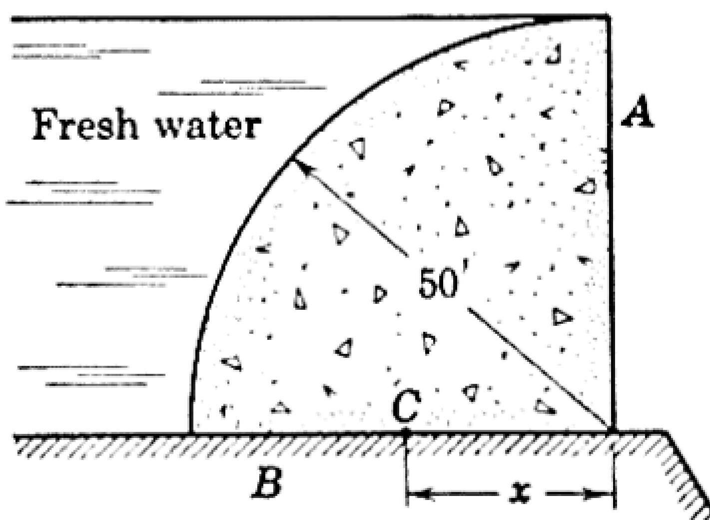
418. The cover plate for an access opening in a fresh-water tank is bolted in place with negligible initial tensions in the bolts, and the tank is filled to the level shown. Find the tension in each bolt at A and at B due to the water pressure.



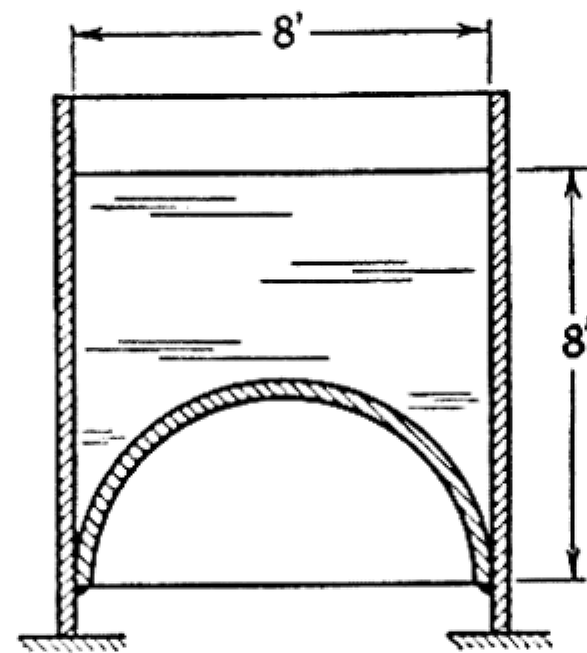
PROB. 418

419. Find the resultant force R exerted by the circular concrete dam section A on the foundation B. Also find the coordinate x of the point C through which the resultant acts Concrete weighs 150 lb./ft.^{3} and the dam is 100 ft. long.

Ans. R = 16,900 tons, x = 19.05 ft.



PROB. 419

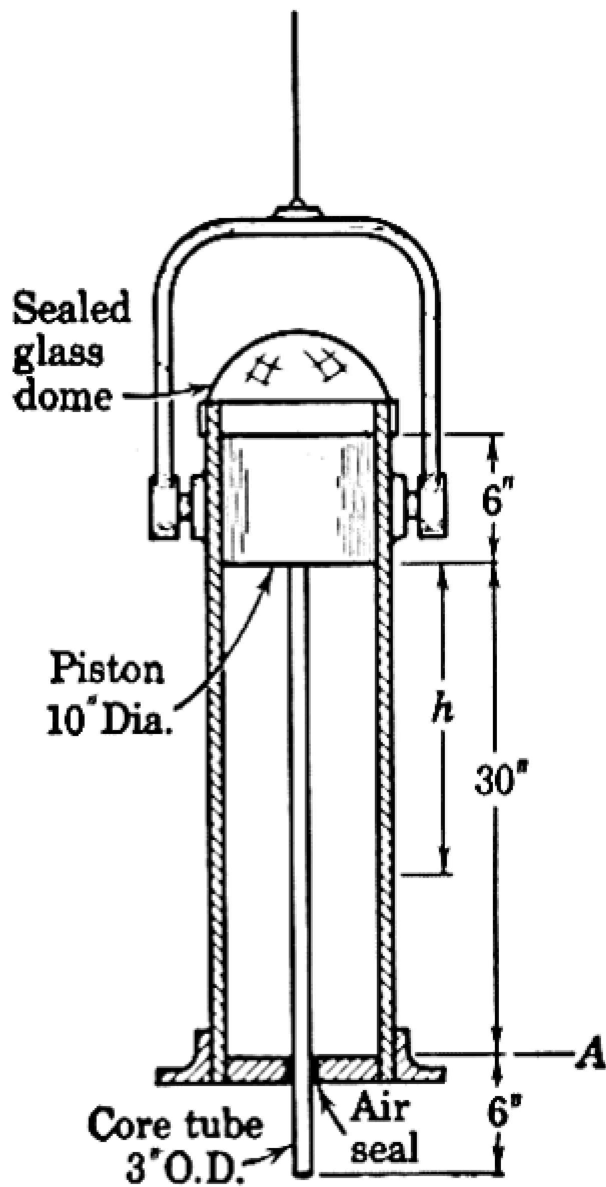


PROB. 420

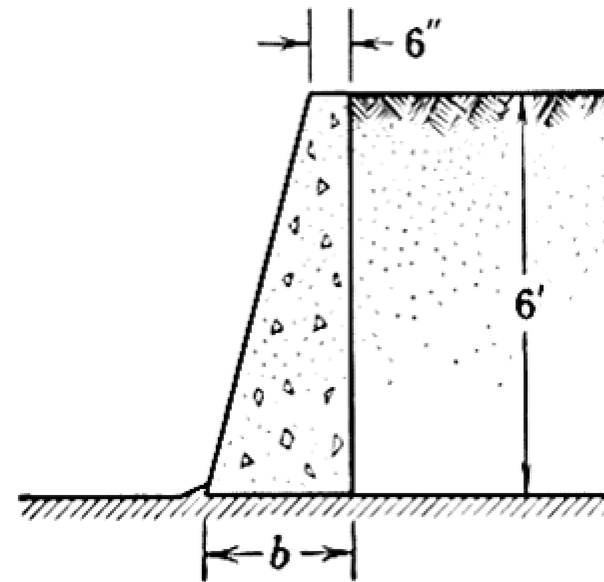
420. The bottom of the fresh-water tank is a hemispherical steel shell ½ in. thick and welded to the cylindrical section around the circular periphery. Determine the vertical force F supported by the weld per unit length. The density of steel is 0.283 lb./in.^{3}

421. A device for taking samples of the ocean's bottom consists of a coring tube which is driven into the bottom by the action of water pressure on the piston. The piston is subject to the pressure of atmospheric air on both sides while the instrument is being lowered to the bottom. To operate the core a

mechanical trip simultaneously releases the piston from its locked position shown and breaks the glass dome above the piston. Water under high pressure moves the piston, and the air in the space below the piston is compressed. Determine the piston travel h for a test run at 50 ft. submergence of section A if the coring tube is closed at its end and does not engage the bottom. Neglect frictional resistance offered by the air seal, the weight of the piston and tube, and any temperature changes. *Ans.* $h = 18$ in.



PROB. 421

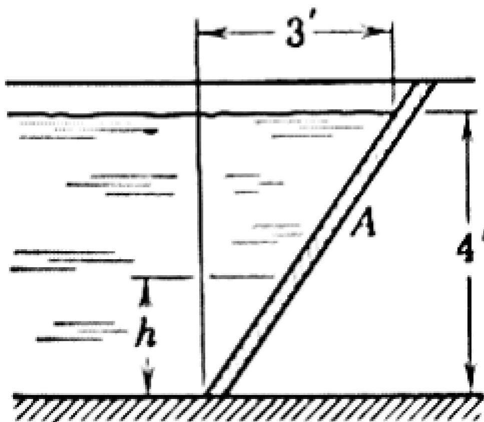


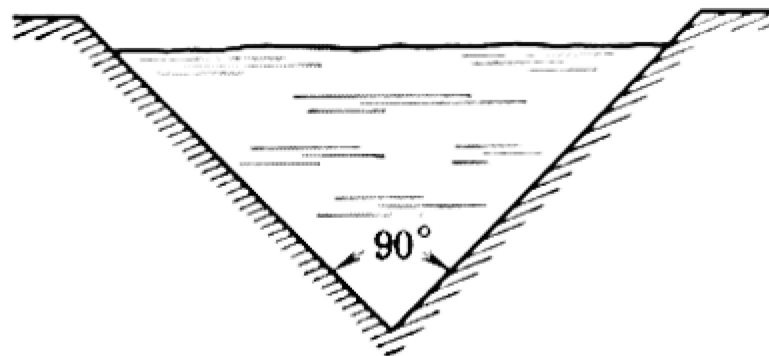
PROB. 422

422. Determine the minimum width b of the footing for the concrete retaining wall shown to prevent tipping of the wall. Concrete weighs 150 lb./ft.^{3}, and the soil above the foundation level may be assumed to be a fluid whose density is 110 lb./ft.^{3}

423. The gate A is fixed in the triangular water channel. Determine the magnitude of the resultant force R exerted on A by the water and the height h of the point on the gate through which the resultant acts.

Ans. $R = 1665$ lb., $h = 2$ ft.



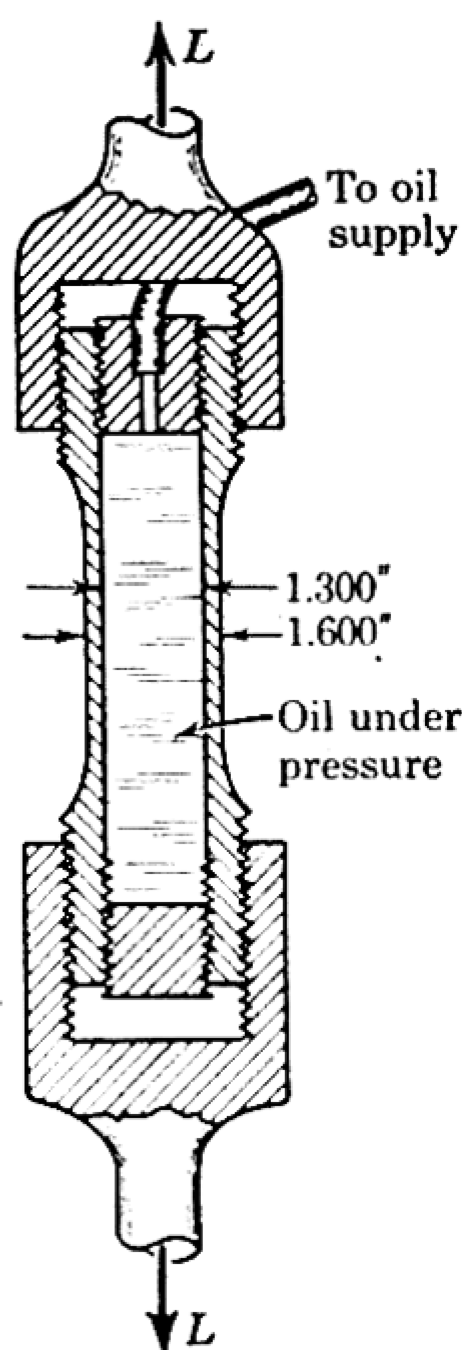


PROB. 423

* **424.** A specimen for measuring the breaking strength of metal under combined stresses (stresses acting on any element in more than one direction) is shown in section. The ratio of longitudinal stress σ_l to tangential stress σ_t is controlled by varying both the oil pressure p and the applied tensile load L. If the ratio of σ_l to σ_t is to be maintained at 4:1 for a certain test, find the neces-

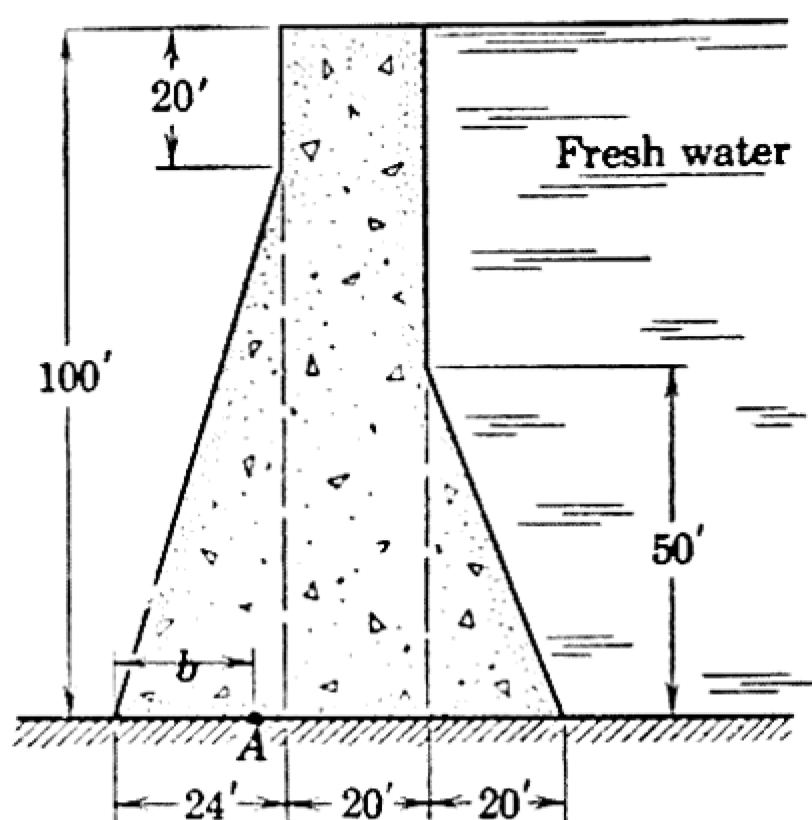
sary variation of p in pounds per square inch gage with L in pounds. If the specimen breaks when L = 32,000 lb., what is the breaking stress σ_l? Assume that the specimen is sufficiently long so that the shape of the ends does not affect the test section.

Ans. $p = 0.0952L$, σ_l = 52,800 lb./in.^{2}



PROB. 424

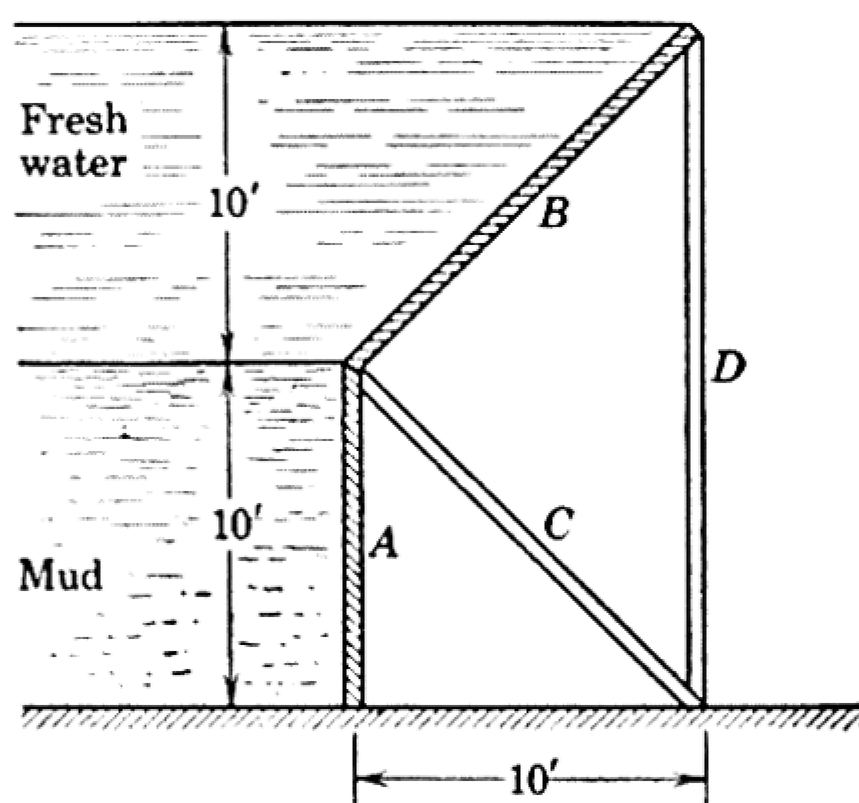
* **425.** A concrete gravity dam has the cross section shown. Determine the resultant force R per foot of dam section exerted by the dam on its foundation and the distance b to the point A through which the resultant acts. Concrete weighs 150 lb./ft.^{3}



PROB. 425

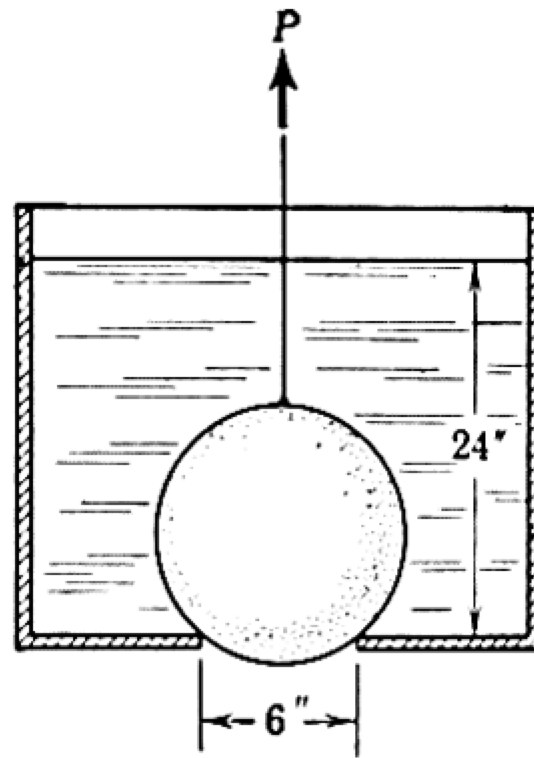
* **426.** A dam consists of the flat plate barriers A and B whose weights are small. Supporting struts C and D are placed every 10 ft. of dam section. A mud sample drawn up to the surface weighs 100 lb./ft.^{3} Determine the compression in C and D. All joints may be assumed to be hinged.

Ans. C = 111,900 lb., D = 20,800 lb.



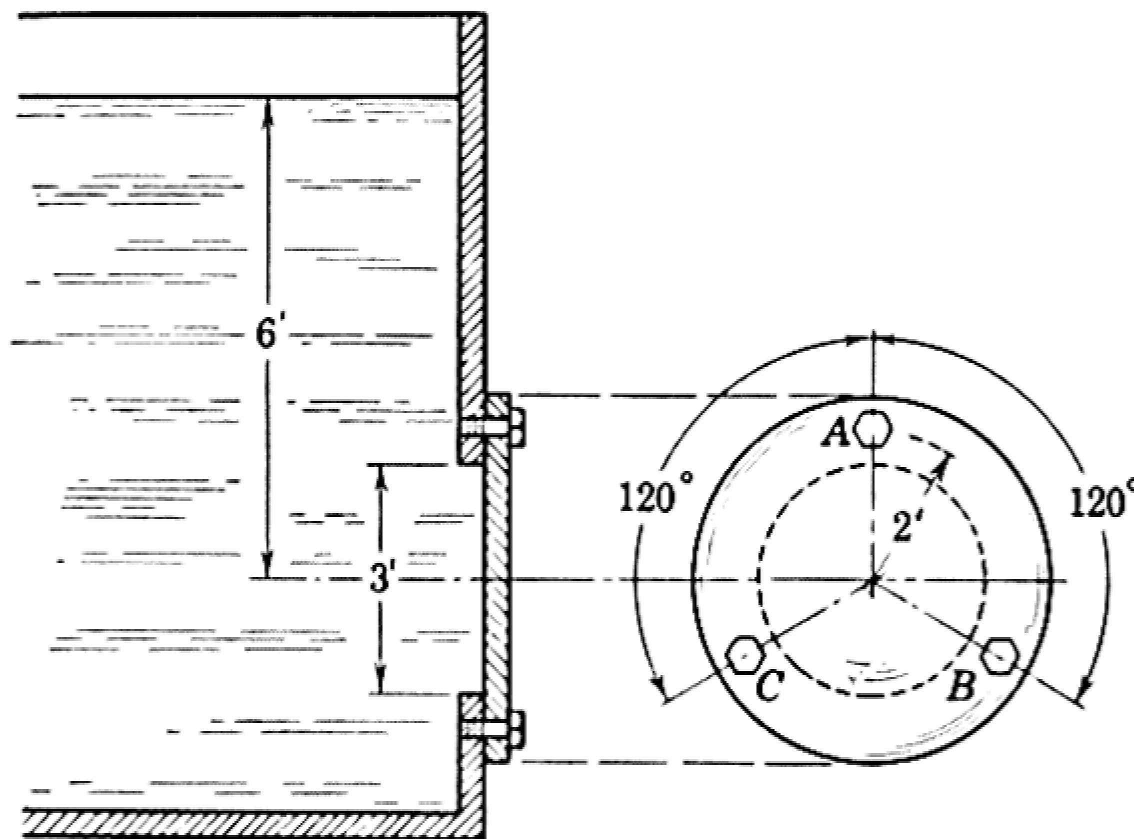
PROB. 426

* **427.** A hollow metal sphere with a diameter of 10 in. and a weight of 8 lb. is used as a valve to close the hole in the fresh-water tank. What force P is required to raise the valve? *Ans.* $P = 14.12$ lb.



PROB. 427

* **428.** The circular cover plate is bolted to the side of a fresh-water tank to seal the 3 ft. diameter opening. Calculate the increase in the bolt tensions at A, B, and C when the tank is filled to the level shown. Assume negligible initial bolt tensions. *Ans.* $A = 800$ lb., $B = C = 925$ lb.



PROB. 428

42. Buoyancy. The principle of buoyancy, the discovery of which is accredited to Archimedes, can be explained in the following manner for any fluid, gaseous or liquid, in equilibrium. Consider a portion of the fluid defined by an imaginary closed surface, as illustrated by the irregular dotted boundary in Fig. 57*a*. If the body of the fluid could be sucked off from within the closed cavity and replaced simultaneously by the forces which it exerted on the boundary of the cavity, Fig. 57*b*, there

would be no disturbance of the equilibrium of the surrounding fluid. Furthermore, a free-body diagram of the fluid portion before removal, Fig. 57*c*, shows that the resultant of the pressure forces distributed over its surface must be equal and opposite to its weight W and must pass through the center of gravity of the fluid element. If the element is replaced by a body of the same dimensions, the surface forces acting on the body held in this position will be identical with those acting on the fluid element. Thus the resultant force exerted on the surface of an object immersed in a fluid is equal and opposite to the weight of fluid displaced

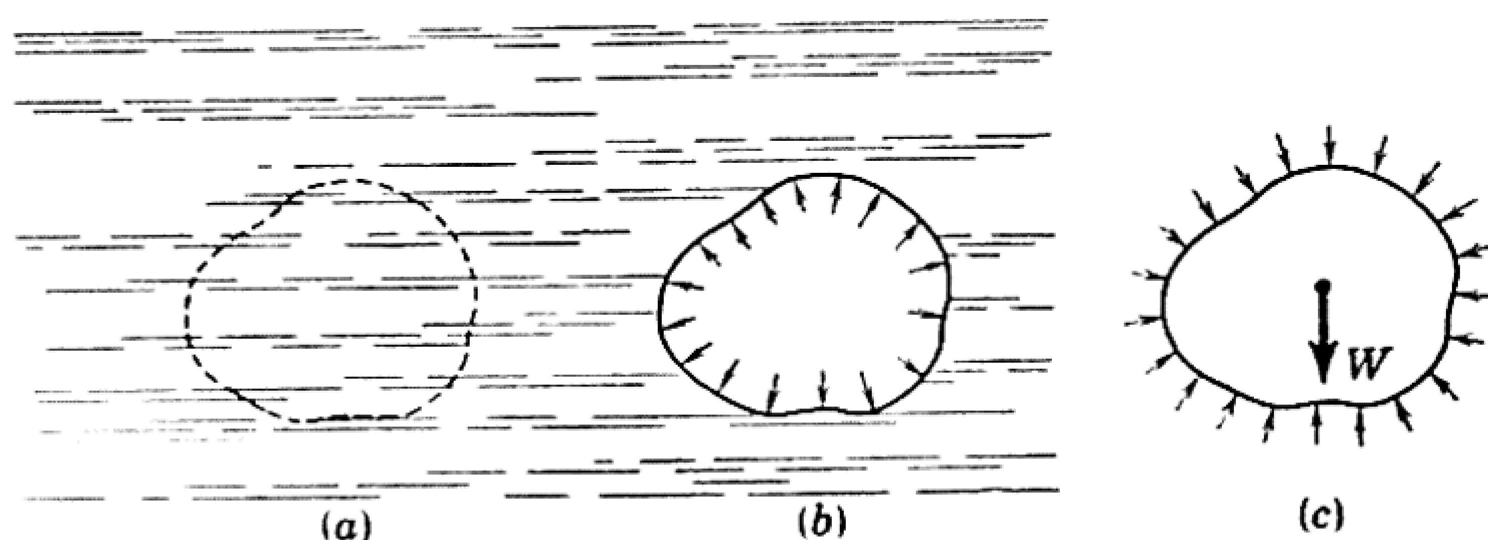


FIG. 57

and passes through the center of gravity of the displaced fluid. This resultant force is the force of buoyancy. In the case of a liquid whose density is constant the center of gravity of the displaced liquid coincides with the centroid of the displaced volume.

It follows from the foregoing discussion that, when the density of an object is less than the density of the fluid in which it is immersed, there will be an unbalance of force in the vertical direction, and the object will rise. The object continues to rise until it comes to the surface of the fluid and then comes to rest in an equilibrium position, assuming that the density of the new fluid above the surface is less than the density of the object. In the case of the surface boundary between a liquid and a gas, such as water and air, the effect of the gas pressure on that portion of the floating object above the liquid is balanced by the added pressure in the liquid due to the action of the gas on its surface.

43. Stability of Floating Bodies. The relations between hull shape and weight distribution in a ship are basic design considerations. The ship shown in Fig. 58*a* is floating in an upright position. Point B is the centroid of the displaced volume and is known as the *center of buoyancy*. The resultant of the forces exerted on the hull by the water is the resultant buoyant force F. Force F passes through B and is equal and opposite to the weight W of the ship. If the ship is caused to roll through an angle α, Fig. 58*b*, the shape of the displaced volume changes, and the

center of buoyancy will shift to some new position such as B'. The point of intersection of the vertical line through B' with the centerline of the ship is called the *metacenter* M, and the distance h of M above the center of gravity G is known as the *metacentric height.* For most hull shapes it is found that the metacentric height remains practically constant for angles of roll up to about 20 deg. When M is above G, as in Fig. 58*b*, there is clearly a righting moment which tends to bring the ship back to its original position. The magnitude of this moment for any particular angle of

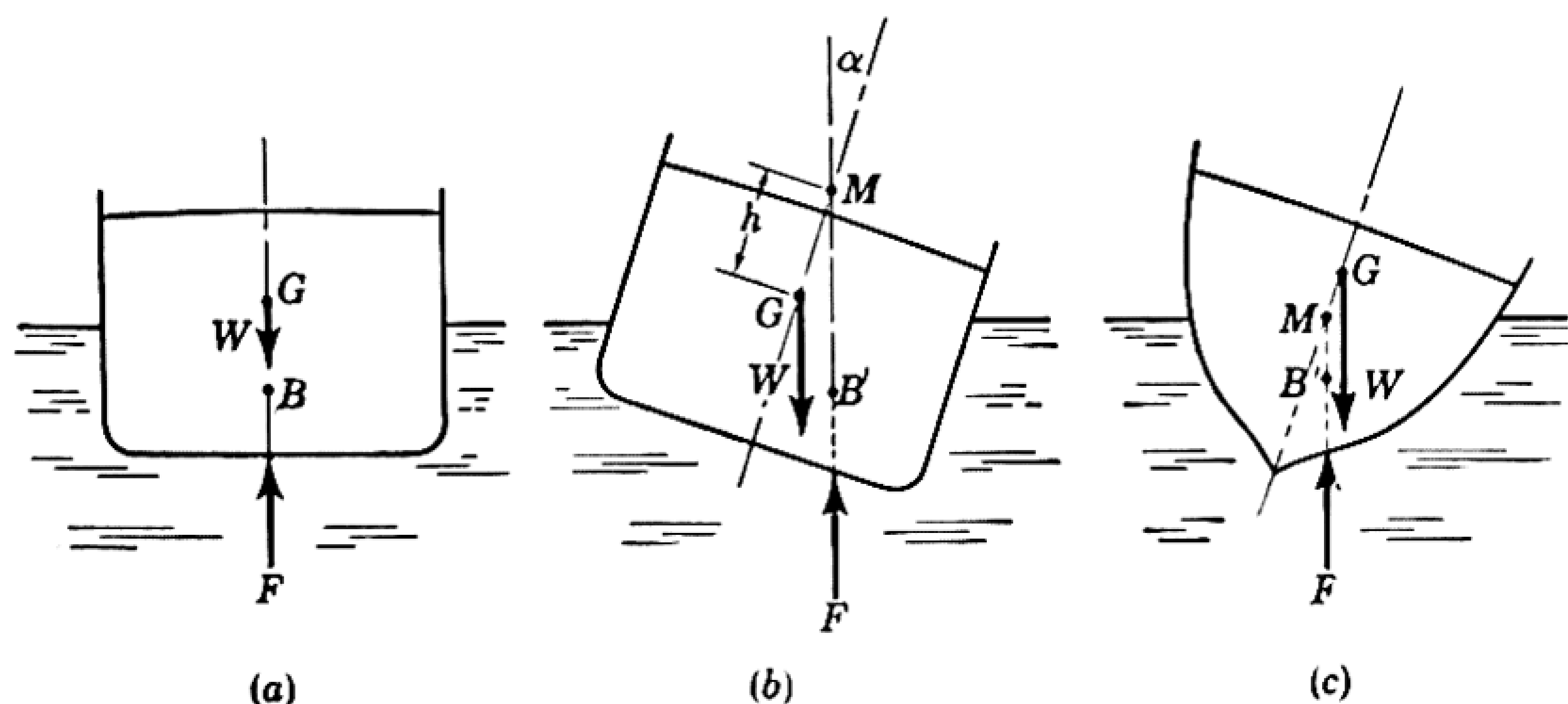


FIG. 58

roll is a measure of the stability of the ship. If M is below G, as for the hull shape of Fig. 58*c*, the moment accompanying any roll is in the direction to increase the roll. This is clearly a condition of instability and must be avoided in the design of any ship.

PROBLEMS

429. A spherical steel buoy 4 ft. in diameter and weighing 350 lb. is to be anchored in salt water 50 ft. below the surface. Find the weight W of scrap iron to be sealed inside the buoy in order that the force on its anchor chain will not exceed 1000 lb. For negligible change in water density, does the buoyancy force depend on the depth of submergence? *Ans.* $W = 795$ lb., no

430. A body whose volume is 8 in.³ weighs exactly 1 lb. when weighed in air at standard conditions. What would be its weight W when weighed in a vacuum if the density of air is 0.0766 lb./ft.³? *Ans.* $W = 1.000355$ lb.

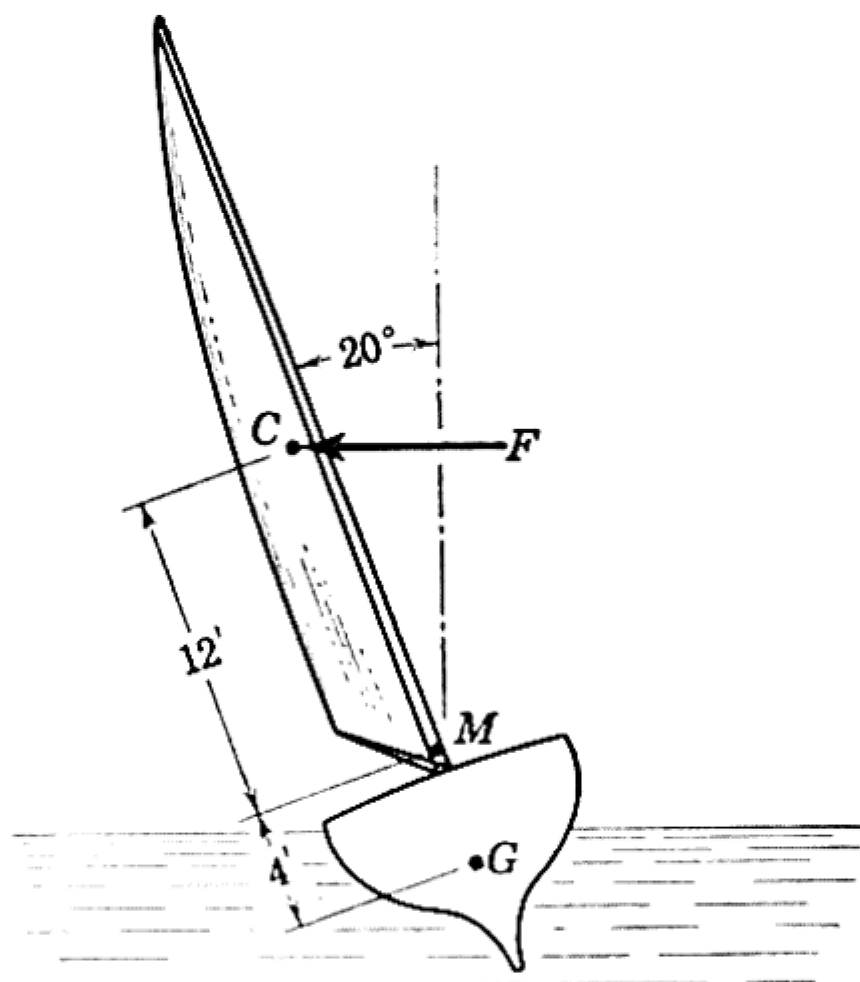
431. A body weighs 50 lb. in air and when immersed in fresh water "weighs" 25 lb. Find its volume V and density μ.

432. The density of ice is 56 lb./ft.³ Determine the ratio n of depth below water to height above water for an iceberg floating in salt water and having rectangular proportions.

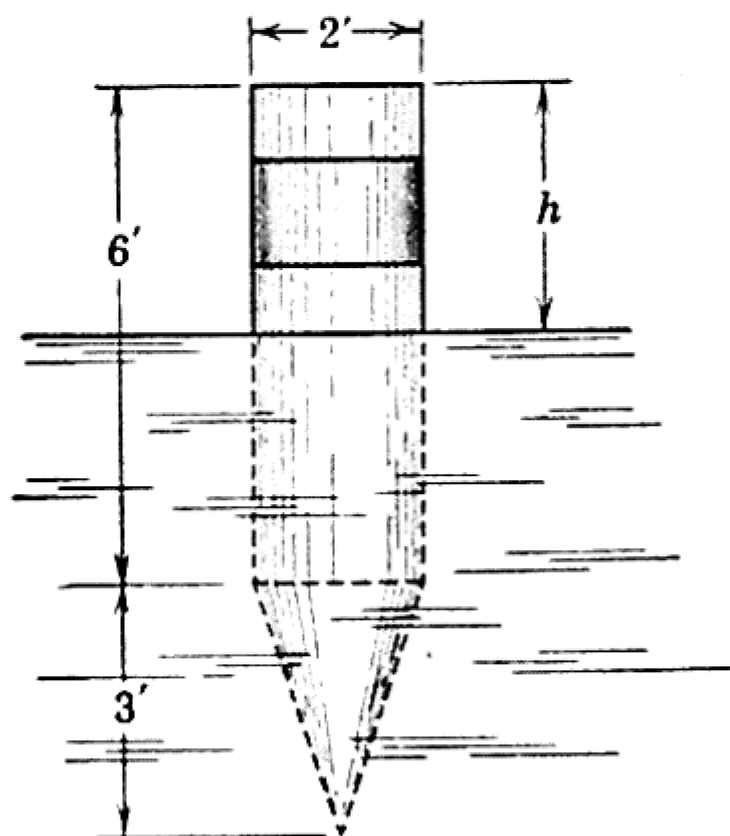
433. A sailboat which displaces 180 ft.³ of salt water is heeled to an angle of 20 deg. under the action of a beam wind as shown. This action is represented

by a resultant force F acting at the center of effort C. Determine F if C and the metacenter M are located as shown in the figure. Assume the resultant lateral resistance of the water on the hull to act through G.

Ans. $F = 1048$ lb.



Prob. 433



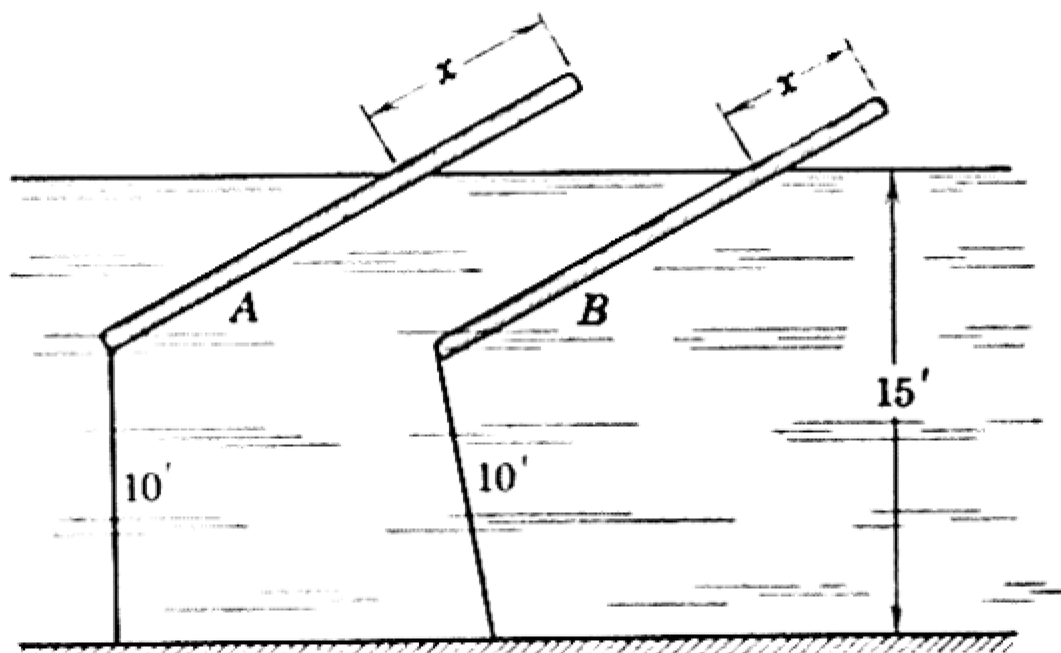
Prob. 434

434. A marker buoy consisting of a cone and cylinder has the dimensions shown and weighs 625 lb. out of water. Determine the protrusion h when the buoy is floating in salt water. The buoy is weighted so that a low center of gravity insures stability.

435. A ship moves from a fresh-water anchorage to a berth in salt water. After the ship takes on 50,000 gal. of fuel oil (specific gravity 0.89), it is noted that the draft marks read the same as previously in fresh water. Determine the displacement W (total weight) of the ship in long tons before fueling.

Ans. $W = 6460$ long tons

436. One end of a 25 lb. wooden pole 3 in. in diameter and 18 ft. long is anchored with 10 ft. of light cable to the bottom of a fresh-water lake where the depth is 15 ft. Which of the two configurations shown describes the equilibrium position? Find the length x of the pole protruding above the water.



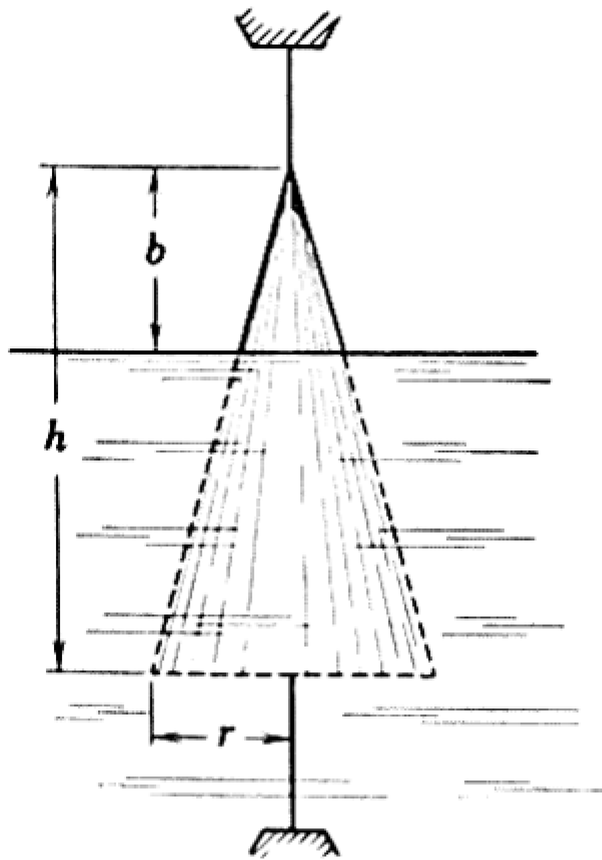
Prob. 436

437. A steel chain weighs 0.78 lb./ft. and has a breaking strength of 4500 lb. If it is lowered into the ocean, determine the depth h which could be reached before the chain breaks. *Ans.* $h = 6630$ ft.

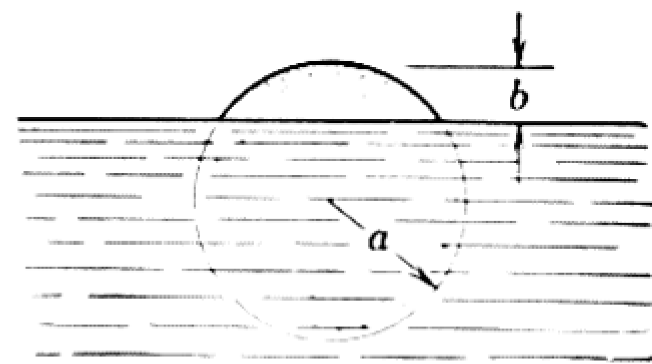
438. A captive balloon weighing 1 lb. empty is inflated with helium to a diameter of 6 ft. and secured at an altitude of 50 ft. by a wire of that length weighing 0.01 lb./ft. Determine the weight W of the helium in the balloon if the tension in the wire at the ground is 5.90 lb. The density of the air is 0.0766 lb./ft.^{3}

439. As a demonstration of the principle of buoyancy it is proposed to construct a light spherical shell made from aluminum $\frac{1}{16}$ in. thick and containing helium so that it will "float" in air. In order not to strain the shell, the helium will be introduced at atmospheric pressure, at which condition its density is $\frac{4}{29}$ that of the surrounding air. The density of air at atmospheric conditions is 0.0766 lb./ft.^{3}, and aluminum weighs 168 lb./ft.^{3} Determine the required radius r of the shell. *Ans.* $r = 39.7$ ft., impractical

440. A solid right circular cone of base radius r, altitude h, and density μ_c is mounted on a fixed vertical wire through a small axial hole so that the cone is able to slide freely along the wire. If the cone is allowed to float in a liquid of density μ_l, find its height b above the surface.



PROB. 440



PROB. 441

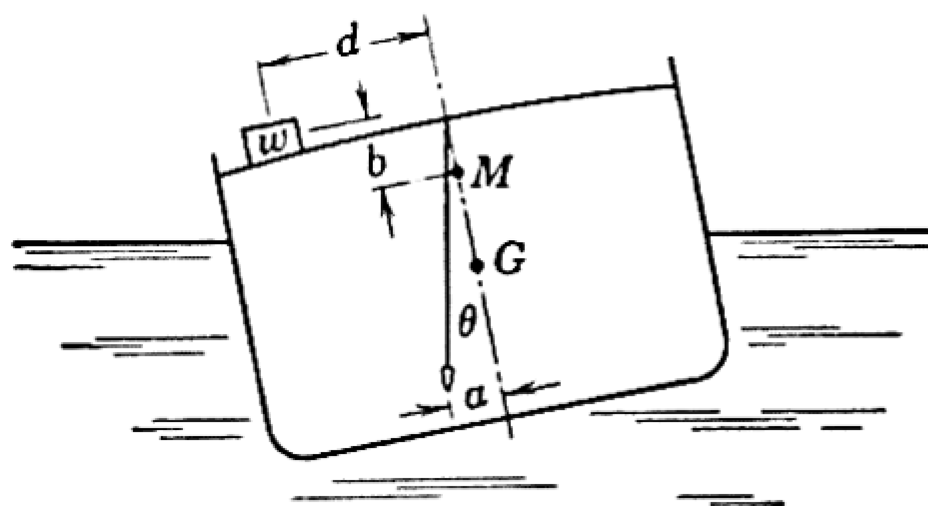
441. A sphere of radius a and density μ_s floats in a liquid of density μ_l. Find the relation between the radius, the densities, and the height b which the sphere protrudes above the surface.

Ans. $1 - \dfrac{\mu_s}{\mu_l} = \dfrac{3}{4}\dfrac{b^2}{a^2}\left(1 - \dfrac{b}{3a}\right)$

* **442.** The accurate determination of the vertical position of the center of gravity G of a ship is difficult to achieve by calculation. It is more easily obtained by a simple inclining experiment on the completed ship. With reference to the figure, a known external weight w is placed a distance d from the center line, and the angle of list θ is measured by means of the deflection of a plumb bob. The displacement of the ship and the location of the metacenter M are known. Calculate the metacentric height GM for a 7500 ton ship (long tons)

inclined by a 70,000 lb. weight placed 22 ft. from the center line if a 20 ft. plumb bob is deflected a distance $a = 12$ in. The weight w is at a distance $b = 5$ ft. above M.

Ans. $GM = 1.85$ ft.



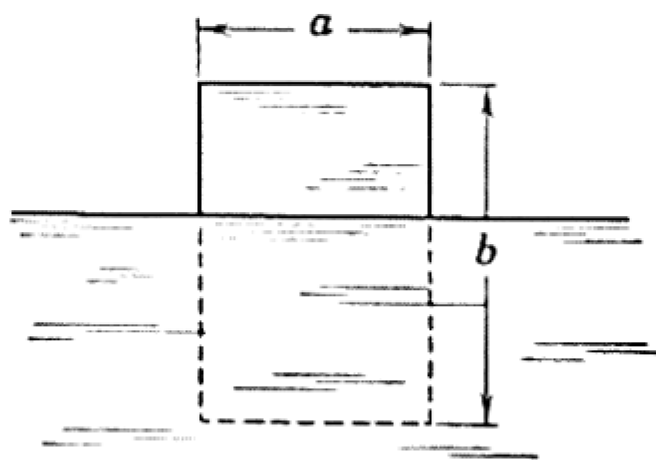
PROB. 442

* **443.** A spherical balloon is made from material which stretches in proportion to the force applied to it. Since the amount of stretch of any element depends directly on the original length and the force acting, the equation $u = Krp$ may be written, where u is the increase in radius of the balloon, r is the radius of the unstretched balloon, p is the internal gas pressure (gage), and K is the constant of proportionality. As the balloon is filled with gas, such as helium, under pressure, the sphere expands and creates greater buoyancy in the air. At the same time the weight of the balloon is increasing as more gas is injected. Find the pressure p for which the balloon has a maximum lift. The density of air is μ_a, and that of the gas is $\mu_g = k(p + p_0)$ for constant temperature, where k is a constant and where p_0 is the atmospheric pressure.

Ans. $p = \frac{1}{4}\left(\frac{3\mu_a}{k} - \frac{1}{K} - 3p_0\right)$

* **444.** Determine the minimum ratio of a to b for stability of the rectangular block of density μ_b for very small angles of list. The density of the liquid is μ_l.

Ans. $\frac{a}{b} = \sqrt{6\frac{\mu_b}{\mu_l}\left(1 - \frac{\mu_b}{\mu_l}\right)}$



PROB. 444

CHAPTER VI

Beams

44. Introduction.- Structural members which offer resistance to bending caused by applied loads are known as beams. Most beams are long prismatical bars, and the loads are usually applied normal to the axes of the bars. Beams are undoubtedly the most important of all structural members, and the basic theory underlying their design must be thoroughly understood. The analysis of the load-carrying capacities of beams consists, first, in establishing the equilibrium requirements of the beam as a whole and any portion of it considered separately. Second, the relations between the resulting forces and the accompanying internal resistance of the beam to support these forces are established. The first part of this analysis requires the application of the principles of statics, while the second part of the problem involves the strength characteristics of the material and is usually treated in the study of strength of materials. This chapter concerns the first aspect of the problem only.

Beams supported in such a way that the external support reactions can be calculated by the methods of statics alone are called *statically determinate* beams. A beam that has more supports than are necessary to provide equilibrium is said to be *statically indeterminate,* and it is necessary to consider the load-deformation properties of the beam in addition to the equations of statical equilibrium to determine the support reactions. In Fig. 59 are shown examples of both types of beams. In this chapter only statically determinate beams are analyzed.

In the analysis of simple trusses the resultant of the forces acting on the section of a cut member was a single force, either tension or compression, in the direction of the member. Bending either was neglected or was absent. In the analysis of beams the resultant of the forces acting on a transverse section of the beam cannot, in general, be represented in terms of a single force but must be expressed by a force and a couple. It should be recalled from Chapter II that the resultant of any coplanar force system is expressible as a resultant force acting at any point and a corresponding couple. In some beams the actual distribution of force over a cross section is exceedingly complex and requires for a complete description extended analysis involving the load-deformation properties

of the beam materials. In the present discussion only the *resultant* of the force distribution across any section of a statically determinate beam will be considered. This resultant, required for equilibrium of the portion of the beam on each side of the section, may be expressed in terms of a force and a couple.

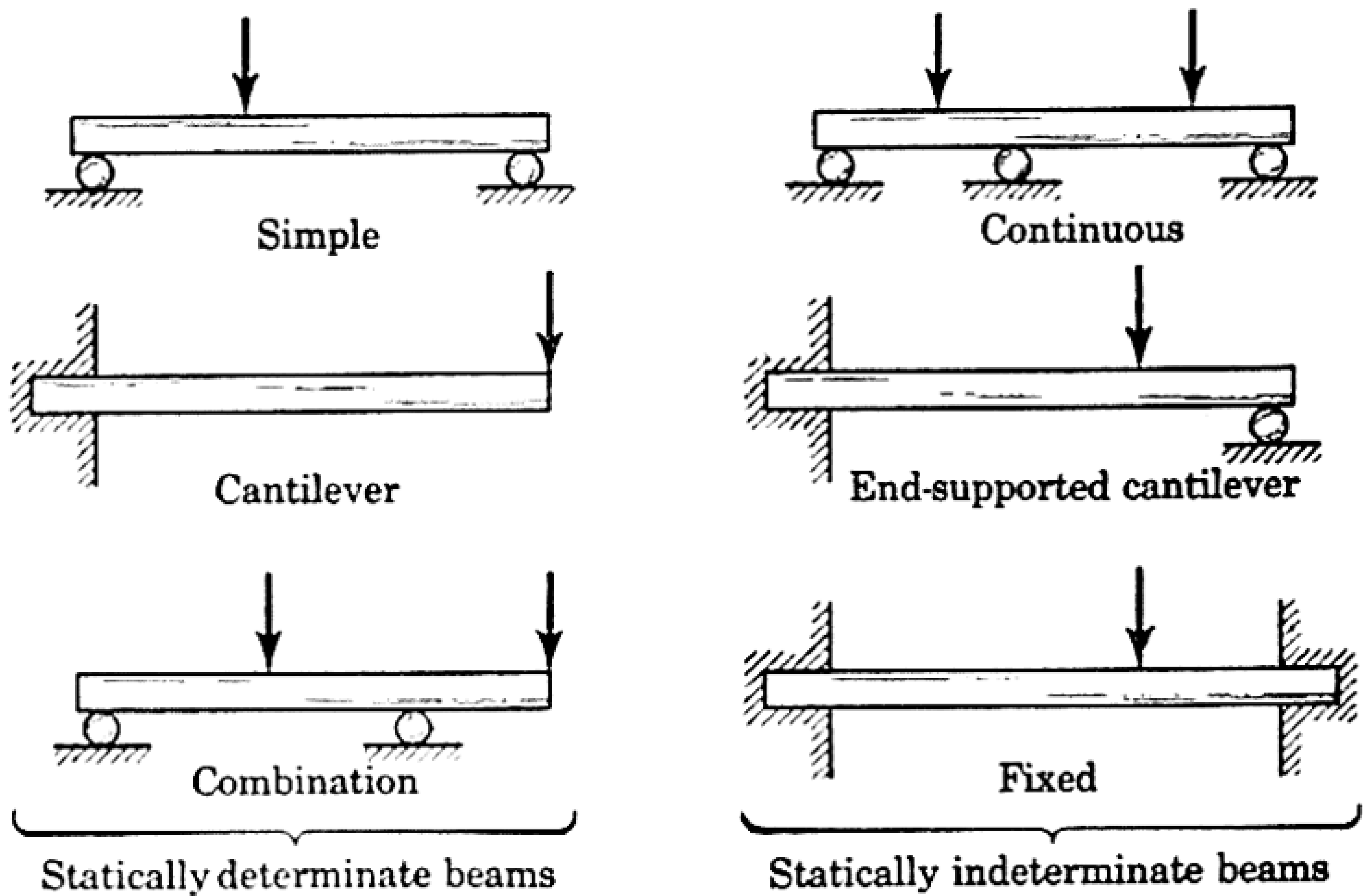


Fig. 59

In Fig. 60 are shown two portions of a beam cut by an imaginary section. The resultant force acting on either section is represented in terms of its two components Q and F and is shown as acting at the center line of the beam. The force Q, tangent to the section, is a *shear* force. The *average shear stress* over the cross section is Q/A, where A is the area of the section. The force F as shown is a tensile force, and the *average tensile stress* over the section is F/A. The couple M is known as the *bending moment* at this section. It should be noted that the directions of Q, F, and M are reversed on the two sections by the principle of action and reaction. Most beams are supported and loaded by forces normal to the beam axis, and in this case the tensile force F or a compression force C is either zero or negligible.

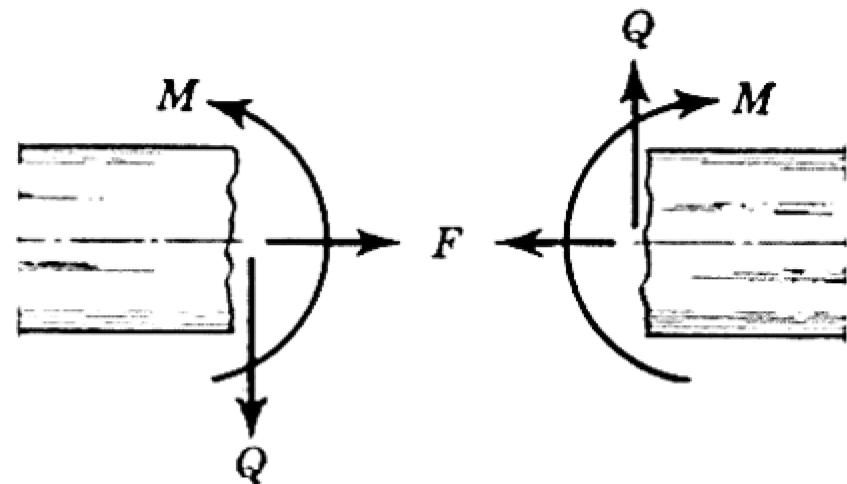


Fig. 60

It is of advantage to adopt conventions for the positive directions of Q and M. Figure 61 shows the conventions generally used, and it is seen that the representation in Fig. 60 is in the positive sense. It is frequently impossible to ascertain at a glance whether the shear and moment on a

certain section of a loaded beam are positive or negative. For this reason it will be found advisable to represent Q and M in their positive directions on the free-body diagrams and let the algebraic signs of the calculated values indicate the proper direction.

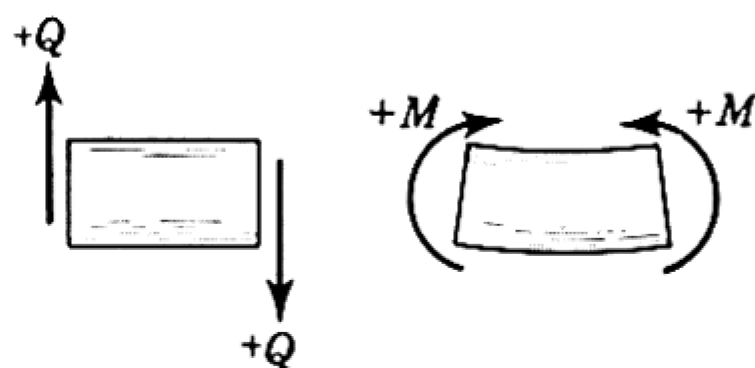


Fig. 61

As an aid to the physical interpretation of the bending couple M consider the beam shown in Fig. 62 bent by the two equal and opposite moments applied at the ends. The cross section of the beam is taken to be that of an H-section with a very narrow center web and heavy top and bottom flanges. For this beam the load carried by the small web may be neglected compared with that carried by the two flanges. It should be perfectly clear that the upper flange of the beam is shortened and is under compression while the lower flange is lengthened and is under tension. The resultant of the two forces, one tension and the other compression, acting on any section is a couple and is the value of the bending moment on the section. If a beam of some other cross section were loaded in the same way, the distribution of force over the cross section would be different, but the resultant would be the same couple.

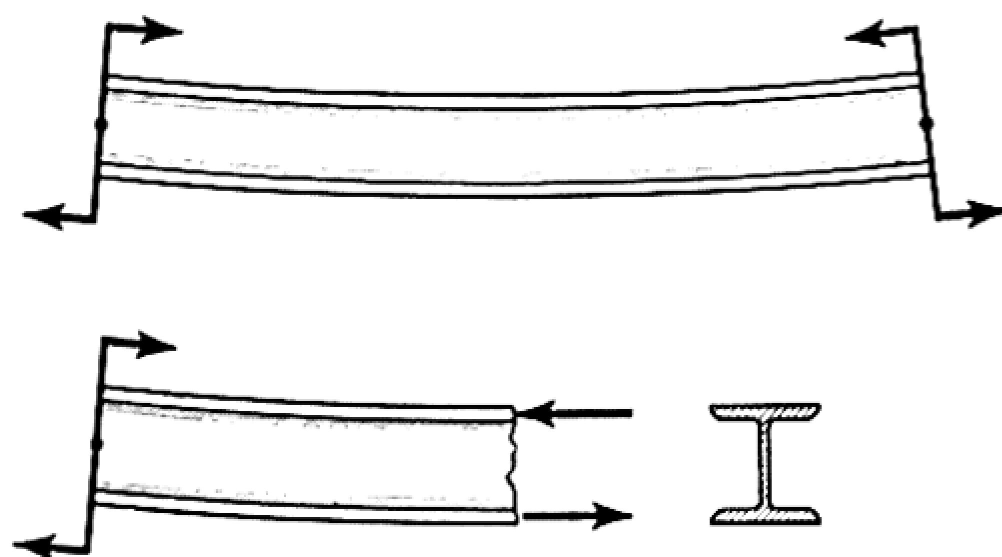


Fig. 62

45. Beams with Concentrated Loads. Consider a simple beam, Fig. 63*a*, whose weight is negligible compared with the applied concentrated load P. The support reactions R_1 and R_2 are easily computed by the requirements of equilibrium for the beam as a whole and are

$$R_1 = \frac{b}{l} P \qquad \text{and} \qquad R_2 = \frac{a}{l} P.$$

Now consider a section A, Fig. 63*b*, which divides the beam into two portions, each of which is in equilibrium. This condition is shown by the free-body diagrams in the *c*-part of the figure. Since no applied hori-

zontal force acts on either portion, there is no resultant tension or compression on the cut section, and the resultant force is shear only. The shear Q and the bending moment M are shown in their positive directions on each of the sections illustrated. Equilibrium in the vertical

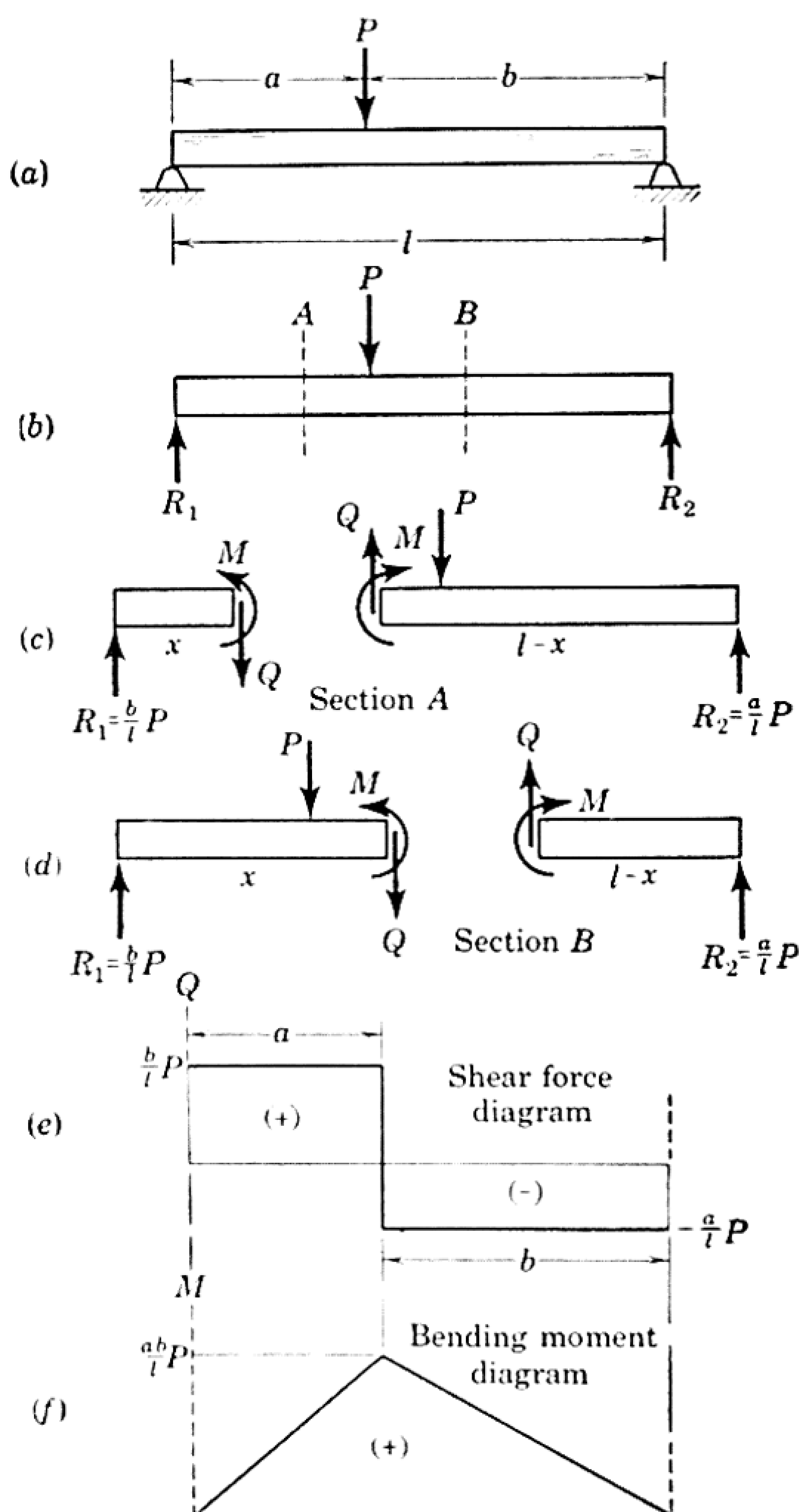


Fig. 63

direction for the left-hand portion of the beam requires that $Q = +\frac{b}{l}P$. Equilibrium of moments for the same body may be expressed by equating to zero the moments about a point on the cut section. Thus $M = +\frac{b}{l}Px$. The identical results for Q and M may be obtained from the

equilibrium of the right-hand member. Consequently it should be observed that the values of Q and M at any section may be determined from the equilibrium of either portion of the beam.

To obtain the shear and bending moment to the right of P a section B is considered. The free-body diagrams of both portions of the beam for section B are shown in Fig. 63*d*. Both Q and M are assigned in their positive sense. It should be noted that P now acts on the left-hand section. Equilibrium of the vertical forces for the left-hand section gives $Q + P = \frac{b}{l} P$, or $Q = -P\left(1 - \frac{b}{l}\right) = -\frac{a}{l} P$. Thus the shear changes sign abruptly at the position of the load P. Equilibrium of moments about a point on section B for the same body gives

$$M + P(x - a) - \frac{b}{l} Px = 0,$$

or

$$M = +Pa\left(1 - \frac{x}{l}\right).$$

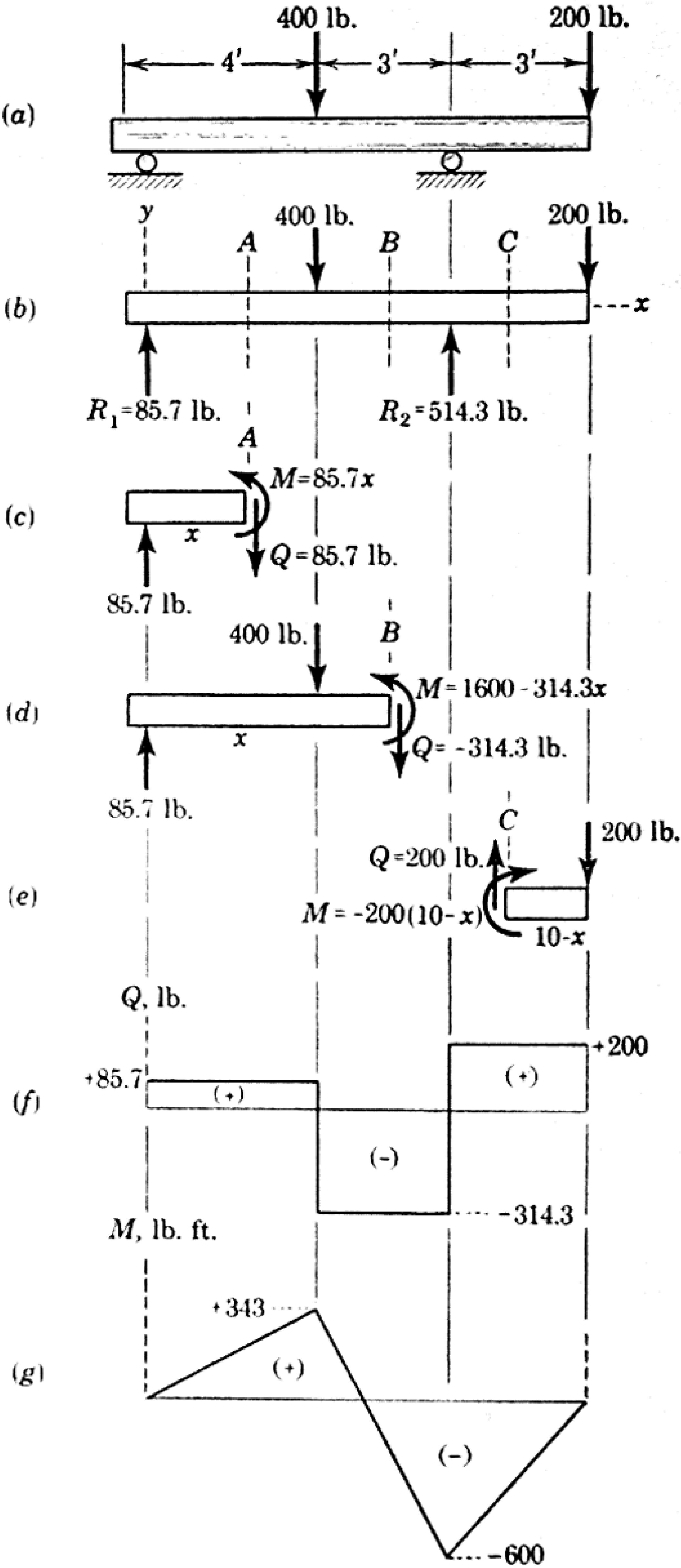
Again the same results may be obtained from the right-hand section.

The variation of shear and bending moment over the length of the beam is best shown graphically. The expressions for Q and M just obtained, when plotted against the horizontal dimension x, give the *shear-force* and *bending-moment diagrams* for the beam as shown in Figs. 63*e* and *f*. The maximum magnitude of the bending moment is usually the primary consideration in the design or selection of a beam, and its value and position should be determined.

There are many possible combinations of loading and supports. In each case the procedure outlined in the preceding problem is followed. The beam is divided into two parts, and the equilibrium requirements for either of the parts will yield the relations between shear force Q and bending moment M as functions of the position along the beam. The section of the beam which involves the least number of forces usually yields the simpler solution. The values of shear and bending moment are discontinuous at the positions of concentrated loads, and sections must be selected on one side or the other of a concentrated load but not at the load.

SAMPLE PROBLEMS

445. Draw the shear-force and bending-moment diagrams for the beam shown in the *a*-part of the illustration and determine the maximum magnitude of the bending moment and its location.



Prob. 445

Solution: From the free-body diagram of the beam in part *b* the reactions are

$[\Sigma M_{R_1} = 0]$ $7R_2 - 4 \times 400 - 10 \times 200 = 0,$ $R_2 = 514.3$ lb.,

$[\Sigma F_y = 0]$ $R_1 + 514.3 - 400 - 200 = 0,$ $R_1 = 85.7$ lb.

The free-body diagram of the part of the beam to the left of section A is shown in part *c* of the figure. The shear Q and moment M are shown in their positive directions. Equilibrium requires

$[\Sigma F_y = 0]$ $Q = 85.7$ lb.,

$[\Sigma M_A = 0]$ $M = 85.7x.$

These values hold between $x = 0$ and $x = 4$ ft.

The next interval is analyzed by the free-body diagram of the entire portion of the beam to the left of section B. Again Q and M are shown in their positive directions, part *d*. Equilibrium gives

$[\Sigma F_y = 0]$ $Q + 400 - 85.7 = 0,$ $Q = -314.3$ lb.,

$[\Sigma M_B = 0]$ $M + 400(x - 4) - 85.7x = 0,$ $M = 1600 - 314.3x.$

Lastly, from the simpler of the two diagrams for section C, shown in part *e* of the figure, literal application of the equilibrium equations gives

$[\Sigma F_y = 0]$ $Q = 200$ lb.,

$[\Sigma M_C = 0]$ $M + 200(10 - x) = 0,$ $M = -200(10 - x).$

It should be noted that M and Q are shown in their positive directions on the free-body diagram.

In parts *f* and *g* are plotted the shear-force and bending-moment diagrams for the three intervals of the beam. The maximum magnitude of the moment and its position are clearly

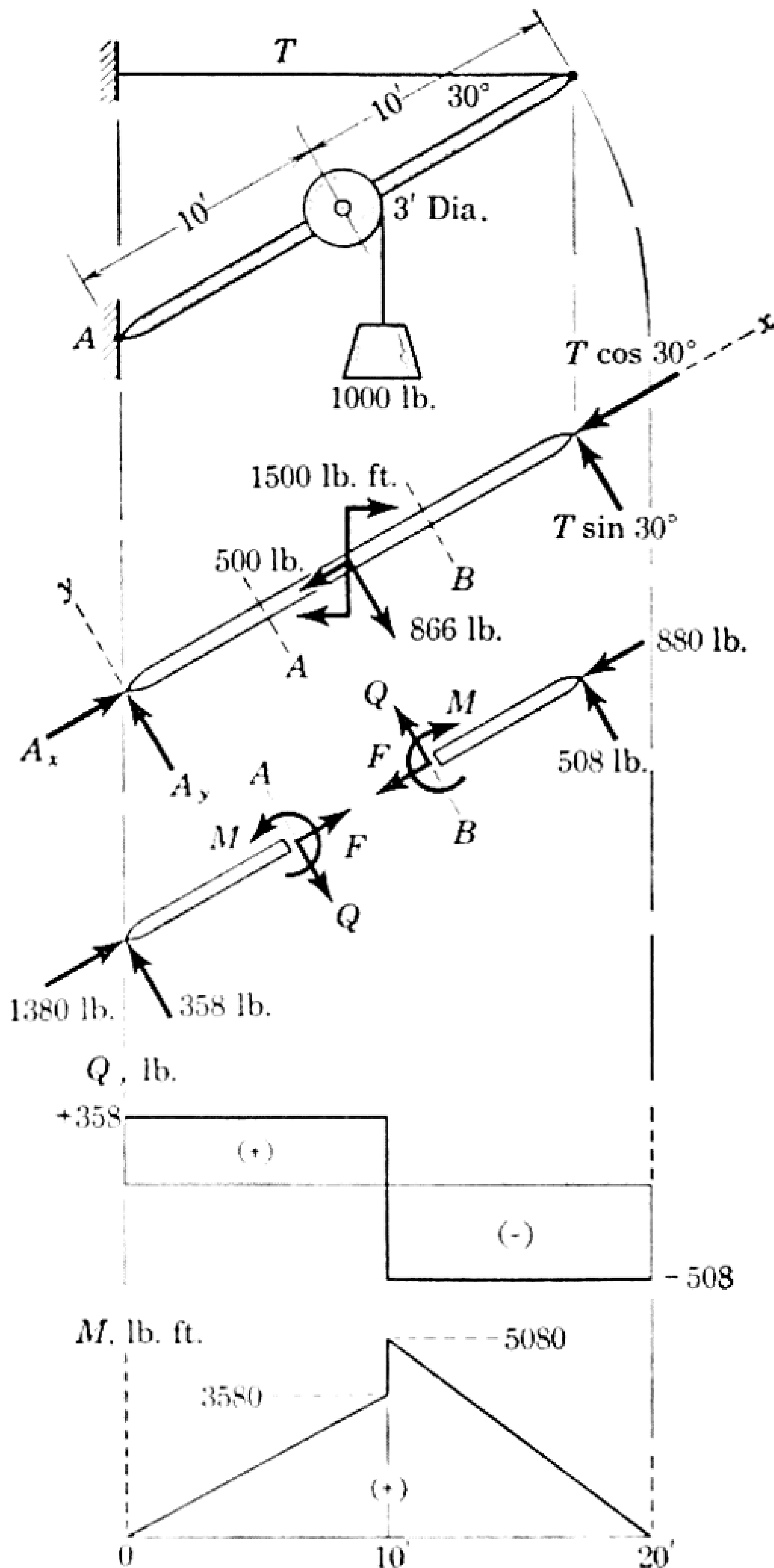
$$M = 600 \text{ lb. ft. at } x = 7 \text{ ft.} \qquad \textit{Ans.}$$

446. The hoisting drum is temporarily locked to its shaft which in turn is rigidly attached to the boom. Determine the variation of shear force (normal to the boom axis) and bending moment over the length of the boom.

Solution: The forces external to the boom are first determined with the aid of the free-body diagram shown. The tension T and the reaction at A are represented by their components along and normal to the boom. Isolation of the pulley alone enables the reactions on the boom at the pulley shaft to be correctly drawn. The 1000 lb. bearing force is represented by its components along and normal to the boom. The 1500 lb. ft. twist is applied through the shaft fixed in the boom.

Two sections A and B are required. Isolation of the part of the boom below section A and isolation of the part above section B are separately shown with

all forces and moments external to both parts indicated. On each of the two free-body diagrams Q and M are shown in their positive directions, and, to be



PROB. 446

consistent, the axial force F is also shown positive as tension. The values of Q, M, and F for the two sections are *not* equal since they act on different sections and do not constitute actions and reactions.

For the lower member,

$[\Sigma F_y = 0]$ $\qquad Q = 358$ lb.,

$[\Sigma M_A = 0]$ $\qquad M = 358x$,

$[\Sigma F_x = 0]$ $\qquad F = -1380$ lb.

For the upper member,

$[\Sigma F_y = 0]$ $Q = -508$ lb.,

$[\Sigma M_B = 0]$ $M = 508(20 - x)$,

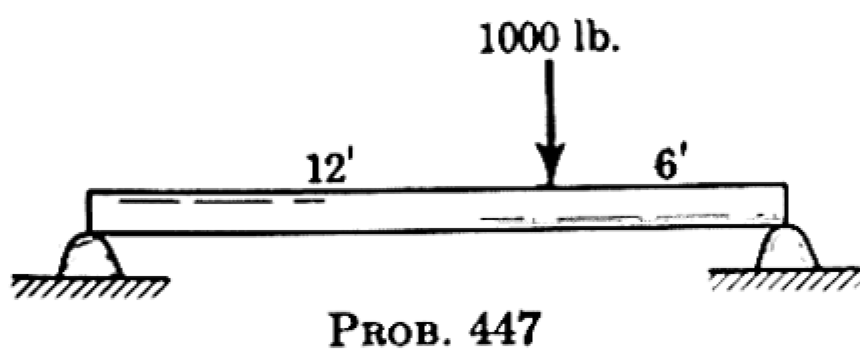
$[\Sigma F_x = 0]$ $F = -880$ lb.

The shear-force and bending-moment diagrams are shown at the bottom of the figure. The discontinuity in M at $x = 10$ ft. is due to the applied moment at that point. The increment of this discontinuity, 5080 minus 3580, equals this applied moment. Below the pulley the axial force is 1380 lb. compression, and above the pulley the axial force is 880 lb. compression.

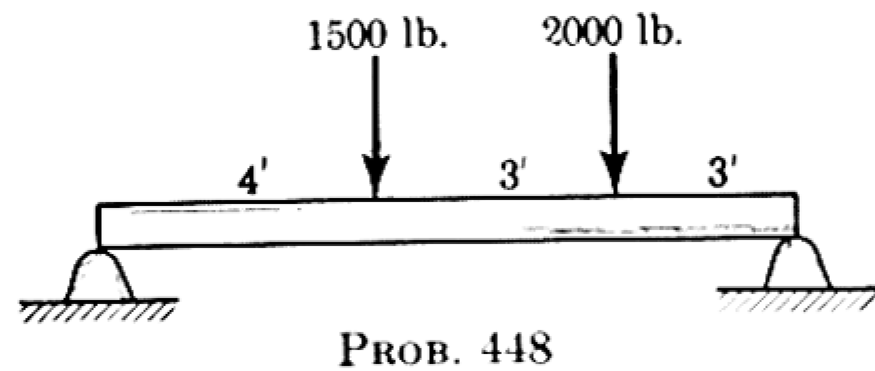
PROBLEMS

Neglect the weight of the beams in the following problems.

447. Draw the shear and moment diagrams for the simply supported beam and determine the maximum moment M.



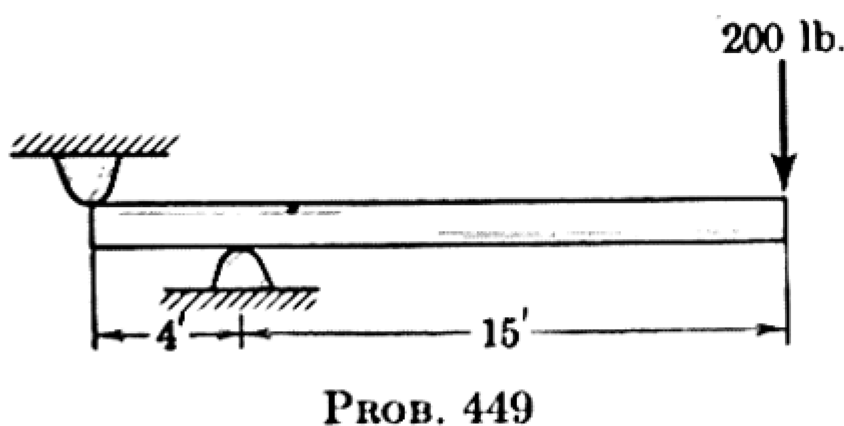
Prob. 447



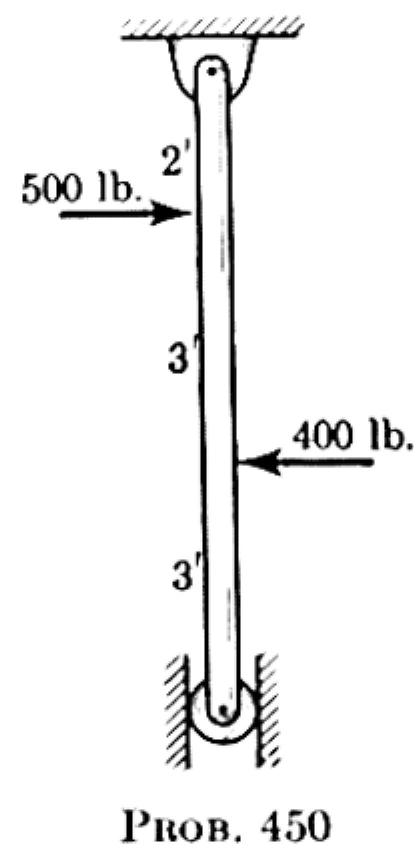
Prob. 448

448. Draw the shear and moment diagrams for the combination beam and determine the maximum moment M. *Ans.* $M = 6000$ lb. ft.

449. Draw the shear and moment diagrams for the cantilever beam.



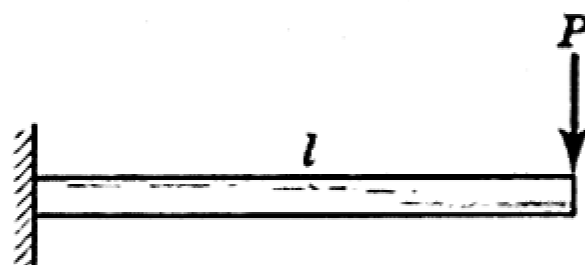
Prob. 449



Prob. 450

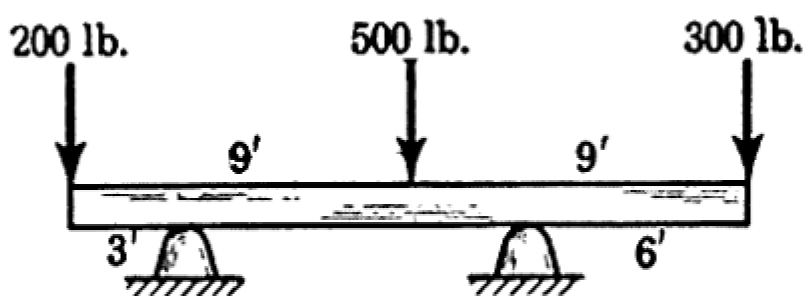
450. Draw the shear and moment diagrams for the vertical member and find the maximum moment M.

451. Draw the shear and moment diagrams for the cantilever beam.

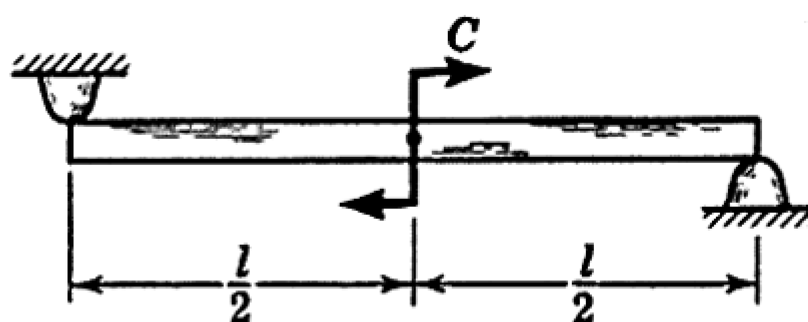


PROB. 451

452. Draw the shear and moment diagrams for the loaded beam and find the maximum magnitude of the moment M. *Ans.* $M = 1800$ lb. ft.



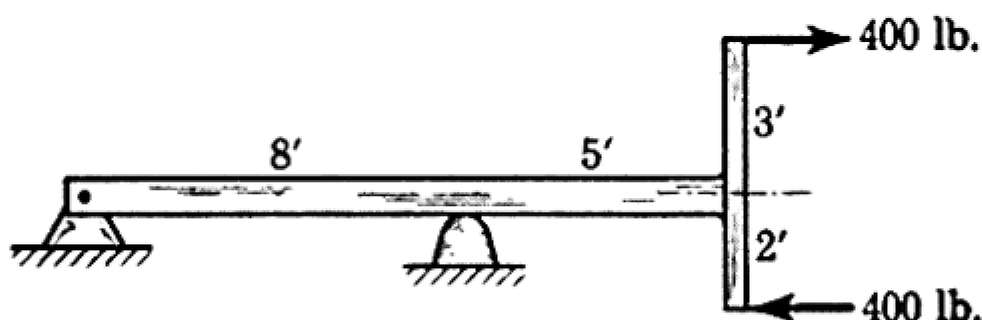
PROB. 452



PROB. 453

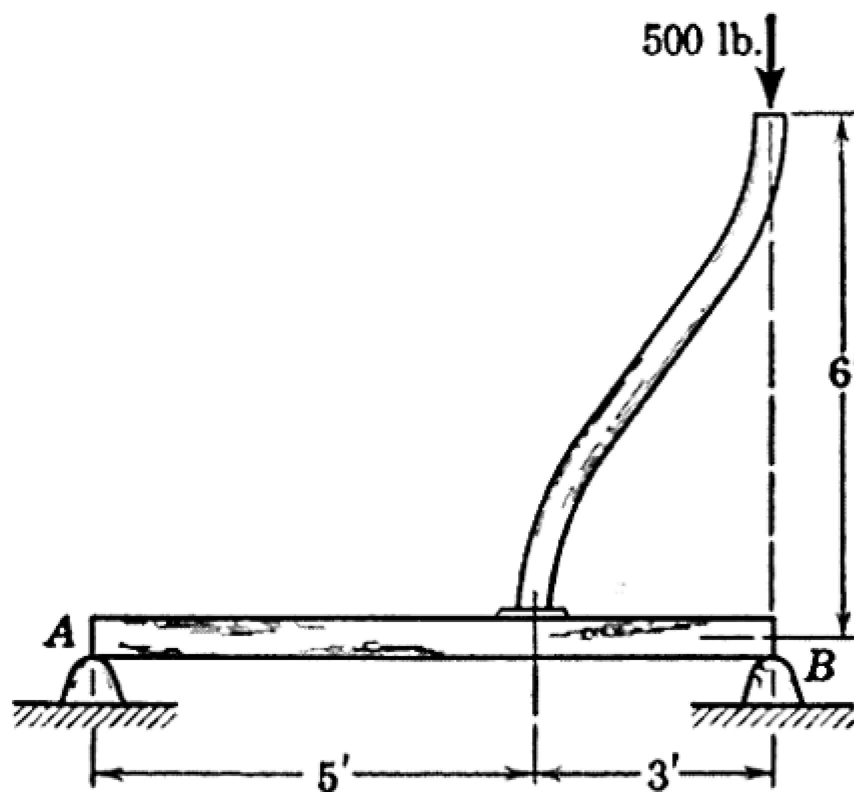
453. Draw the shear and moment diagrams for the beam loaded in its center by the couple C. What is the magnitude of the bending moment M on a section of the beam which approaches the midpoint?

454. Draw the shear and moment diagrams for the beam loaded as shown.



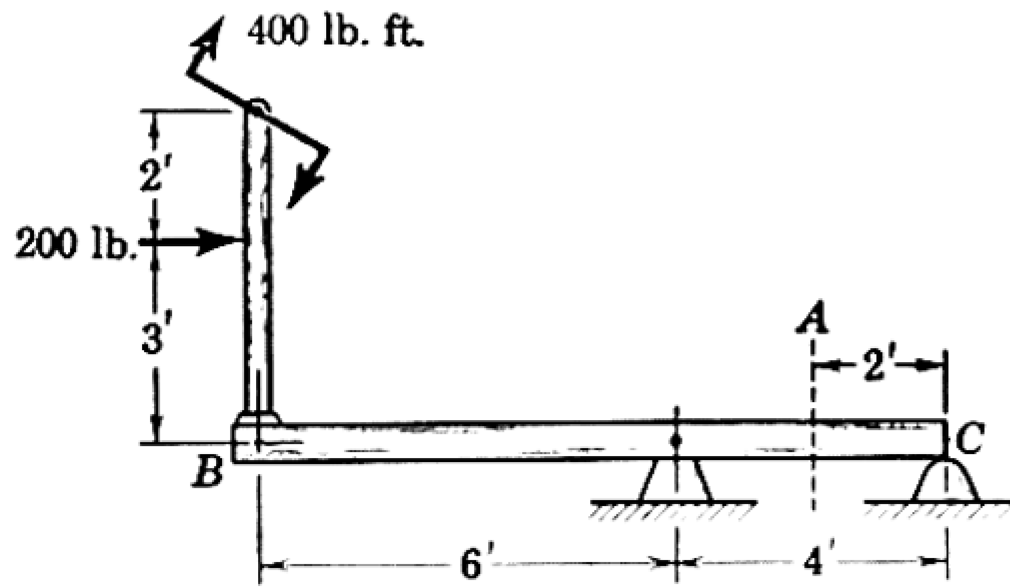
PROB. 454

455. Draw the shear and moment diagrams for the beam AB loaded through the attached curved strut.



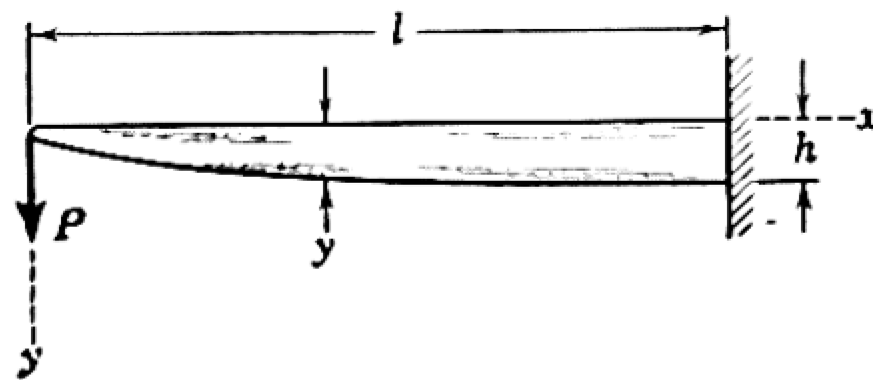
PROB. 455

456. Determine the bending moment M at section A from the moment diagram of the beam BC. *Ans.* $M = 500$ lb. ft.



PROB. 456

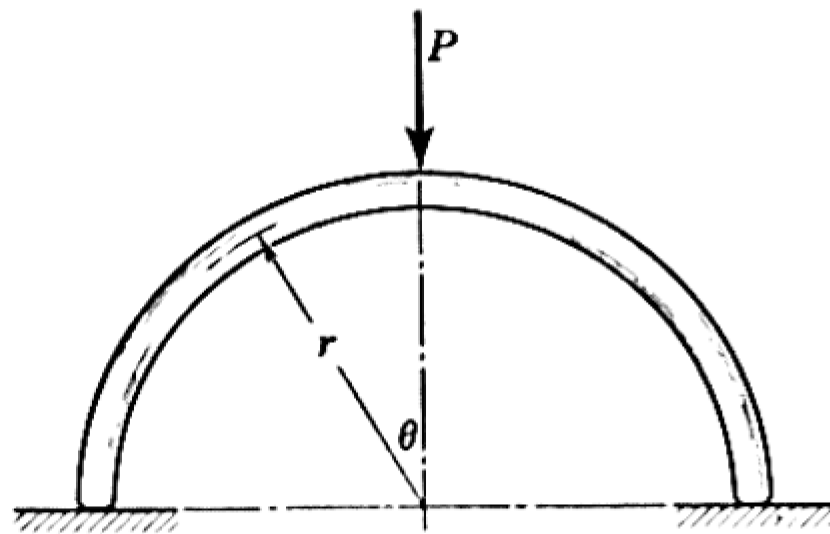
457. The resistance of a beam of uniform width to bending is found to be proportional to the square of the beam depth y. For the cantilever beam shown the depth is h at the support. Find the required depth y as a function of the



PROB. 457

length x in order that the resistance to bending offered by the beam at the support will be the same for all sections.

458. A curved beam has the form of a semicircular arc and is centrally loaded as shown. Determine the shear Q (perpendicular to the arc) and the bending

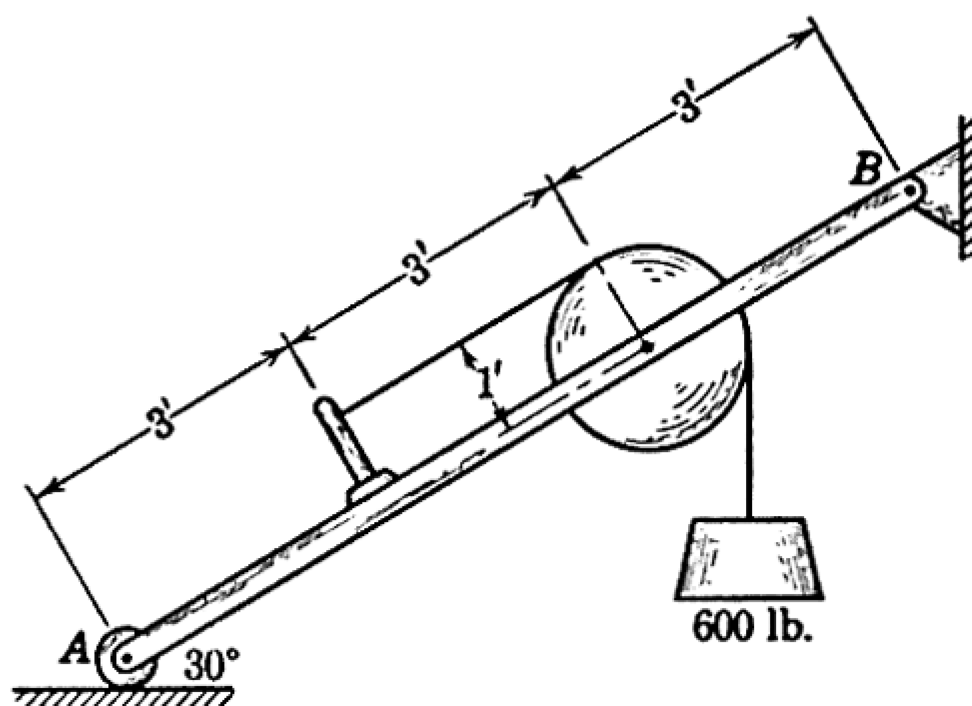


PROB. 458

moment M as functions of θ. Assume that the horizontal surface is perfectly smooth.

Ans. $Q = \frac{P}{2}\cos\theta,\ M = \frac{Pr}{2}(1 - \sin\theta)$

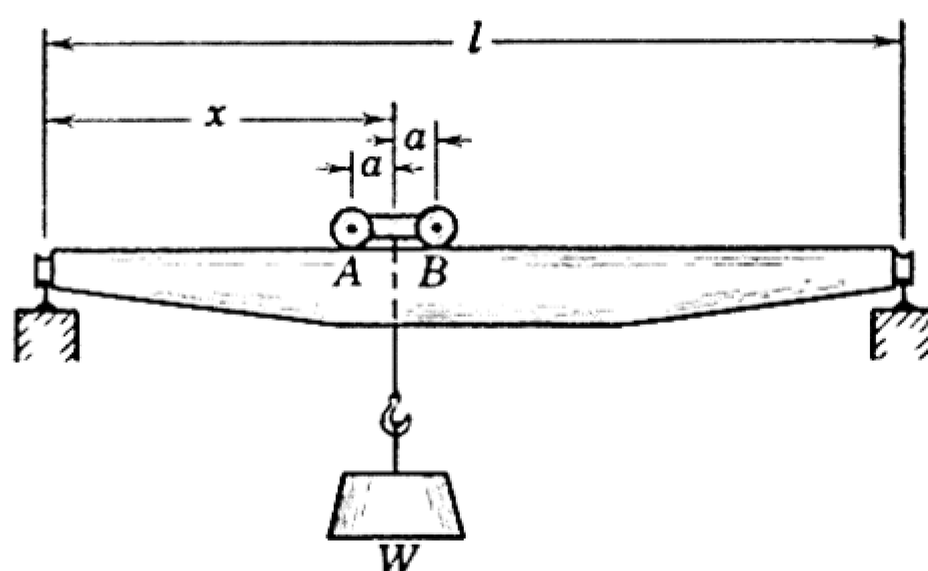
459. Find the maximum bending moment M in the arm AB and its distance x from A.



PROB. 459

* **460.** Determine the maximum bending moment M and the corresponding value of x in the crane beam and indicate the section where this moment acts.

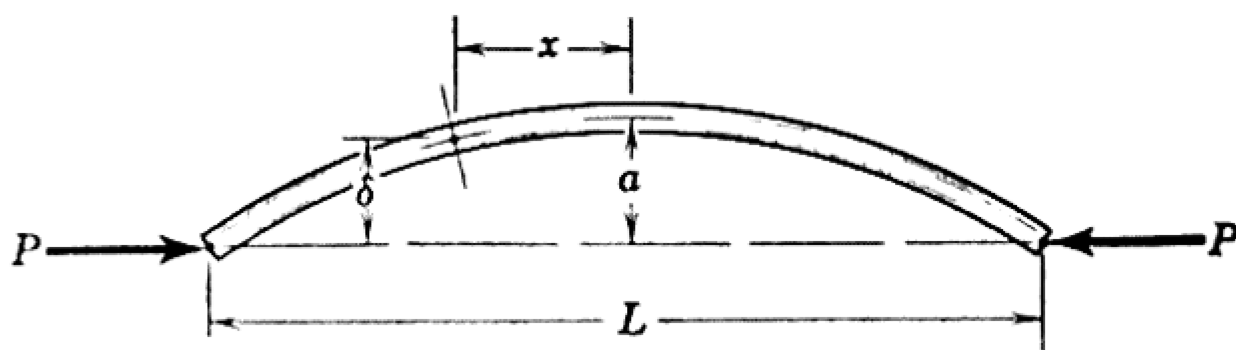
Ans. $M_A = \dfrac{W}{4l}(l - a)^2,\ x = \dfrac{a + l}{2}$



PROB. 460

* **461.** A curved bar with the dimensions shown is in the form of a circular segment. Determine the shear Q normal to the axis of the bar and the bending moment M as functions of x.

Ans. $Q = \dfrac{2Pax}{a^2 + L^2/4},\ M = P\left[\sqrt{\left(\dfrac{a^2 + L^2/4}{2a}\right)^2 - x^2} + \dfrac{4a^2 - L^2}{8a}\right]$



PROB. 461

46. Beams with Distributed Loads. When the weight of the beam is not negligible or when a distributed load is supported by the beam, the

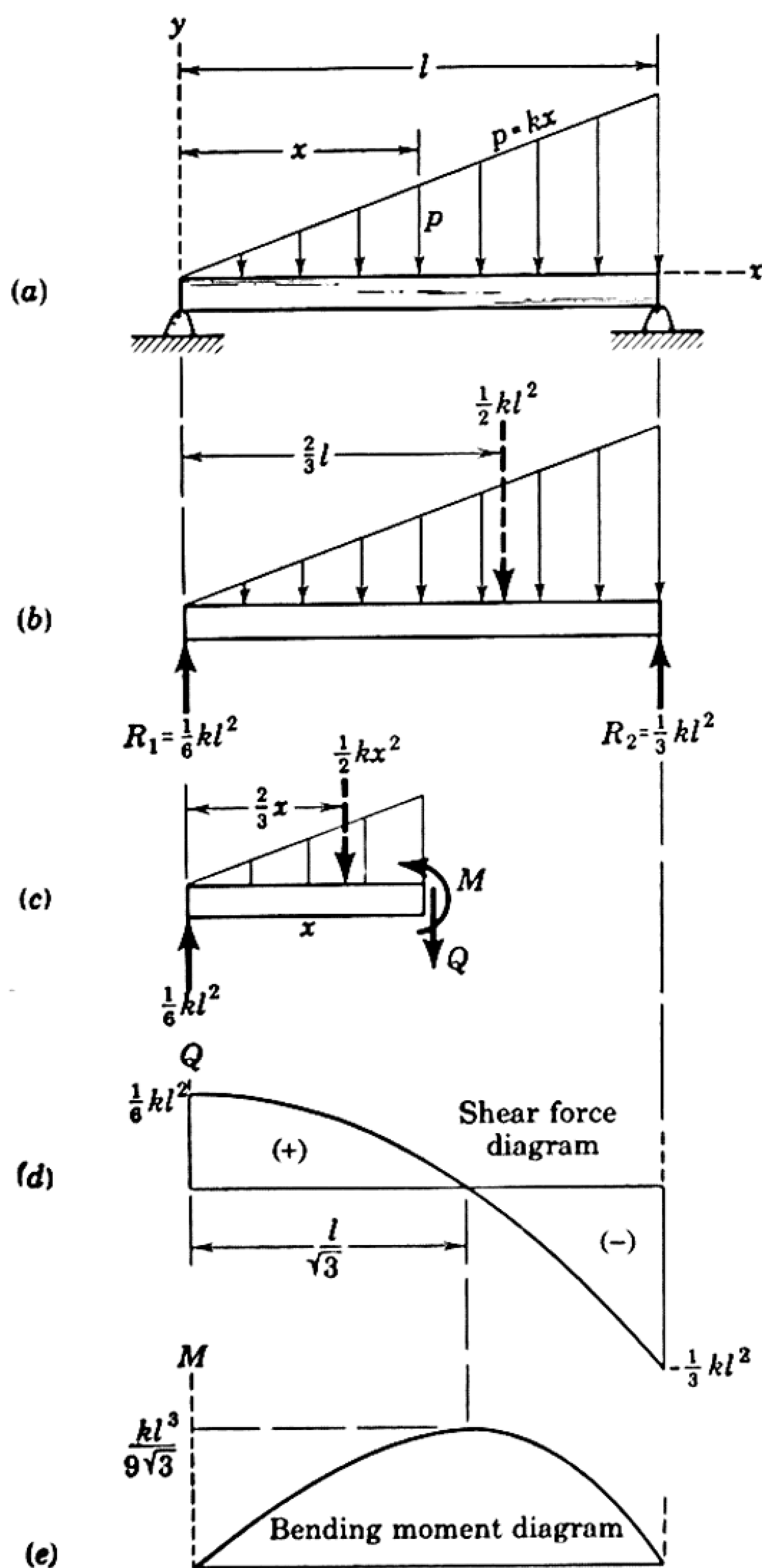


Fig. 64

shear and moment diagrams are somewhat different from the case of concentrated loading. Consider the beam shown in Fig. 64*a*, where the weight of the beam is negligible but a uniformly increasing load $p = kx$ is applied to the upper surface of the beam. The applied load p is the

force per unit length of the beam at any section. The resultant of the p-distribution is the area under $p = kx$, which is $\frac{1}{2}kx^2$, and the resultant passes through the centroid of this area. The support reactions are easily computed from this resultant by the familiar moment principle and are shown in Fig. 64*b* on the free-body diagram of the beam as a whole.

The equilibrium of a portion of the beam of length x is established from its free-body diagram, shown in Fig. 64*c*. The resultant of the applied p-distribution on this section is $\frac{1}{2}kx^2$ and again acts through the centroid of the triangular area. The shear Q and moment M are both shown in their positive directions. Equilibrium of forces in the vertical direction gives

$$Q = \tfrac{1}{6}kl^2 - \tfrac{1}{2}kx^2.$$

Equilibrium of moments about the cut section requires that

$$M + \tfrac{1}{2}kx^2(x - \tfrac{2}{3}x) - \tfrac{1}{6}kl^2x = 0,$$

or

$$M = \tfrac{1}{6}kl^2x - \tfrac{1}{6}kx^3.$$

The variations of Q and M are shown in Figs. 64*d* and *e*. The maximum moment may be obtained by equating dM/dx to zero. The magnitude of this moment is shown in the figure.

There is no fundamental difference between the analysis of beams with distributed loads and those with concentrated loads. In each case the equilibrium of a portion of the beam is established from the free-body diagram, using the familiar requirements of equilibrium.

The illustrative example of Fig. 64 involves the very simple linear distribution $p = kx$. The resultant of this loading and its position are easily remembered from the properties of the triangle. When more complex nonlinear distributions are involved, it is usually easier to integrate the effect of $p\,dx$ over the element under consideration than to determine and use the resultant of the p-distribution.

Many beams involve combinations of distributed and concentrated loadings or loadings which are distributed only over sections of the beam. In such cases sections of the beam between discontinuities at the concentrated loads or at the breaks in the distributed load curve should be analyzed as bodies in equilibrium in the manner already described. Or, when combined distributed and concentrated loads are present, the resulting shear and moment at any section may be treated as the sum of the respective shears and moments for each of the two loadings considered as separate problems in equilibrium.

47. Loading, Shear, and Moment Relations. There are several relations involving Q and M which may be established for any beam in general and which will aid greatly in the construction of the shear and moment diagrams. Figure 65 represents a portion of a loaded beam, and an element dx of the beam is isolated. At the location x the shear Q and moment M acting on the element are drawn in their positive directions. On the opposite side of the element these quantities are also shown in their positive directions but must be labeled $Q + dQ$ and $M + dM$

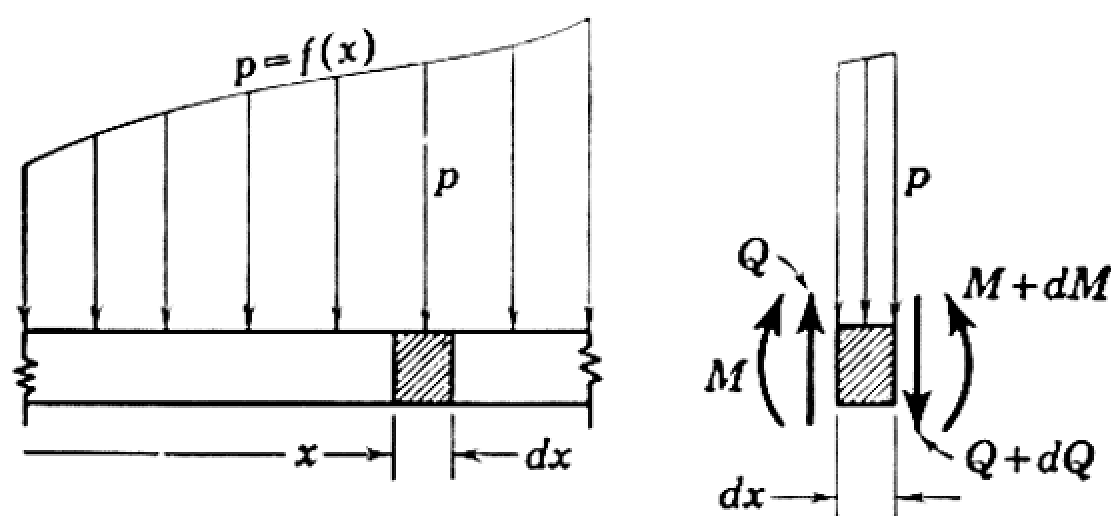


FIG. 65

since the change in Q and M with x is required. The applied loading p may be considered constant over the length of the element since this length is a differential quantity. Equilibrium of the element requires that the sum of the vertical forces be zero. Thus

$$Q = p\,dx + (Q + dQ),$$

or

$$p = -\frac{dQ}{dx}. \tag{34}$$

It is clear from Eq. (34) that the slope of the shear diagram must everywhere be equal to the negative of the applied loading. This may be seen to be the case in Fig. 64*d* and Fig. 63*e*. In the latter figure the slope is zero since p is also zero. Equation (34) holds on either side of a concentrated load but not at the concentrated load by reason of the discontinuity due to the abrupt change in shear.

Equilibrium of the element in Fig. 65 also requires that the moment sum must be zero. Taking moments about the left end gives

$$M + p\,dx\frac{dx}{2} + (Q + dQ)\,dx = M + dM.$$

The two M's cancel, and the terms $p\dfrac{(dx)^2}{2}$ and $dQ\,dx$ may be dropped

since they are differentials of higher order than those that remain. This leaves merely

$$Q = \frac{dM}{dx}. \tag{35}$$

which expresses the fact that the shear everywhere is equal to the slope of the moment curve. Or the moment M may be considered as being equal to the integral of the shear curve. In integral form Eq. (35) may be written

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} Q\,dx,$$

or

$$M = M_0 + (\text{area under shear diagram from } x_0 \text{ to } x).$$

In this expression M_0 is the bending moment at x_0 and M is the bending moment at x. For beams similar to those of Figs. 63 and 64, where there is no externally applied moment M_0 at $x_0 = 0$, the total moment at any section equals the area under the shear diagram up to that section. Summing up the area under the shear diagram is usually the simplest way to construct the moment diagram.

From Eq. (35) it is now clear why the maximum bending moment M occurs where the shear Q is zero. Mathematically $dM/dx = 0$ is the condition for a maximum or minimum value of M. Thus the critical locations for bending moments may be spotted from the shear diagram wherever $Q = 0$.

It should be noted from Eqs. (34) and (35) that the degree of Q in x is one higher than that of p. Also M is of one higher degree in x than is Q. Furthermore M is two degrees in x higher than p. Thus for the beam in Fig. 64, $p = kx$ is of the first degree in x, so that Q is of the second degree in x and M is of the third degree in x.

Equations (34) and (35) may be combined to yield

$$\frac{d^2M}{dx^2} = -p.$$

Thus, if p is a known function of x, the moment M may be obtained by two integrations, provided the limits of integration are properly evaluated each time. This method is usable only if p is a continuous function of x. It is usually simpler to construct the variation of M directly from the areas under the shear diagram.

SAMPLE PROBLEM

462. Draw the shear-force and bending-moment diagrams for the loaded beam shown at the top of the figure and determine the maximum moment M and its location x from the left end.

Solution: The support reactions are most easily obtained by considering the resultants of the distributed loads as shown on the free-body diagram of the beam as a whole. The first interval of the beam is analyzed from the free-body diagram of the section for $x < 4$ ft. A vertical summation of forces for this section gives

$$[\Sigma F_y = 0] \qquad Q = 247 - 12.5x^2,$$

and a moment summation about the cut section yields

$$[\Sigma M = 0] \qquad M + (12.5x^2)\frac{x}{3} - 247x = 0,$$

$$M = 247x - 4.167x^3.$$

These values of Q and M hold for $0 < x < 4$ ft. and are plotted for that interval in the shear and moment diagrams shown.

From the free-body diagram covering the section for which $4 < x < 8$ ft., equilibrium in the vertical direction requires

$$[\Sigma F_y = 0] \qquad Q + 100(x - 4) + 200 - 247 = 0,$$

$$Q = 447 - 100x.$$

A moment sum about the cut section gives

$$[\Sigma M = 0] \qquad M + 100(x - 4)\frac{x - 4}{2} + 200\left(x - \frac{2}{3} \times 4\right) - 247x = 0,$$

$$M = -266.7 + 447x - 50x^2.$$

These values of Q and M are plotted on the shear and moment diagrams for the interval $4 < x < 8$ ft.

The analysis of the remainder of the beam is continued from the free-body diagram of the portion of the beam to the right of a section in the next interval. It should be noted that Q and M are represented in their positive directions. A vertical force summation requires

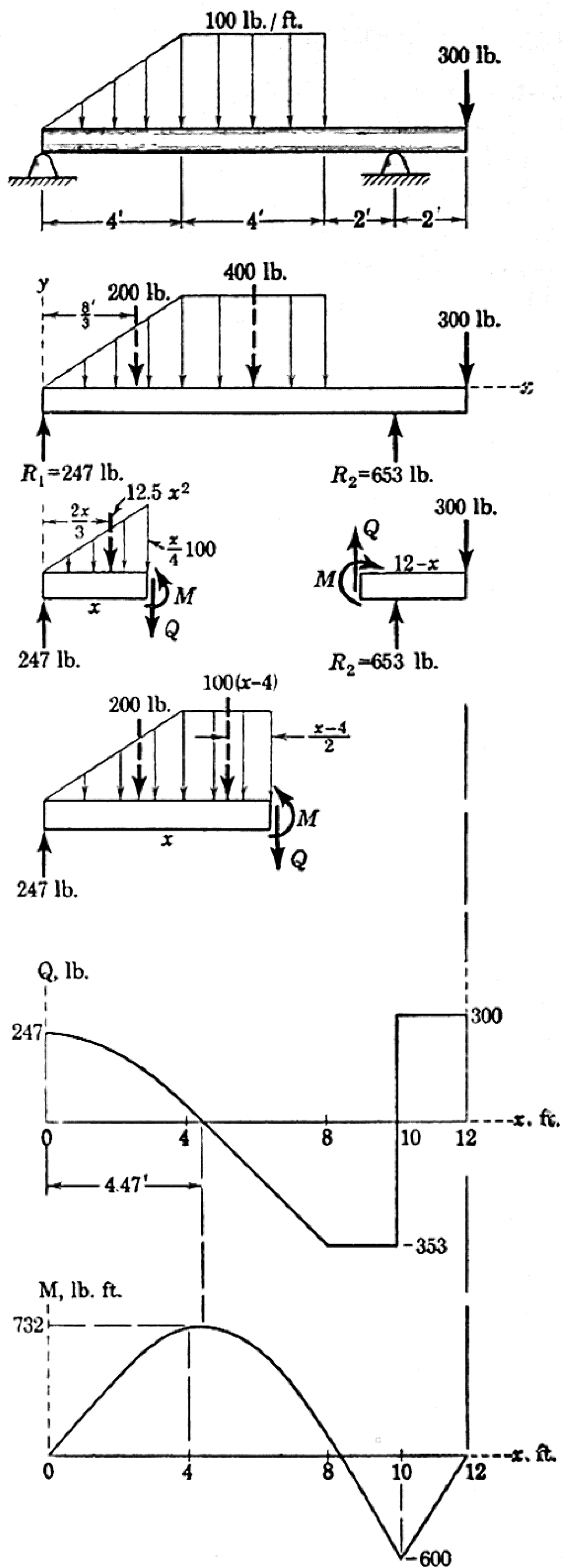
$$[\Sigma F_y = 0] \qquad Q + 653 - 300 = 0, \qquad Q = -353 \text{ lb.},$$

and a moment summation about the section yields

$$[\Sigma M = 0] \qquad M + 300(12 - x) - 653(12 - x - 2) = 0,$$

$$M = 2930 - 353x.$$

These values of Q and M are plotted on the shear and moment diagrams for the interval $8 < x < 10$ ft.


100 lb./ft.
300 lb.
4'
4'
2'
2'
y
200 lb.
400 lb.
8'/3
300 lb.
x
R₁=247 lb.
R₂=653 lb.
12.5 x²
2x/3
x/4 100
M
Q
x
247 lb.
300 lb.
Q
M
12−x
R₂=653 lb.
100(x−4)
200 lb.
(x−4)/2
M
Q
x
247 lb.
Q, lb.
247
300
x, ft.
0
4
8
10
12
4.47'
−353
M, lb. ft.
732
x, ft.
0
4
8
10
12
−600

Prob. 462

The last interval may be analyzed by inspection. The shear is constant at +300 lb., and the moment follows a straight line relation beginning with zero at the right end of the beam.

The maximum moment occurs at $x = 4.47$ ft., where the shear curve crosses the zero axis, and the magnitude of M is obtained for this value of x by substitution into the expression for M for the second interval. The maximum moment is

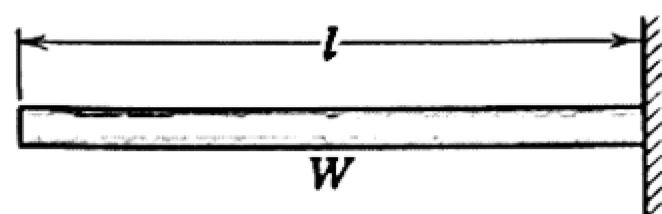
$$M = 732 \text{ lb. ft.} \qquad \textit{Ans.}$$

The derivative $dM/dx = 0$ for this second interval will check the value $x = 4.47$ ft. Although the shear crosses the zero axis again at $x = 10$ ft., the magnitude of the bending moment is less than that at $x = 4.47$ ft.

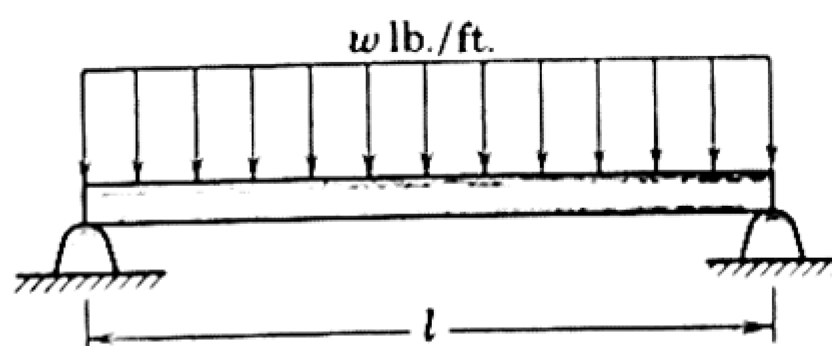
The moment diagram may also be constructed directly from the shear diagram. The value of the moment M at any section x is the net positive area under the shear diagram to the left of x.

PROBLEMS

463. Draw the shear and moment diagrams for the uniform cantilever beam of weight W.



Prob. 463

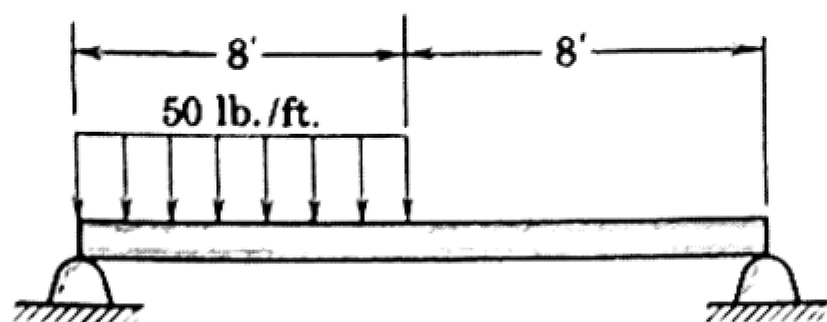


Prob. 464

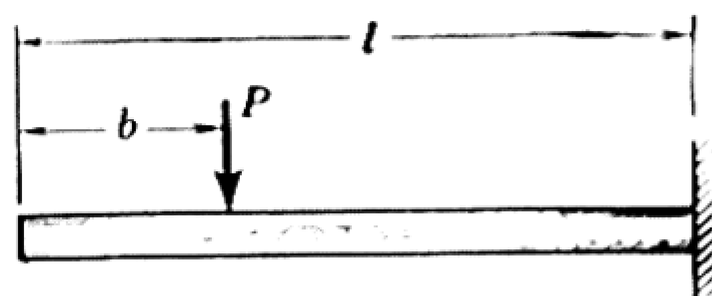
464. Draw the shear and moment diagrams for the uniformly loaded beam and find the maximum bending moment M. *Ans.* $M = \frac{wl^2}{8}$

465. For the beam of Prob. 464 let the length l be 20 ft. and the unit load w be 100 lb./ft. Approximate the shear and moment diagrams by dividing the beam into, say, five equal parts and replacing the distributed load on each part by an equivalent concentrated load.

466. Find the maximum magnitude M of the bending moment. *Ans.* $M = 900$ lb. ft.



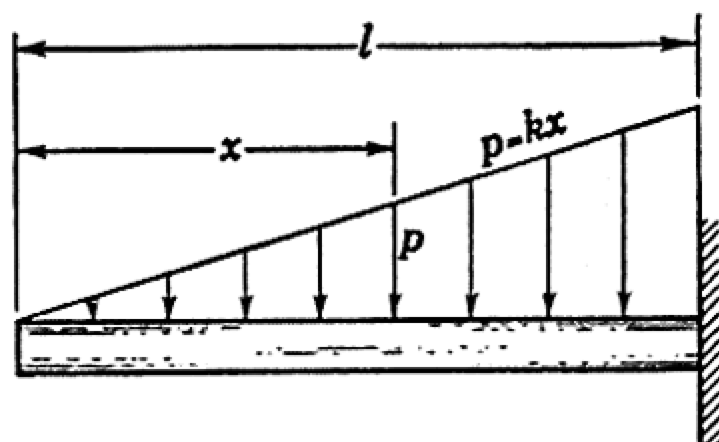
Prob. 466



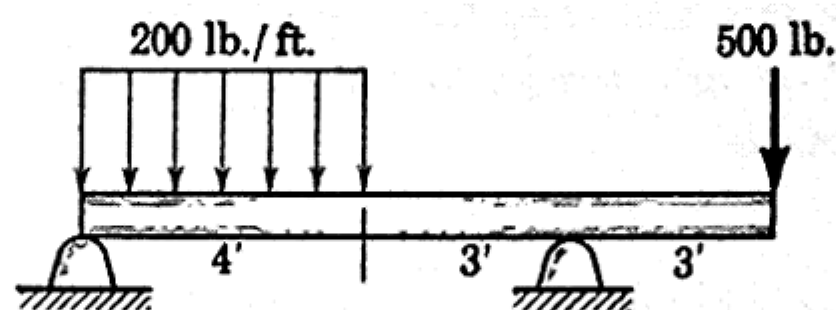
Prob. 467

467. The force P is applied to the uniform cantilever beam of weight W. Draw the shear and moment diagrams.

468. Draw the shear and moment diagrams for the cantilever beam with the linear loading and find the maximum magnitude M of the bending moment.



Prob. 468

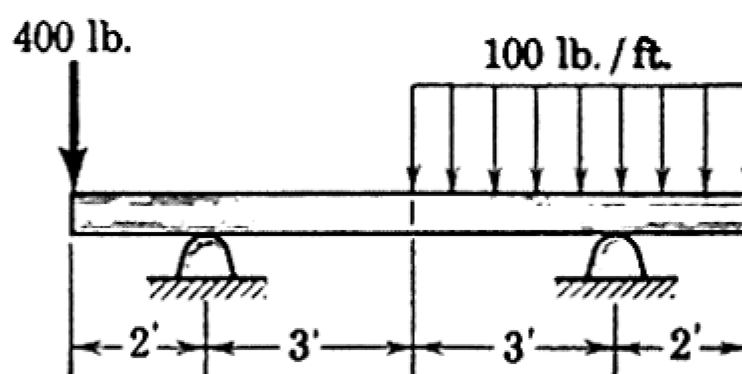


Prob. 469

469. Draw the shear and moment diagrams for the loaded beam and determine the maximum magnitude M of the bending moment.

Ans. $M = 1500$ lb. ft.

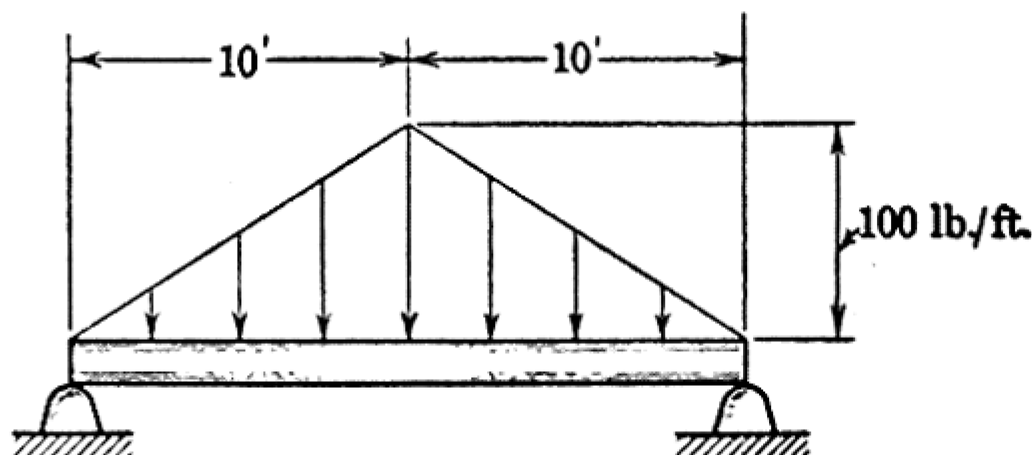
470. Draw the shear and moment diagrams for the loaded beam and determine the maximum magnitude M of the bending moment.



Prob. 470

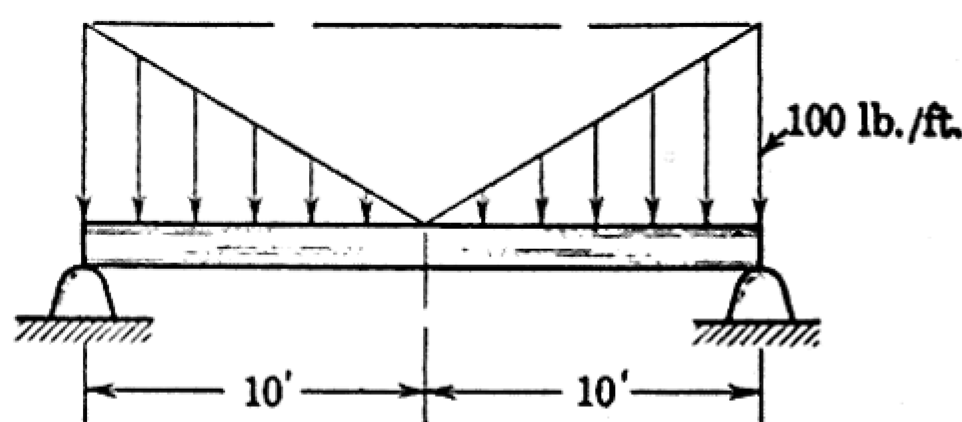
471. Draw the shear and moment diagrams for the beam and find the maximum bending moment M.

Ans. $M = 3333$ lb. ft.



Prob. 471

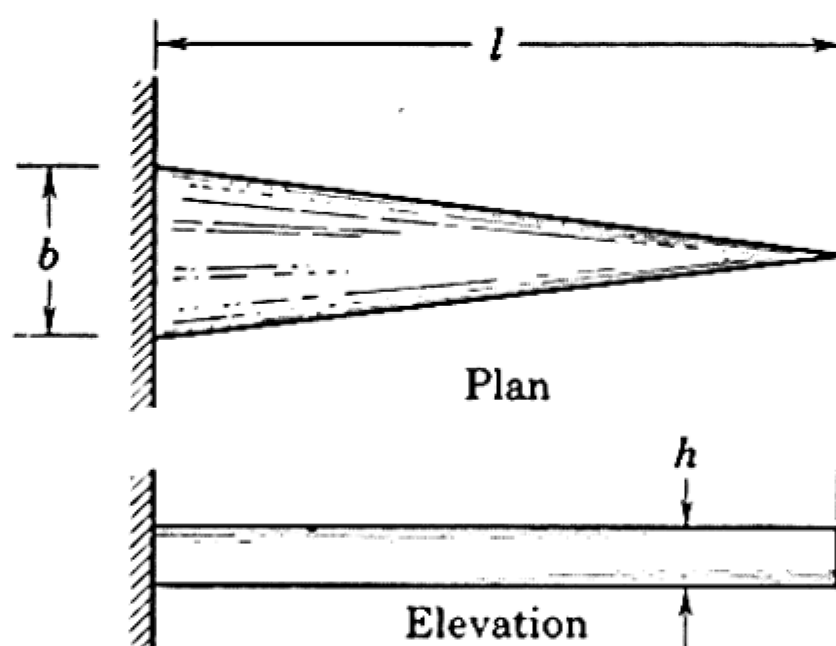
472. Draw the shear and moment diagrams for the beam with the linearly varying loads and find the maximum bending moment M.



Prob. 472

473. Determine the shear Q and moment M as functions of x for the prismatic cantilever beam made from homogeneous material with a weight density μ.

Ans. $Q = \frac{\mu bh}{2l}x^2,\ M = -\frac{\mu bh}{6l}x^3$

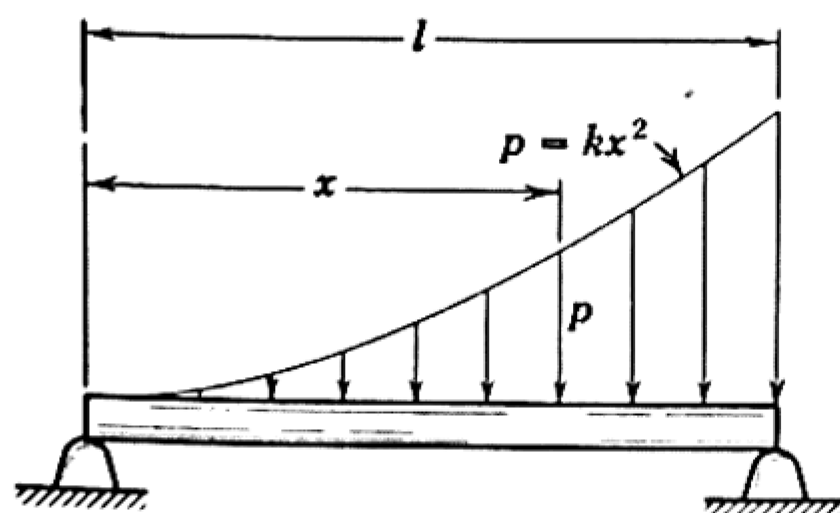


PROB. 473

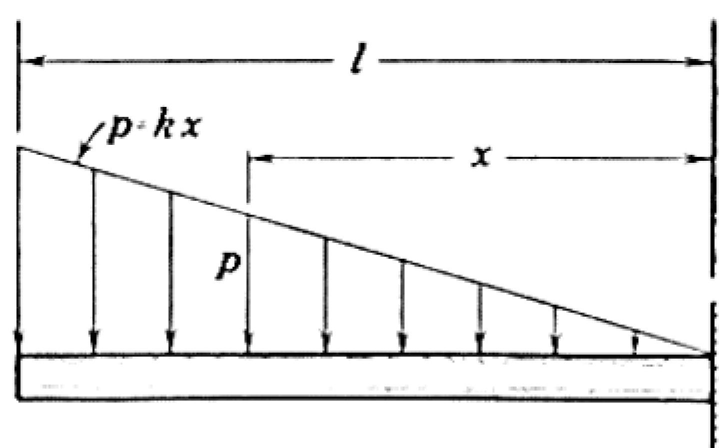
474. Find the maximum bending moment M per foot of width in the horizontal direction for the reservoir gate of Prob. 410.

475. Determine the expression for the shear Q as a function of x and obtain the moment M by the integration of this function.

Ans. $Q = \frac{k}{12}(l^3 - 4x^3),\ M = \frac{k}{12}(l^3x - x^4)$



PROB. 475

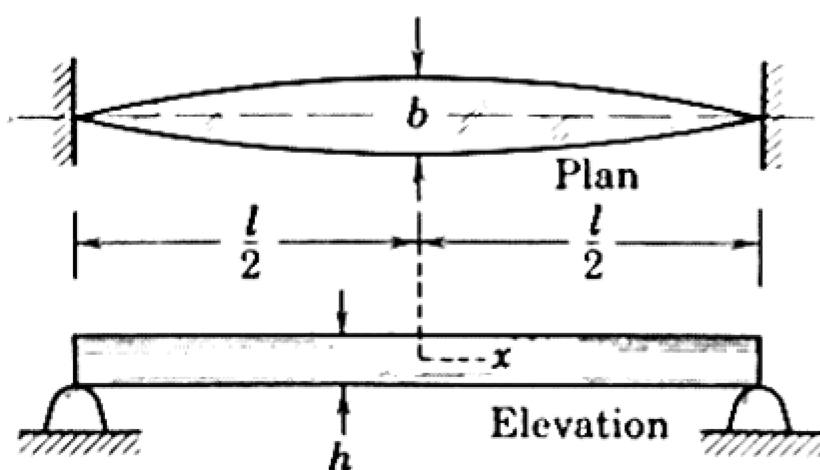


PROB. 476

476. Draw the shear and moment diagrams for the cantilever beam with the reversed linear loading and find the maximum magnitude M of the bending moment.

Ans. $M = \frac{1}{3}kl^3$

* **477.** The end-supported beam has a constant depth h and a variable width y such that the two symmetrical contours in the plan view are defined by two parabolas with vertices at the center of the span. Determine the shear Q and moment M, induced by the weight of the beam, as functions of the distance x from the center of the span. The weight density of the beam material is μ.

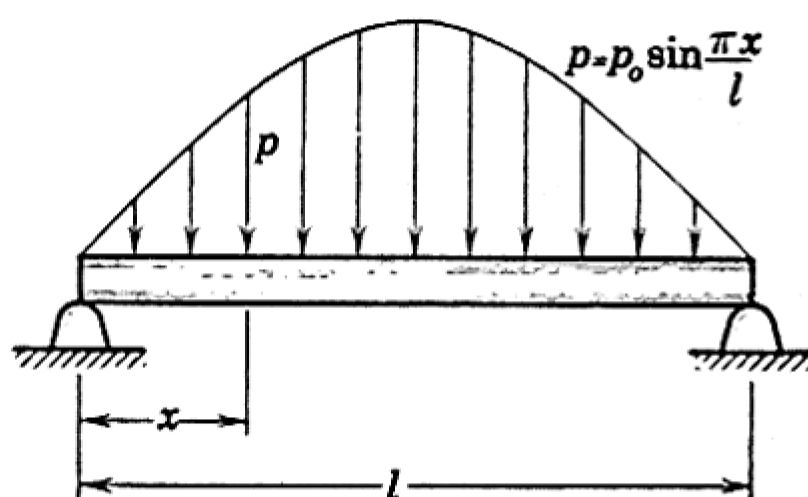


PROB. 477

Ans. $Q = \mu hbx\left(\frac{4x^2}{3l^2} - 1\right),\ M = \mu hb\left(\frac{5}{48}l^2 - \frac{x^2}{2} + \frac{x^4}{3l^2}\right)$

* **478.** Determine the shear Q and moment M as functions of x for the beam whose loading is described by the sine function. Find the maximum magnitude of M.

Ans. $Q = \frac{p_0 l}{\pi} \cos \frac{\pi x}{l}, M = \frac{p_0 l^2}{\pi^2} \sin \frac{\pi x}{l}$



Prob. 478

48. Graphical Solution for Bending Moments. The bending-moment distribution may be determined graphically with the aid of the funicular polygon. A simple beam with two vertical applied loads, Fig. 66, is used to illustrate the method. Although the reactions CD and DA may

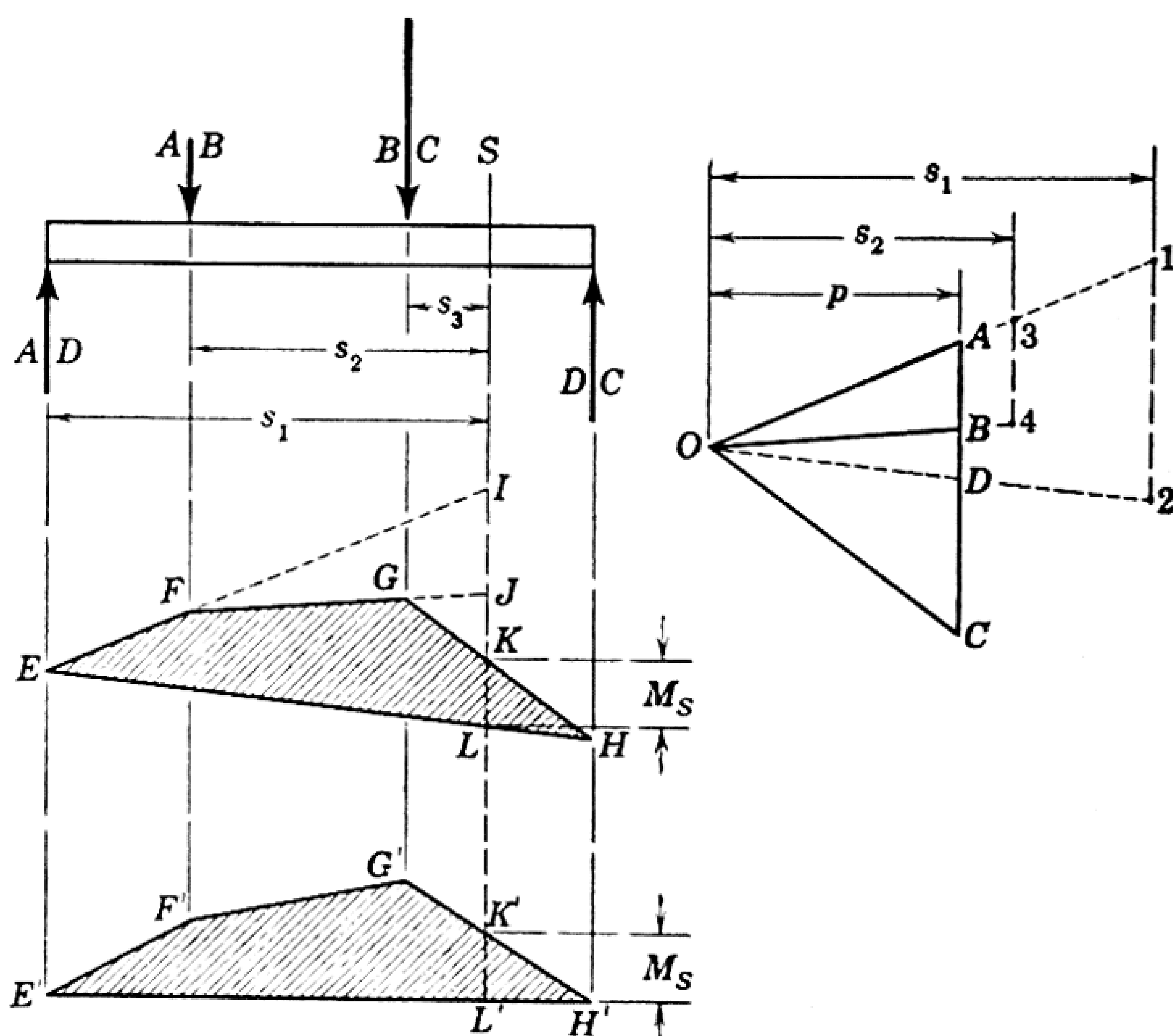


Fig. 66

be computed, they will be determined as a part of the graphical process. The force diagram is constructed first by laying off the vectors AB and BC and selecting a convenient pole O. The pole is chosen to the left of

the load line ABC in order that the resulting diagram will be in accord with the previous representation of bending moments. The full meaning of this statement will become clear when the construction which follows is explained. After the rays OA, OB, and OC are drawn, the funicular curve $EFGH$ is constructed on the lines of action of the respective forces in the manner previously explained in Art. 23. The resulting line HE gives the direction of the initially unknown ray OD on the force diagram. Thus point D is located, and the reactions CD and DA are determined.

The resulting funicular polygon $EFGHE$ may be interpreted as the moment diagram. In order to see this, consider any section such as S on the beam. The bending moment at S is found by isolating the portion of the beam to the left of S and finding the moment about S of all forces acting on that portion of the beam. Clearly, the moment is

$$M_S = (DA)s_1 - (AB)s_2 - (BC)s_3.$$

The moment of a force about a point varies directly as the distance from the point to the line of action of the force. Thus the moment of DA about any point on the beam may be represented to some scale by the linearly varying vertical intercept of the triangle EIL. Hence the moment of DA about section S is the intercept distance IL to some scale. The triangle EIL is also shown in the force diagram and is labeled 0–1–2. Next, the moment of AB about any point will vary linearly with the distance from the point to the line of action of AB and likewise is represented by a vertical intercept of triangle FIJ. At section S the moment of AB is the intercept IJ. The triangle FIJ is also shown on the force diagram and is labeled 0–3–4. The two intercepts IL and IJ represent the moments of AD and AB to the same scale as can be seen from the force diagram, where the same distances 1–2 and 3–4 have been constructed with force and distance scales common to each. Thus the difference JL of the two intercepts represents the net moment at section S due to the two forces AD and AB. In the same manner the moment at S due to BC is represented by the intercept JK. The remaining distance KL, then, represents the net moment M_S in the beam at S. This distance KL is clearly the vertical intercept of the funicular polygon $EFGH$. Consequently the bending moment at any point in the beam will be represented by the vertical intercept of the funicular polygon at that point. If desired, the polygon may be redrawn, as in the lower part of Fig. 66, so that the figure will have a horizontal base but still possess the same vertical dimensions. This last polygon is precisely the bending-moment diagram for the beam.

It is necessary to determine the scale of the resulting bending-moment diagram before it can be used. By similar triangles on the force di-

agram $(\overline{1-2})/s_1 = (AD)/p$ or $\overline{1-2} = IL = (AD)s_1/p$. Thus vertical measurement on the moment diagram will give the proper moment if the measurement is multiplied by the pole distance p. A scale incorporating this factor may be constructed for convenience if desired.

If the pole O had been chosen to the right of line ABC, the resulting moment diagram would have been below the base line instead of above. It is largely a matter of preference which of the two representations is chosen, and in practice both types are in common use.

The graphical solution is not limited to concentrated loads. A beam with distributed loads may be handled by dividing the beam into small increments and representing the resultant of the distributed load for any increment as a force through the centroid of the loading diagram strip. Accuracy will be increased as the number of separate increments is increased, and the resulting moment diagram will be composed of a series of short straight line segments which are tangent to the curve representing the actual variation. The correct curve may be drawn from a surprisingly few increments and corresponding line segments.

The graphical solution for bending moments is commonly used in the design of machine shafts as well as for other beams. The speeds of rotation which give troublesome vibrations are determined from the shaft deflections, which in turn may be obtained from a graphical solution of the bending-moment distribution.

PROBLEMS

In the following problems construct the bending-moment diagram by means of the funicular polygon. In the case of the distributed loads divide the portion of the beam involving the distributed load into a reasonable number of intervals and replace the actual load with equivalent concentrated loads.

479. Prob. 447.
480. Prob. 448.
481. Prob. 452.
482. Prob. 466.
483. Prob. 469.
484. Prob. 472.

49. Bending in Two Orthogonal Planes. Beams are often subjected to transverse forces which are not all in the same plane. In this event the beam may be analyzed by considering the bending action in two perpendicular planes whose line of intersection coincides with the axis of the beam. In the case of circular shafts subjected to such loads it is convenient to consider the vector combination of the bending moments in the two perpendicular planes. This combination is illustrated by the

shaft AB shown in Fig. 67, which is subjected to the transverse loads F_1 and F_2 in the two separate planes. The bending-moment diagram for each of the two planes is constructed independently by considering only the forces in or parallel to the respective plane. In the x–z plane this diagram is labeled M_{xz}, and in the y–z plane it is labeled M_{yz}. The resulting total bending moment M at any section, such as S, is the vector combination of M_{xz} and M_{yz}. The maximum moment as determined

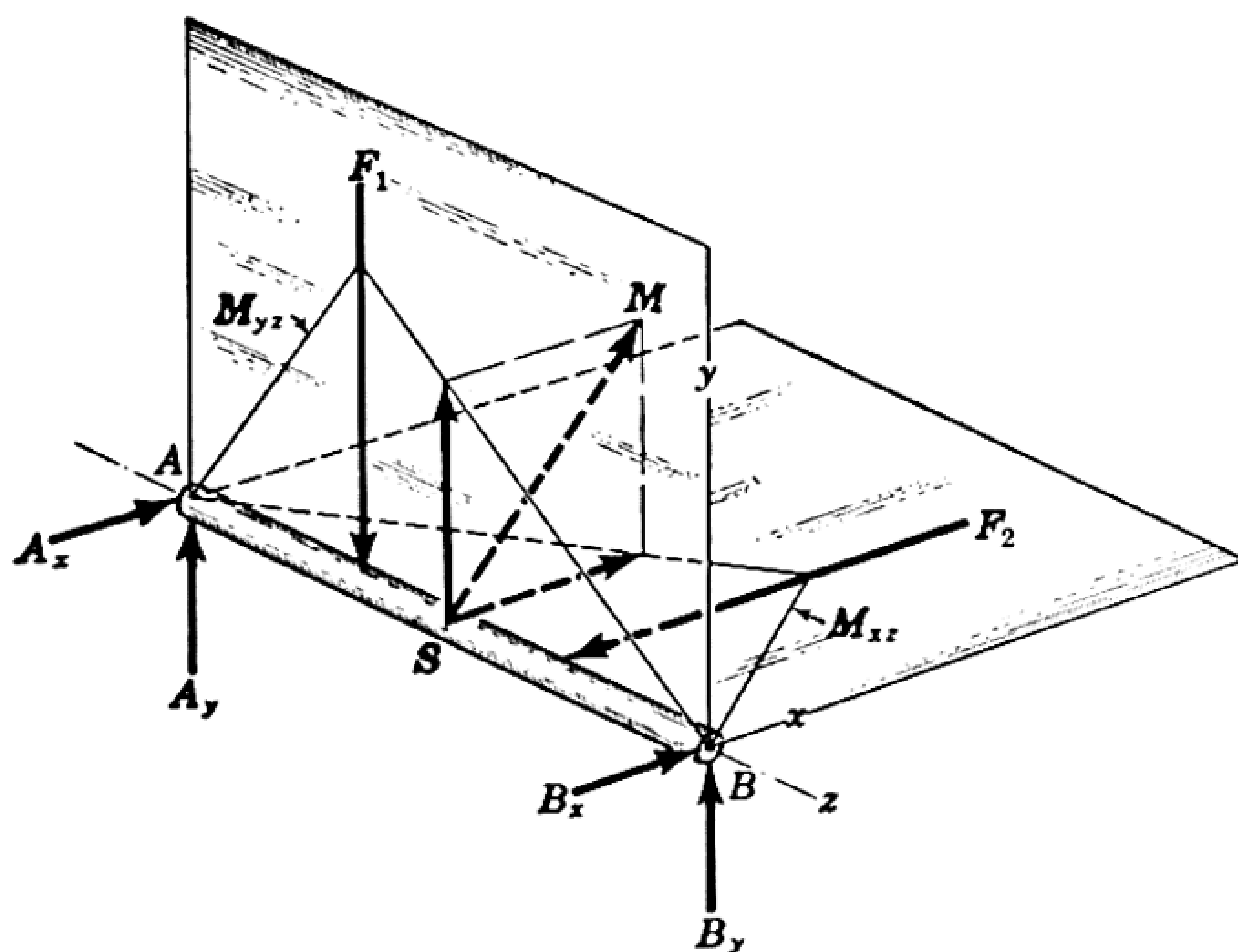


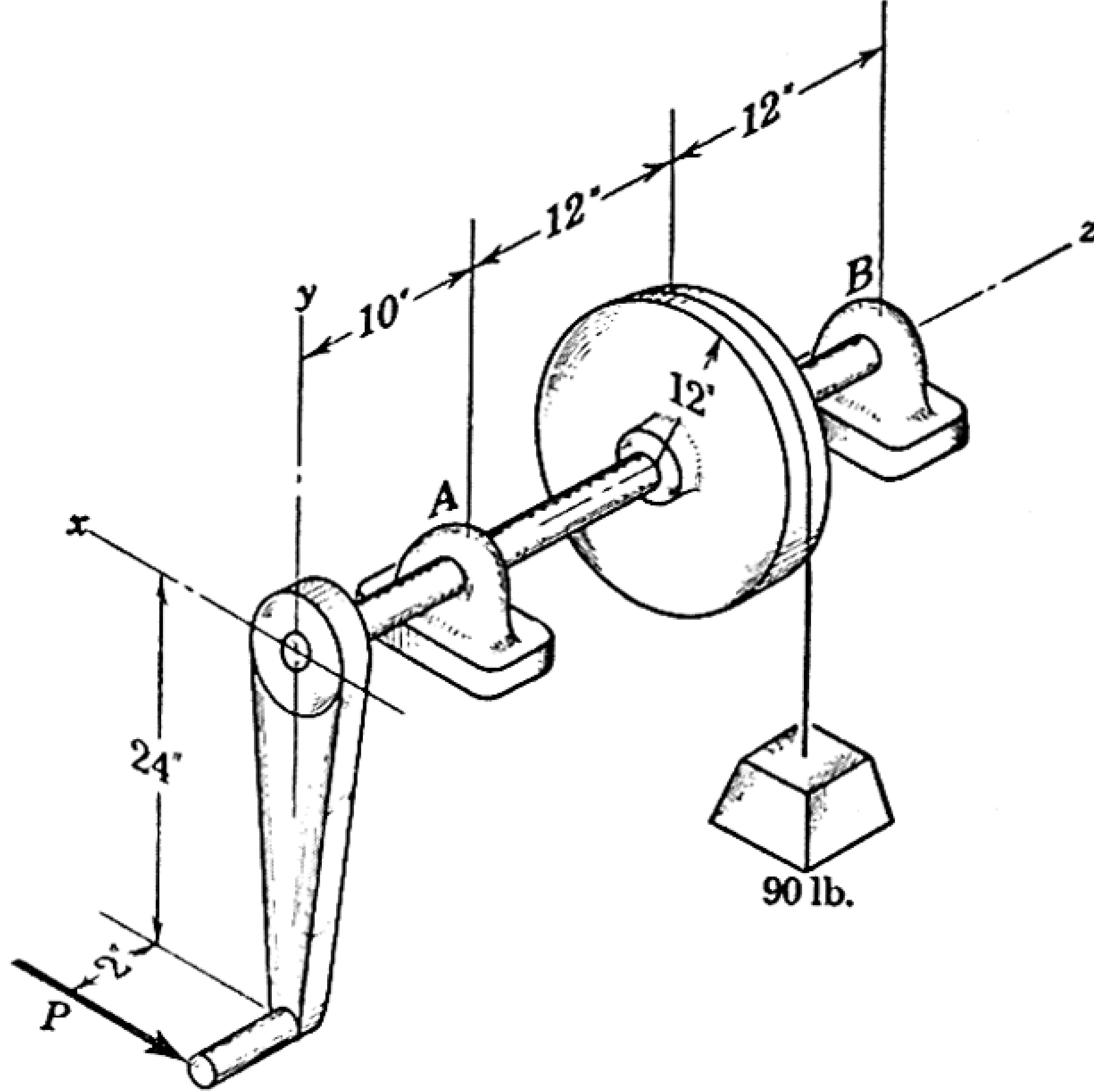
Fig. 67

in one of the planes will, in general, not represent the maximum bending moment in the shaft. Thus it is necessary to determine the location at which the vector sum of the two moment components is a maximum.

A practical method for obtaining the two bending-moment diagrams and their vector resultant for this type of problem is afforded by an adaptation of the graphical procedure described in the previous article. It is possible to construct the two bending-moment distributions for each of the two orthogonal planes and the vector resultant moment all on the same diagram and thereby to locate the maximum magnitude of the bending moment and its position along the shaft. Further explanation of this procedure and sample solutions to typical problems will be found in most books on machine design. This three-dimensional aspect of the problem is very commonly encountered in the design of machine shafts where loads due to the action of cams, gears, bearings, and the like are encountered.

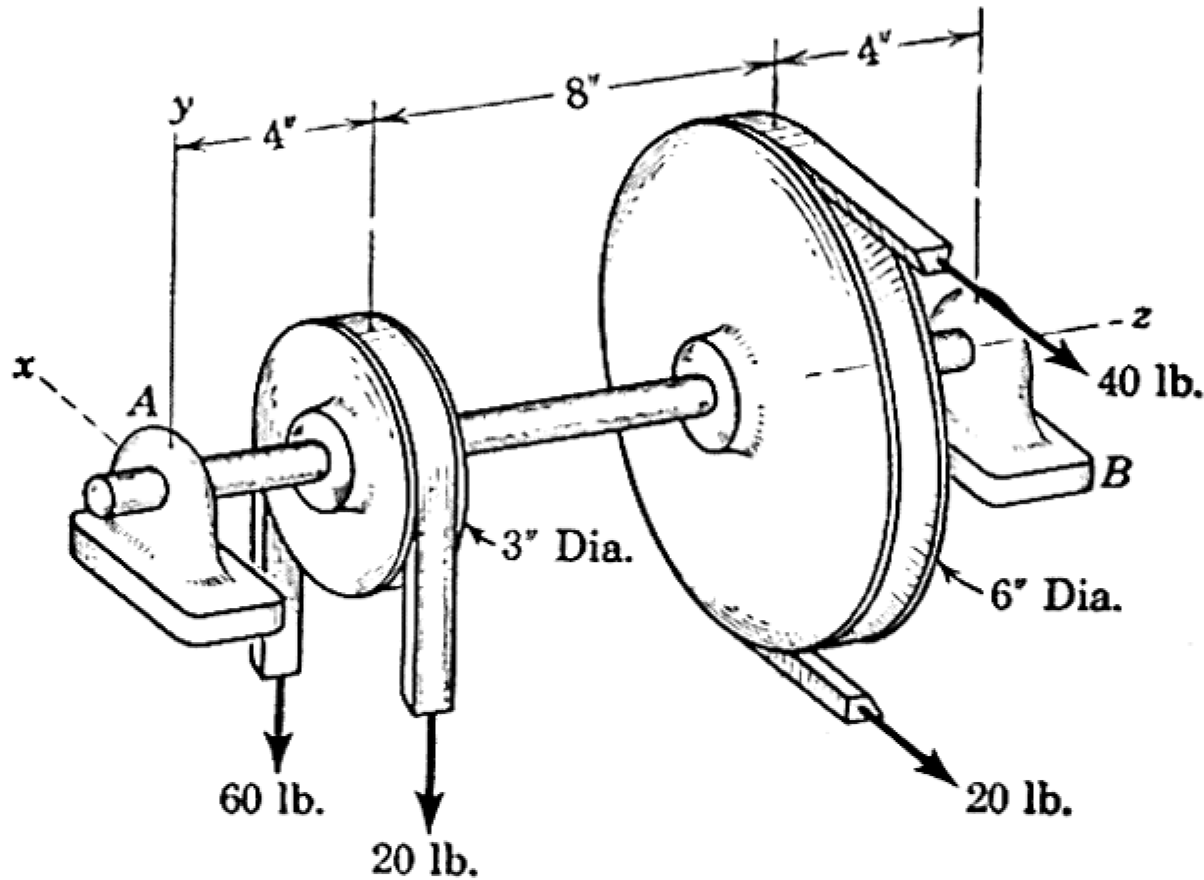
PROBLEMS

485. Locate and determine the maximum bending moment M in the shaft AB held in equilibrium by the force P.



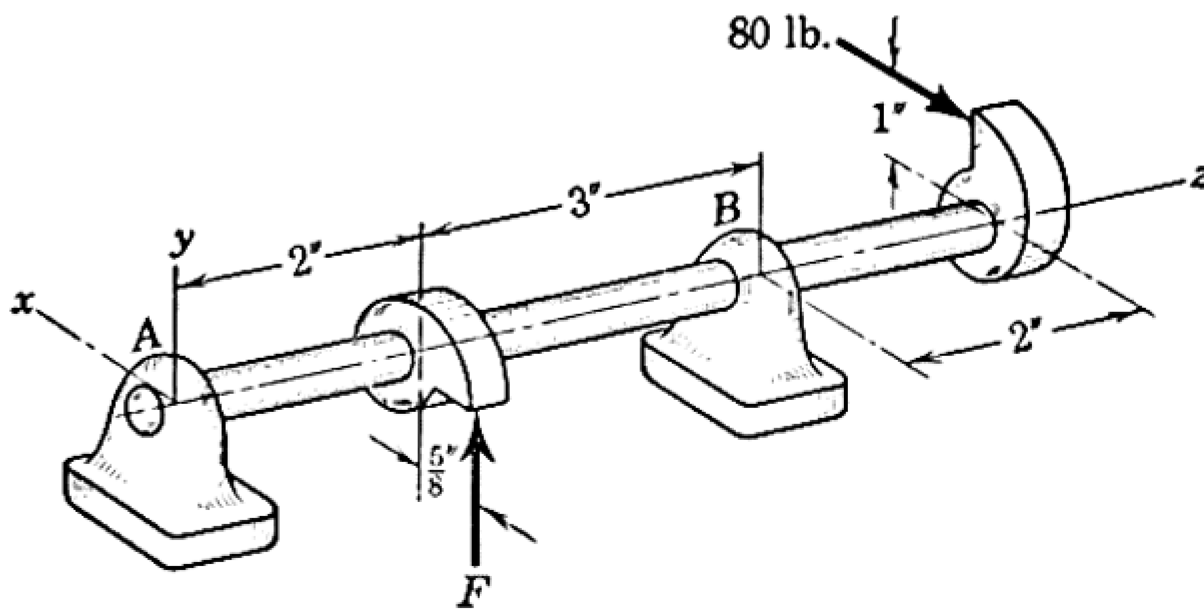
Prob. 485

486. Determine the maximum moment M in the shaft due to the applied belt tensions and find the distance z along the shaft from bearing A to the section at which this moment exists. *Ans.* $M = 247$ lb. in., $z = 4$ in.



Prob. 486

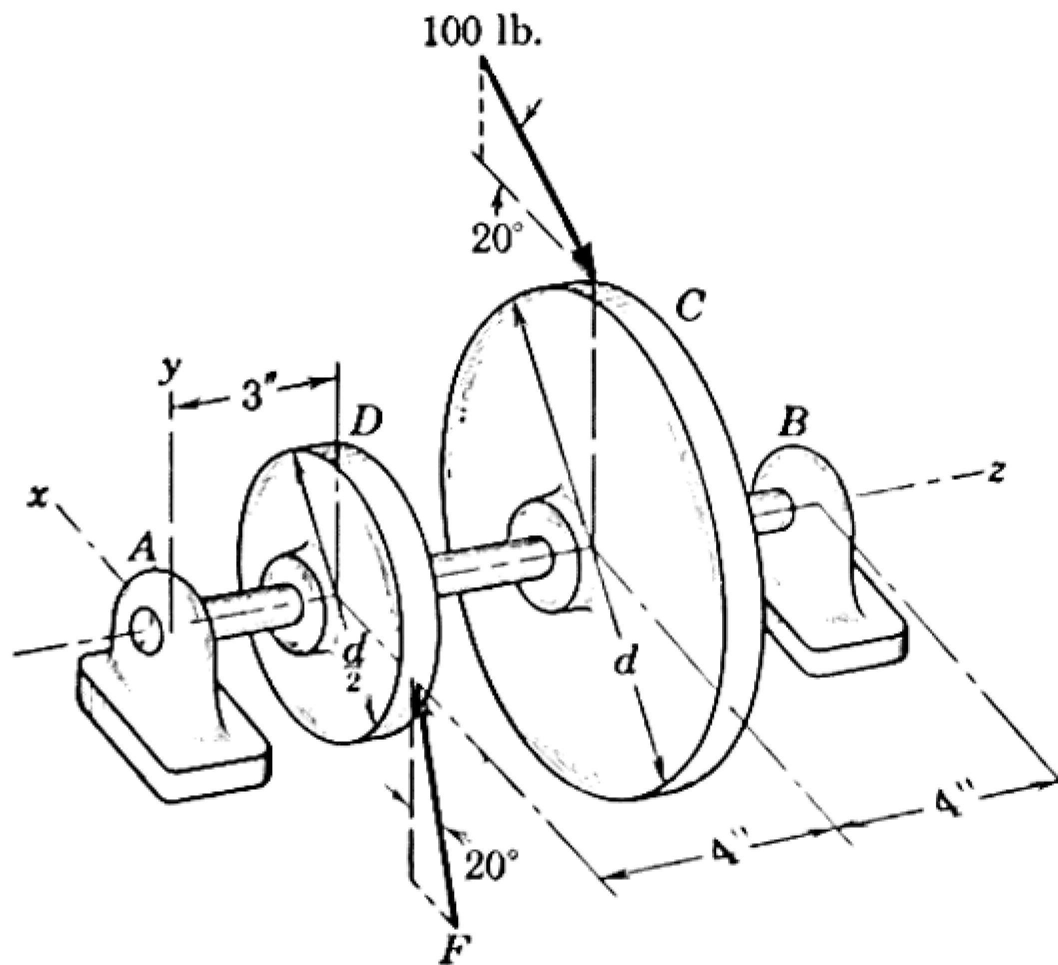
487. Determine the maximum bending moment M in the cam shaft and find the distance z along the shaft to the section at which this moment exists. The force F maintains equilibrium of the shaft.



Prob. 487

488. A speed-reducer shaft is running at constant speed and is driven through the large gear C of diameter d. The pinion D of diameter $d/2$ drives another gear (not shown) at a reduced speed. Determine the maximum bending moment M in the shaft due to the applied loads and find the distance z along the shaft from bearing A to the section at which this moment exists.

Ans. M = 375 lb. in., z = 3 in.



Prob. 488

CHAPTER VII

Friction

50. Introduction. In the preceding chapters contacting surfaces were considered as being perfectly smooth so that the forces of interaction were normal to the surface. Although in many instances this ideal assumption involves only a very small error, there are a great many problems wherein the ability of contacting surfaces to support tangential as well as normal forces must be considered. Forces tangent to contacting surfaces are known as friction forces and are present to some degree with the interaction between all real surfaces. Whenever a tendency for movement between contacting surfaces exists, the friction forces developed are always found to be in a direction to oppose this tendency.

In some types of machines and processes it is desirable to minimize the retarding effect of friction forces. Examples are bearings of all types, flow of fluids in pipes, power screws, and gears. In other machines effort is made to take advantage of friction, as in brakes, clutches, belt drives, and wedges. Friction forces are present throughout nature and exist to a considerable extent in all machines no matter how accurately constructed.

There are two types of friction, *fluid* friction and *dry* or *Coulomb* friction. Fluid friction is developed when layers in a fluid (liquid or gas) are moving at different velocities. This motion gives rise to frictional forces between fluid elements, and these forces depend on the relative velocity between layers. When there is no such velocity there is no friction. Fluid friction depends not only on the velocity gradients but also on the viscosity or resistance to shearing action of the fluid. The analysis of the fluid friction in fully lubricated bearings, centrifugal pumps, ship propulsion, aircraft performance, and projectile flight, to mention only a few examples, is basic to the design of these elements and systems. The investigation of fluid friction is best undertaken in the study of fluid mechanics and will not be developed in this book. It is mentioned, nevertheless, because of the importance of this type of friction in many engineering problems.

51. Dry Friction. Frictional force developed between unlubricated mating surfaces of solids in contact is known as *dry friction*. The theory

of dry friction often bears the name of *Coulomb* friction, although the basic relations involved were known before Coulomb's day. The laws governing the behavior of dry friction * can be explained by means of a very simple experiment. Consider a solid block of weight W resting on a horizontal surface as shown in Fig. 68*a*. The contacting surfaces possess a certain amount of roughness. The experiment will involve the application of a horizontal force P which will vary continuously from zero to a value sufficient to move the block and give it an appreciable

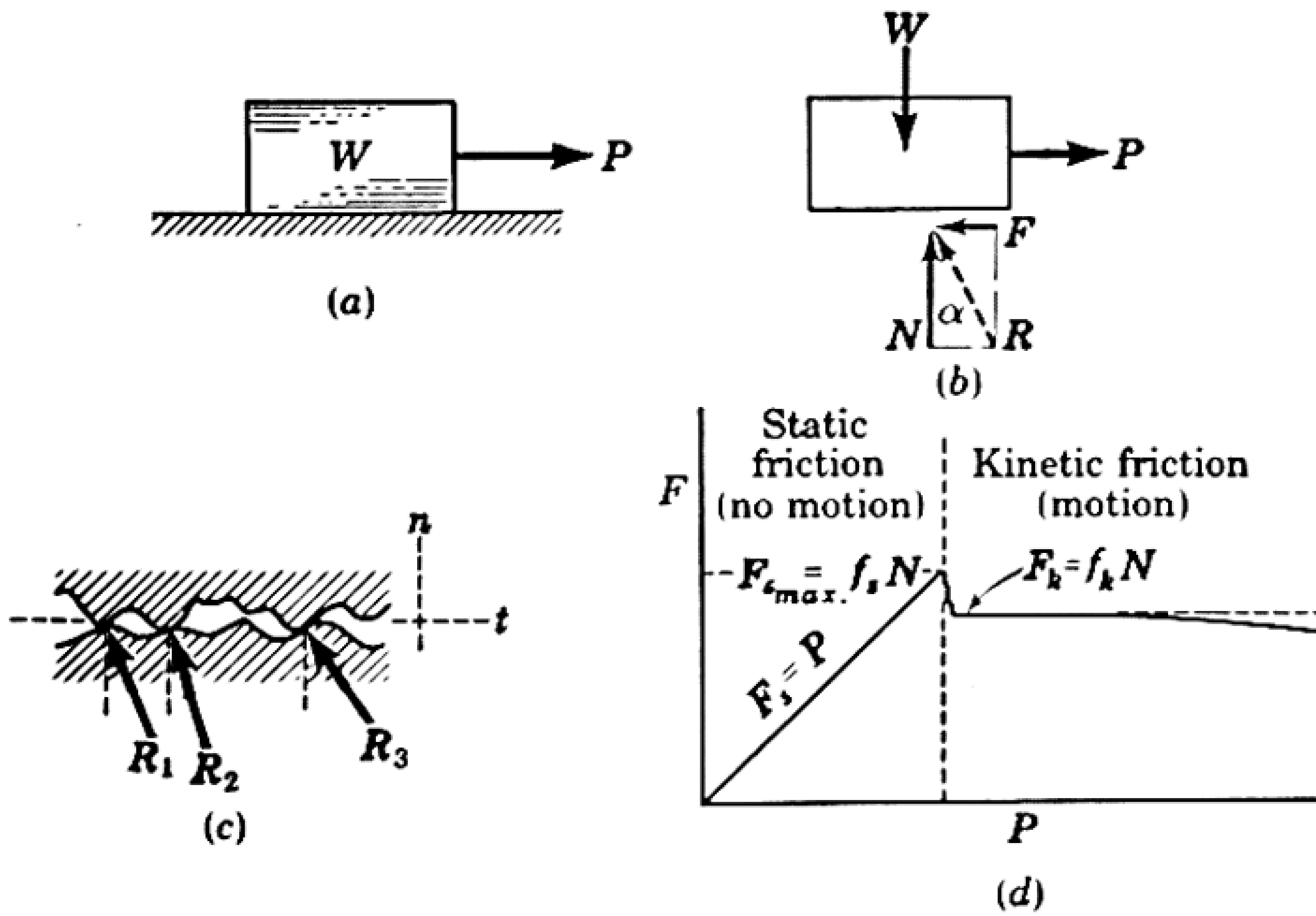


Fig. 68

velocity. The free-body diagram of the block for any value of P is shown in Fig. 68*b*, and the tangential friction force exerted by the plane on the block is labeled F. This friction force will *always* be in a direction to oppose motion or the tendency toward motion of the body on which it acts. There is also a normal force N which in this case equals W, and the total force R exerted by the supporting surface on the block is the resultant of N and F. A magnified view of the irregularities of the mating surfaces, Fig. 68*c*, will aid in visualizing the action of mechanical friction. Support is necessarily intermittent and exists at the mating humps. The direction of each of the reactions on the block R_1, R_2, R_3, etc., may be taken to be along the normal to the contact surface of each respective hump. The total normal force N is merely the sum of the n-components

* The principles of dry friction are based largely on the experiments of Coulomb in 1781 and on the work of Morin from 1831 to 1834. A comprehensive theory of dry friction, not yet available, must go beyond the mechanical explanation given in this chapter. For example, there is some evidence to support the theory that molecular attraction may be an important cause of friction under certain conditions.

of the R's, and the total frictional force F is the sum of the t-components of the R's. When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps and the t-components of the R's will be smaller than when the surfaces are at rest relative to one another. This consideration will explain the well-known fact that the force P necessary to maintain motion is less than that required to start the block when the irregularities are more nearly in mesh.

Assume now that the experiment indicated is performed and the friction force F is measured as a function of P. The resulting experimental relation is indicated in Fig. 68*d*. When P is zero, equilibrium requires that there be no friction force. As P is increased, the friction force must be of equal magnitude to P as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally a value of P is reached which causes the block to slip and to move in the direction of the applied force. At this same time the friction force drops slightly and abruptly to a somewhat lesser value. Here it remains essentially constant for a period but then drops off still more with higher velocities.

The region up to the point of slippage or impending motion is known as the range of *static friction*, and the value of the friction force is determined by the equations of equilibrium. This force may have any value from zero up to and including, in the limit, the maximum value. For a given pair of mating surfaces this maximum value of static friction $F_{s_{\text{max.}}}$ is found to be proportional to the normal force N. Thus

$$F_{s_{\text{max.}}} = f_s N,$$

where f_s is the proportionality constant, known as the *coefficient of static friction*. It must be carefully observed that this equation describes only the *limiting* or *maximum* value of the static friction force and not any lesser value. Thus the equation applies *only* to cases where it is known that motion is impending.

After slippage occurs a condition of *kinetic friction* is involved. Kinetic friction force is always somewhat less than the maximum static friction force. The kinetic friction force F_k is also found to be proportional to the normal force. Thus

$$F_k = f_k N,$$

where f_k is the *coefficient of kinetic friction*. It follows that f_k is somewhat less than f_s. As the velocity of the block increases, the kinetic friction coefficient decreases somewhat, and when high velocities are reached, the effect of lubrication by intervening air film may become appreciable. Coefficients of friction depend greatly on the exact condition of the sur-

faces as well as on the velocity and are subject to a considerable measure of uncertainty.

It is quite customary to write the two friction force equations merely as

$$F = fN. \tag{36}$$

There will be an understanding from the problem whether limiting static friction with its corresponding coefficient of static friction or whether kinetic friction with its corresponding kinetic coefficient is implied. It should be emphasized again that many problems involve a static friction force which is less than the maximum value at impending motion, and therefore the friction equation cannot be used.

From Fig. 68*c* it may be observed that for rough surfaces there is a greater possibility for large angles between the reactions and the n-direction than for smoother surfaces. Thus a friction coefficient measures the roughness of a pair of mating surfaces and incorporates a geometric property of these mating contours. It is meaningless to speak of a coefficient of friction for a single surface.

The direction of the resultant R in Fig. 68*b* measured from the direction of N is specified by $\tan \alpha = F/N$. When the friction force reaches its limiting static value, the angle α reaches a maximum value ϕ_s. Thus

$$\tan \phi_s = f_s.$$

When slipping occurs, the angle α will have a value ϕ_k corresponding to the kinetic friction force. In like manner

$$\tan \phi_k = f_k.$$

It is customary to write merely

$$\tan \phi = f, \tag{37}$$

where application to the limiting static case or to the kinetic case is inferred from the problem at hand. The angle ϕ_s is known as the *angle of static friction*, and the angle ϕ_k is called the *angle of kinetic friction*. This friction angle ϕ for each case clearly defines the limiting position of the total reaction R between two contacting surfaces. If motion is impending, R must be one element of a right circular cone of vertex angle $2\phi_s$, as shown in Fig. 69. If motion is not impending, R will be within the cone. This cone of vertex angle $2\phi_s$ is known as the *cone of static friction* and represents the locus of possible posi-

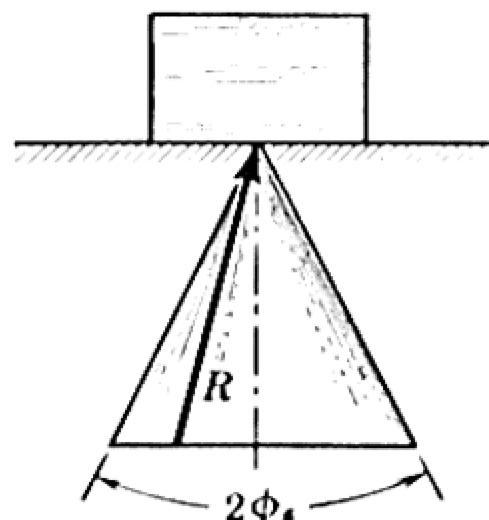


Fig. 69

tions for the reaction R at impending motion. If motion occurs the angle of kinetic friction applies, and the reaction must lie on the surface of a slightly smaller cone of vertex angle $2\phi_k$. This cone is the cone of *kinetic friction*.

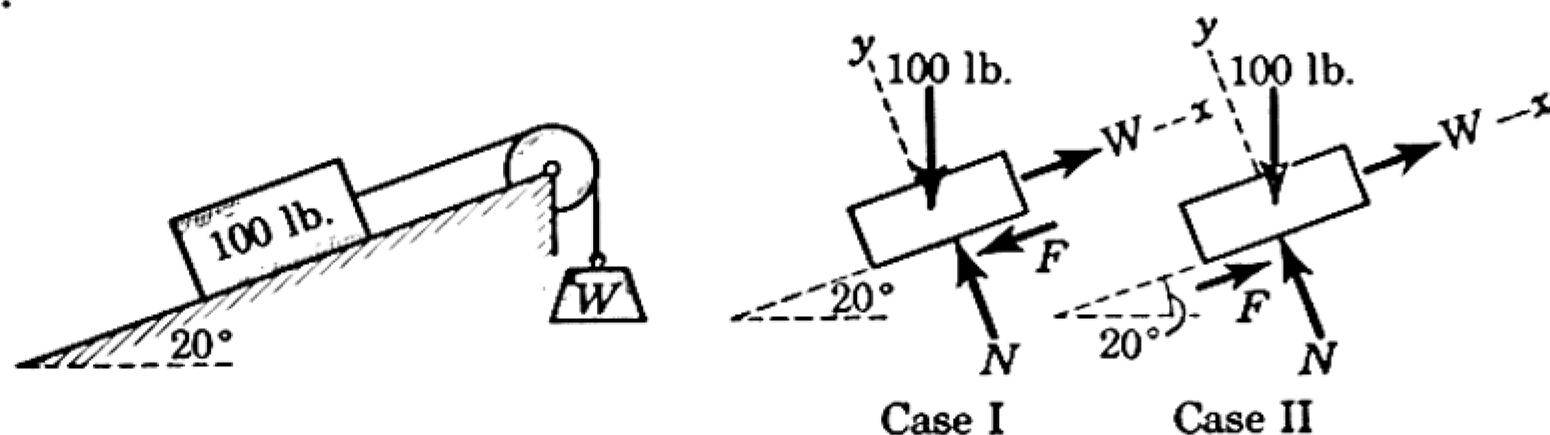
Further experiment shows that the friction force is independent of the area of contact. This is true as long as the pressure is not great. For high pressures the surface characteristics are changed and the frictional coefficient increases.

There are three types of problems in dry friction commonly encountered in mechanics. In the *first* type the condition of impending motion is to be investigated. It will be clear from the wording of the problem that the requirement of limiting static friction should be used. In the *second* type of problem impending motion need not exist, and therefore the friction force may be smaller than that given by Eq. (36) with the static coefficient. In this event the friction force will be determined only by the equations of equilibrium. In such a problem it may be asked whether or not the existing friction is sufficient to maintain the body at rest. To answer the question equilibrium may be assumed, and the corresponding friction force necessary to maintain this state can be calculated from the equations of equilibrium. This friction force may then be compared with the maximum static friction which the surfaces can support as calculated from Eq. (36) with $f = f_s$. If F is less than that given by Eq. (36), it follows that the assumed friction force can be supported and therefore the body is at rest. If the calculated value of F is greater than the limiting value, it follows that the given surfaces cannot support that much friction force, and therefore motion exists and the friction becomes kinetic. The *third* type of problem involves relative motion between the contacting surfaces, and here the kinetic coefficient of friction applies. For this case Eq. (36) with $f = f_k$ will always give the kinetic friction force directly.

The foregoing discussion applies to all dry contacting surfaces and, to a limited extent, to moving surfaces which are partially lubricated. Some typical values of the coefficients of friction are given in Table B2, Appendix B. These values are only approximate and are subject to considerable variation, depending on the exact conditions prevailing. They may be used, however, as typical examples of the magnitudes of frictional effects. When a reliable calculation involving friction is required, it is often desirable to determine the appropriate friction coefficient by experiment wherein the surface conditions of the problem are duplicated as closely as possible.

SAMPLE PROBLEMS

489. Determine the range of values which the weight W may have such that the 100 lb. block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



PROB. 489

Solution: The maximum value of W will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane as shown in the free-body diagram of the block for case I in the figure. Applying the equations of equilibrium gives

$[\Sigma F_y = 0]$ $\qquad N - 100 \cos 20° = 0, \qquad N = 94$ lb.,

$[F = fN]$ $\qquad F = 0.30 \times 94.0 = 28.2$ lb.,

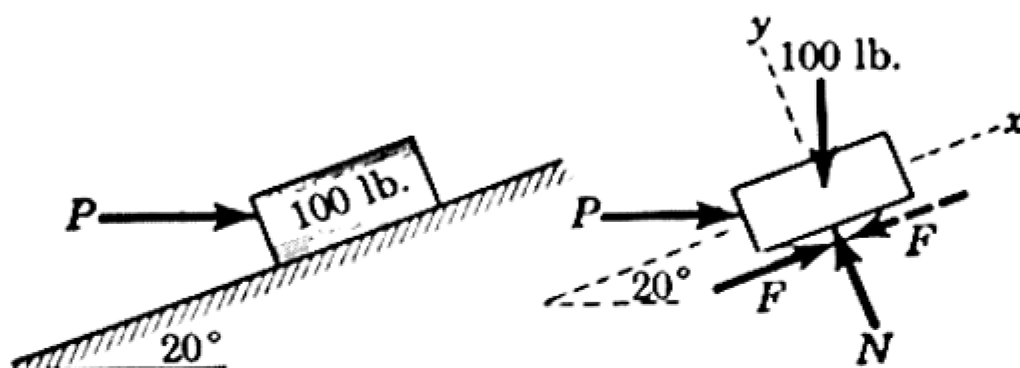
$[\Sigma F_x = 0]$ $\qquad W - 28.2 - 100 \sin 20° = 0, \qquad W = 62.4$ lb. *Ans.*

The minimum value of W is determined when motion is impending down the plane. The friction force on the block will act up the plane to oppose the tendency to move as shown in the free-body diagram for case II. Equilibrium in the x-direction requires

$[\Sigma F_x = 0]$ $\qquad W + 28.2 - 100 \sin 20° = 0, \qquad W = 6.0$ lb. *Ans.*

Thus W may have any value from 6.0 lb. to 62.4 lb. and the block will remain at rest.

490. Determine the amount and direction of the friction force acting on the 100 lb. block shown if, first, $P = 50$ lb., and, second, $P = 10$ lb. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17.



PROB. 490

Solution: There is no way of telling from the statement of the problem whether the block is or is not on the verge of slipping or whether it is in motion

at the position shown. It is therefore necessary to make an assumption. Assume the friction force to be up the plane, as shown by the solid arrow, and the block to be in equilibrium. A balance of forces in both x- and y-directions gives

$[\Sigma F_x = 0]$ $\qquad P \cos 20° + F - 100 \sin 20° = 0,$

$[\Sigma F_y = 0]$ $\qquad N - P \sin 20° - 100 \cos 20° = 0.$

Case I, $P = 50$ lb.

Substitution into the first of the two equations gives

$$F = -12.8 \text{ lb.}$$

The negative sign means that, *if* the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane as represented by the dotted arrow. Conclusion on the magnitude of F cannot be reached, however, until it is verified that the surfaces are capable of supporting 12.8 lb. of friction force. This may be done by substituting $P =$ 50 lb. into the second equation, which gives

$$N = 111.1 \text{ lb.}$$

The maximum static friction force which the surfaces can support is then

$[F = fN]$ $\qquad F = 0.20 \times 111.1 = 22.2$ lb.

Since this force is greater than that required for equilibrium, it follows that the assumption of equilibrium was correct. The answer is, then,

$F = 12.8$ lb. down the plane. *Ans.*

Case II, $P = 10$ lb.

Substitution into the two equilibrium equations gives

$$F = 24.8 \text{ lb.}, \qquad N = 97.4 \text{ lb.}$$

But the maximum possible static friction force is

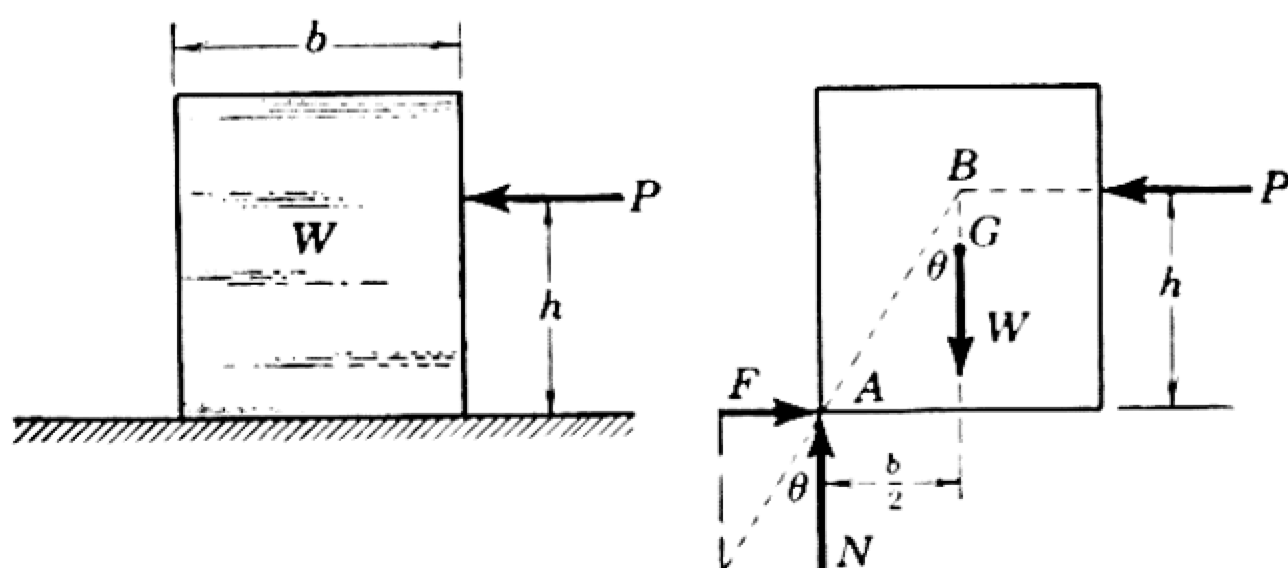
$[F = fN]$ $\qquad F = 0.20 \times 97.4 = 19.5$ lb.

It follows that 24.8 lb. of friction cannot be supported. Therefore equilibrium cannot exist, and the correct value of the friction force is obtained by using the kinetic coefficient of friction accompanying the motion down the plane. Thus the answer is

$[F = fN]$ $\qquad F = 0.17 \times 97.4 = 16.6$ lb. up the plane. *Ans.*

It should be noted that, even though ΣF_x is no longer equal to zero, equilibrium does exist in the y-direction so that $\Sigma F_y = 0$.

491. A homogeneous rectangular block of weight W rests on a horizontal plane and is subjected to the horizontal force P as shown. If the coefficient of friction is f, determine the greatest value which h may have such that the block will slide without tipping.



PROB. 491

Solution: If the block is on the verge of tipping, the entire reaction between the plane and the block will be at A. The free-body diagram of the block for this condition is shown in the right side of the figure. If P is sufficient to cause slipping, the friction force is the limiting value fN, and the angle θ becomes $\theta = \tan^{-1} f$. The resultant of F and N passes through a point B through which P must also pass since three coplanar forces in equilibrium are concurrent. Hence from the geometry of the block

$$\tan \theta = f = \frac{b/2}{h}, \qquad h = \frac{b}{2f}. \qquad \textit{Ans.}$$

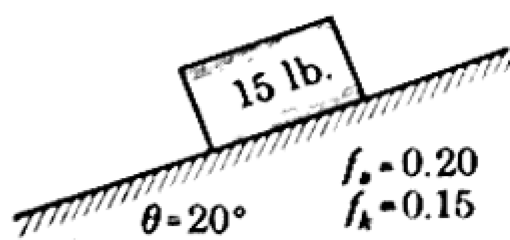
If h were greater than this value, moment equilibrium about A would not be satisfied. For h less than $b/2f$ the resultant of F and N would be concurrent with P and W at a point below B. Thus this resultant would act not at A but at some point to the right of A.

PROBLEMS

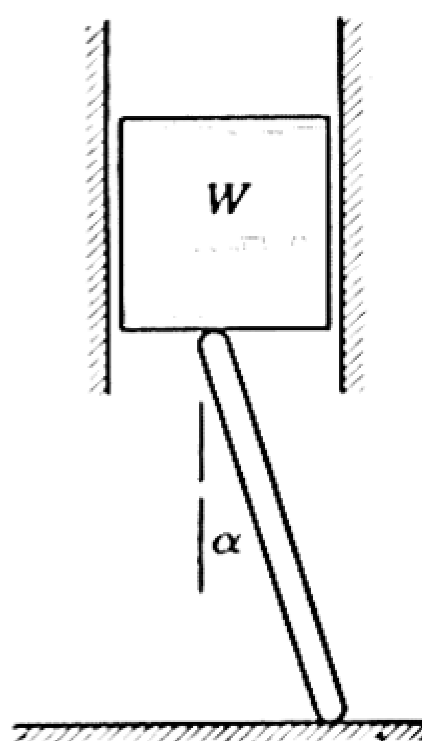
492. Determine the maximum angle θ which an inclined plane may have with the horizontal and not cause a block resting upon it to slide down. The coefficient of static friction is f. (This angle is known as the *angle of repose*.)

493. Prove whether the block is in equilibrium or whether it is sliding. Find the friction force F acting on the block.

494. Determine the maximum angle α for which the bar will not slip. Neglect the weight of the bar and assume that the coefficient of static friction f is the same for both ends of the bar.

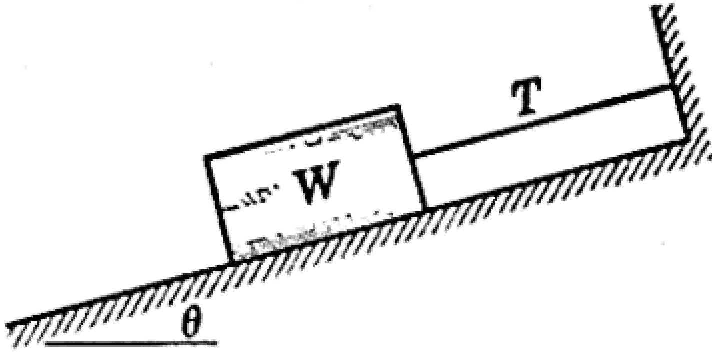


PROB. 493



PROB. 494

495. The block of weight W is placed on the incline and is prevented from sliding by the tension T in the cord. If $f < \tan\theta$, determine the maximum and minimum values which T may have and show that the tension depends on the manner in which the block was placed on the plane.



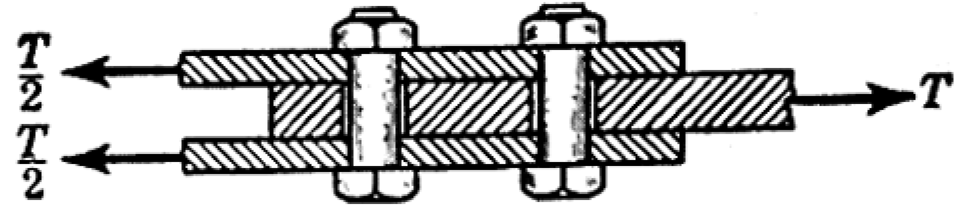
Prob. 495

496. If the angle θ in Prob. 493 is reduced to 10 deg. and the block is released with an initial velocity up the plane, (a) determine what it will do and the friction force F acting on it. (b) If it is given an initial start down the plane, determine its action and the friction force.

Ans. (a) Block will stop and remain at rest with $F = 2.60$ lb., (b) Block will continue to move with $F = 2.21$ lb.

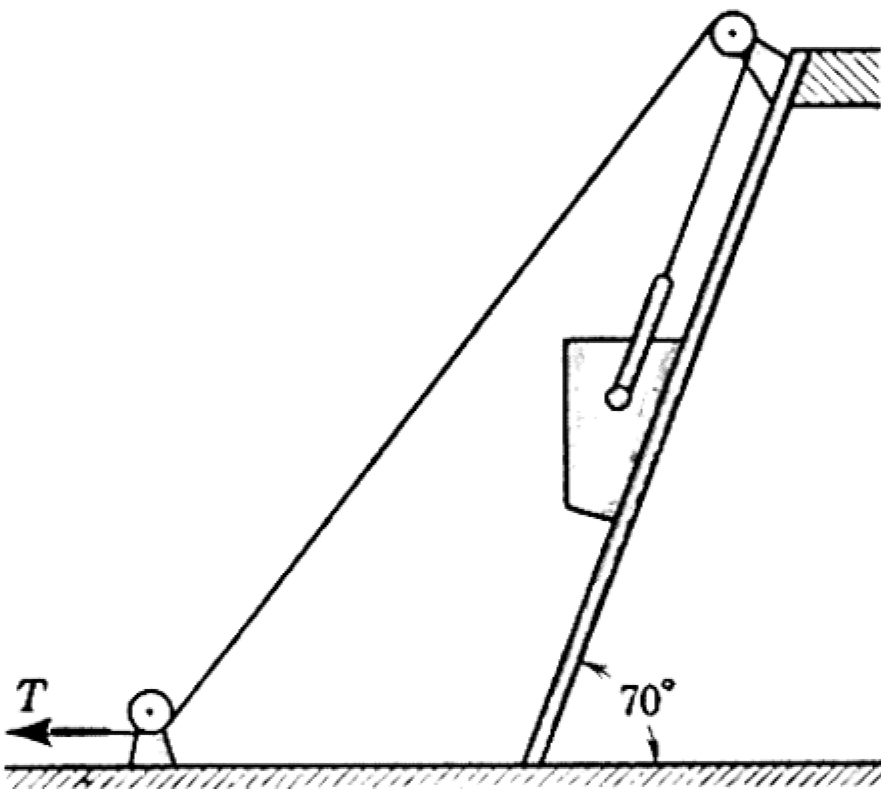
497. For the block in Prob. 491, if $h = 2$ ft., $b = 2$ ft., and $f = 0.4$, determine the distance x to the point on the right of A at which the resultant force between the plane and the block acts when slipping impends. *Ans.* $x = 0.2$ ft.

498. The steel straps shown are held together by two bolts which have been tightened with a torque wrench so that the tension in each bolt is 2000 lb. If the holes in the center strap were drilled slightly oversized, find the tension T which can be supported before the joint slips enough to induce shear in the bolts. Take the coefficient of friction to be 0.2.

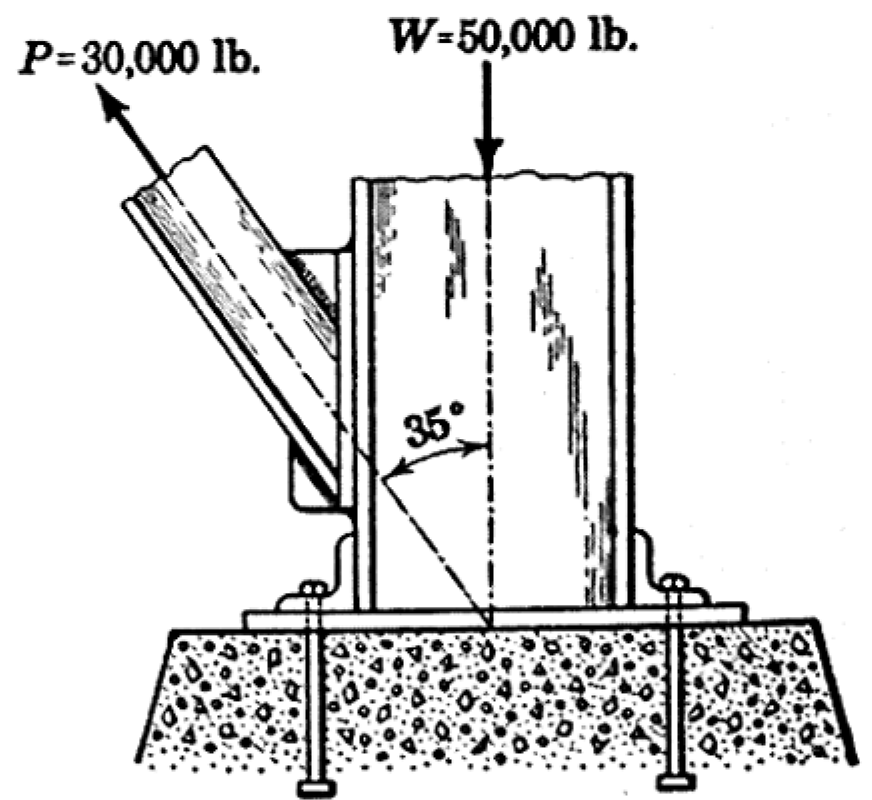


Prob. 498

499. A simple hoist for handling concrete is shown in the figure. If the combined weight of bucket and concrete is 5000 lb. and the coefficient of friction between the bucket and its guide is 0.30, determine the tension T in the cable required (a) to raise the load and (b) to lower the load.



Prob. 499



Prob. 500

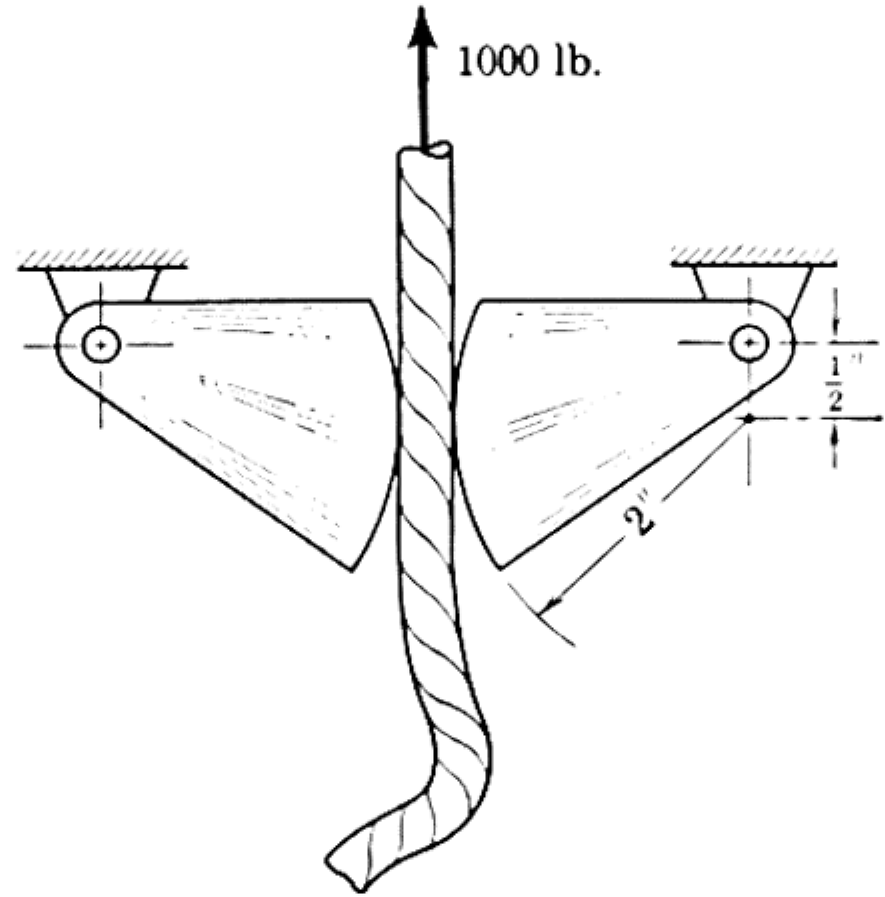
500. Find the total horizontal force S (shear) which the two anchor bolts must exert on the structure to prevent slipping if $f = 0.3$.

Ans. $S = 9590$ lb.

501. A flywheel 2 ft. in diameter is forced onto a 3 in. shaft. If a couple consisting of two 400 lb. forces applied tangentially to the rim is required to slip the wheel on the fixed shaft and the coefficient of friction between the shaft and bearing surface is 0.3, determine the pressure in the bearing. The axial length of the flywheel hub is 4 in.

Ans. $p = 566$ lb./in.^{2}

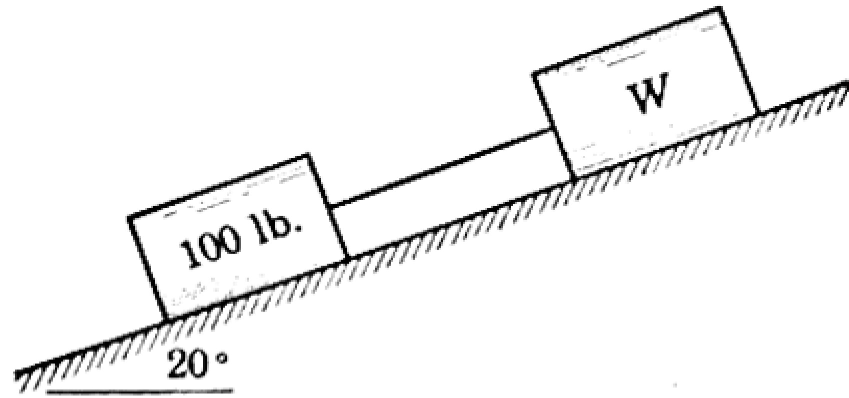
502. In the figure is shown a device which secures a rope or cable under tension by reason of large friction forces developed. For the position shown determine the total reaction R on the cam bearing.



PROB. 502

503. The two blocks shown are connected by a cord and are placed on the incline with zero tension in the cord. The coefficient of static friction for the 100 lb. block is 0.2, and that for the block of weight W is 0.4. Determine (*a*) the minimum weight W such that the blocks will remain at rest on the incline and (*b*) the friction force F acting on the block W if it weighs 800 lb.

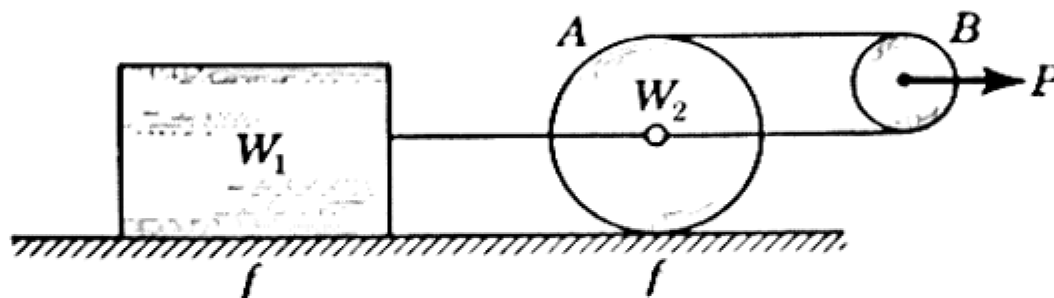
Ans. (*a*) $W = 453$ lb., (*b*) $F = 289$ lb.



PROB. 503

504. The pulley arrangement shown is used to move the block of weight W_1. The weight of A is W_2, and the weight of B is negligible. Determine the force P required to move the block and the maximum value of W_1 such that the wheel A does not slip. The coefficient of static friction for wheel and block is f.

Ans. $P = \dfrac{2fW_1}{3}$, $W_1 = 3W_2$

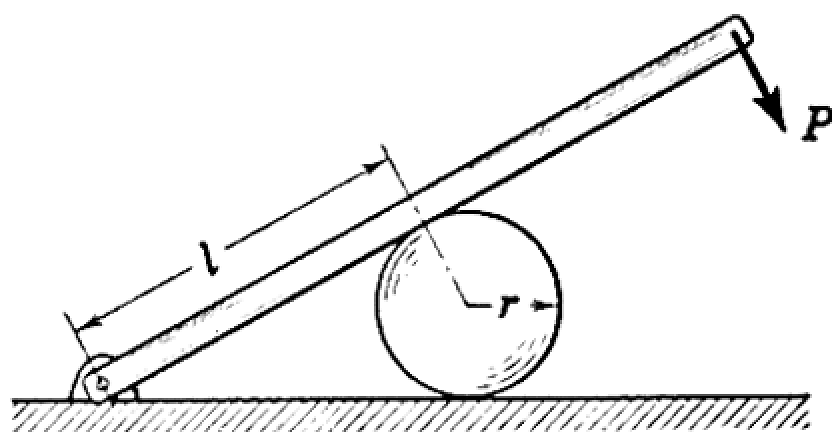


PROB. 504

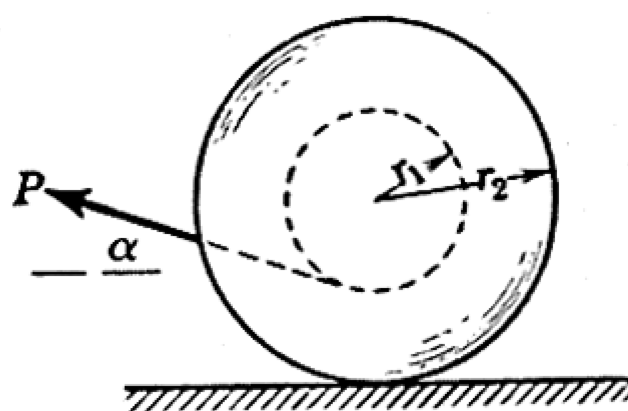
505. A 3000 lb. car with 10 ft. wheelbase has a center of gravity 2 ft. from the road and midway between the axles. If the friction coefficient between tires and road is 0.80, find the angle θ with the horizontal made by the steepest grade which the car can climb at constant speed before the wheels slip. What torque M is applied to each 28 in. diameter rear wheel by the engine under these conditions? Neglect friction under the front wheels.

Ans. $\theta = 25° 28'$, $M = 752$ lb. ft.

506. The lever shown is used to exert a large force on the roller. If the coefficient of friction for both pairs of surfaces is f, determine the minimum distance l at which the roller may be placed and not slip under application of a load P which is large compared with the weight of the roller.



PROB. 506

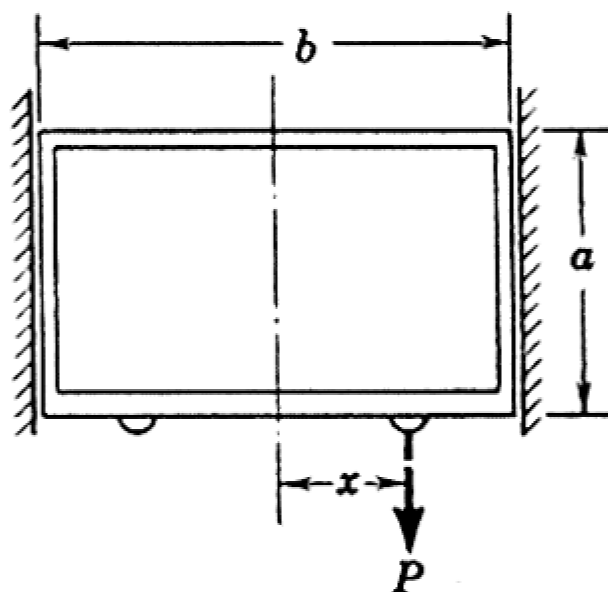


PROB. 507

507. The wheel shown will roll to the left when the angle α of the cord is small. When α is large the wheel rolls to the right. Determine from the geometry of the free-body diagram and without calculation the angle α for which the wheel will not roll in either direction. If the coefficient of friction is f and the weight of the wheel is W, calculate the value of P for which the wheel will slip for the critical value of α.

508. Find the maximum distance x from the horizontal center line of the drawer at which the force P may be applied and still open the drawer without binding at the corners. Neglect friction on the bottom of the drawer and take the coefficient of friction at the corners to be f.

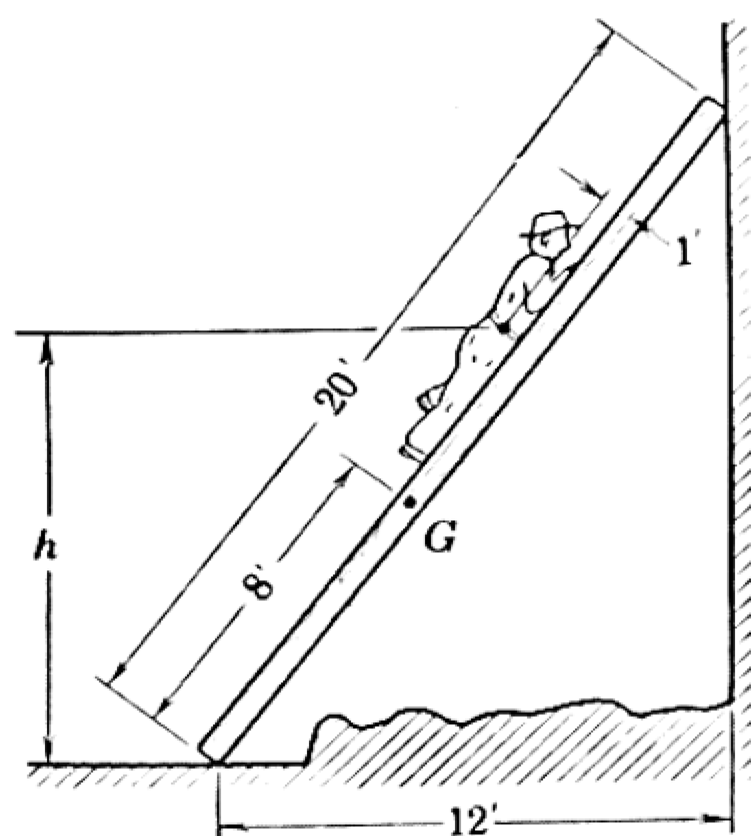
Ans. $x = \dfrac{a}{2f}$



PROB. 508

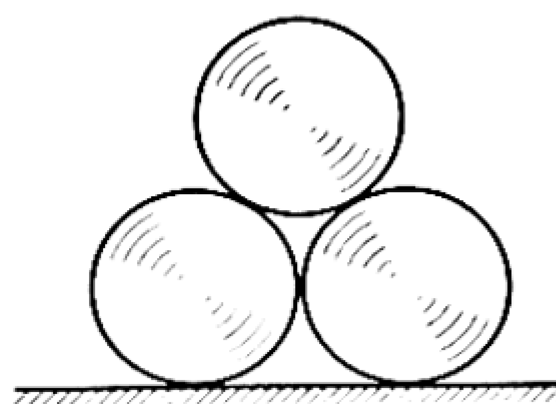
509. If, for the drawing of Prob. 494, $W = 50$ lb., the bar is uniform and weighs 15 lb., and the static coefficient of friction for each end is $f = 0.2$, determine the maximum angle α for equilibrium and find which end of the bar slips first. Neglect friction along the vertical guides. *Ans.* $\alpha = 9.9°$, upper end

510. The center of gravity of the 50 lb. ladder which is 20 ft. long is at G. The coefficient of friction for the upper end of the ladder and the wall is 0.3, and that for the bottom and the ground is 0.4. Determine the maximum safe height h which a 150 lb. man may climb. *Ans.* $h = 12.0$ ft.



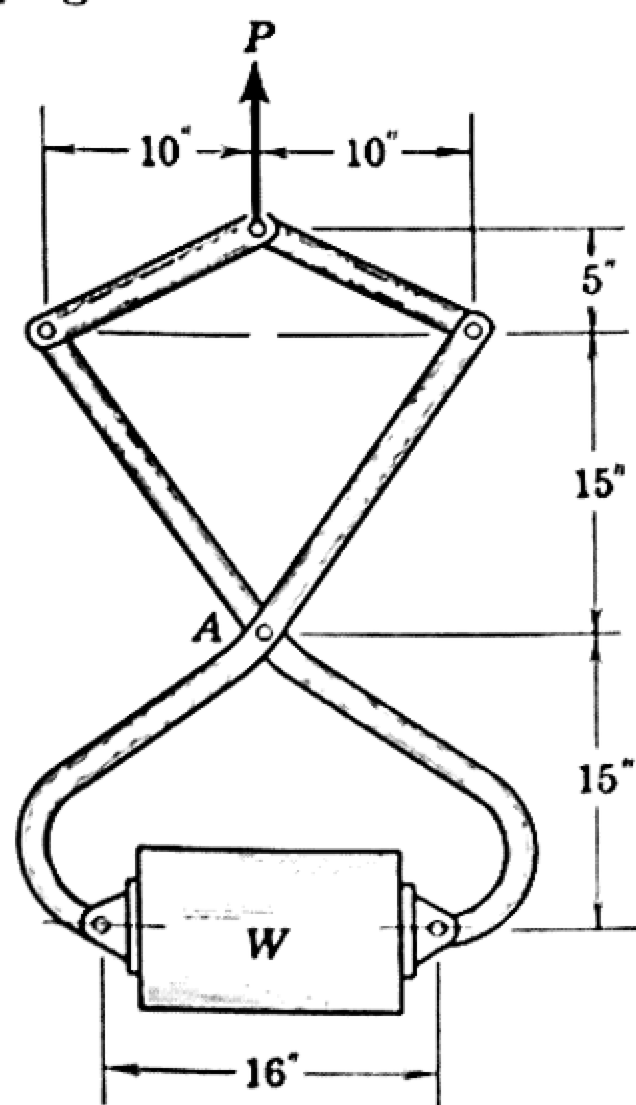
Prob. 510

511. The three identical rollers are stacked as shown. If the friction coefficient is the same for all pairs of surfaces, find the minimum value of f such that the rollers will not slip. *Ans.* $f = 0.268$



Prob. 511

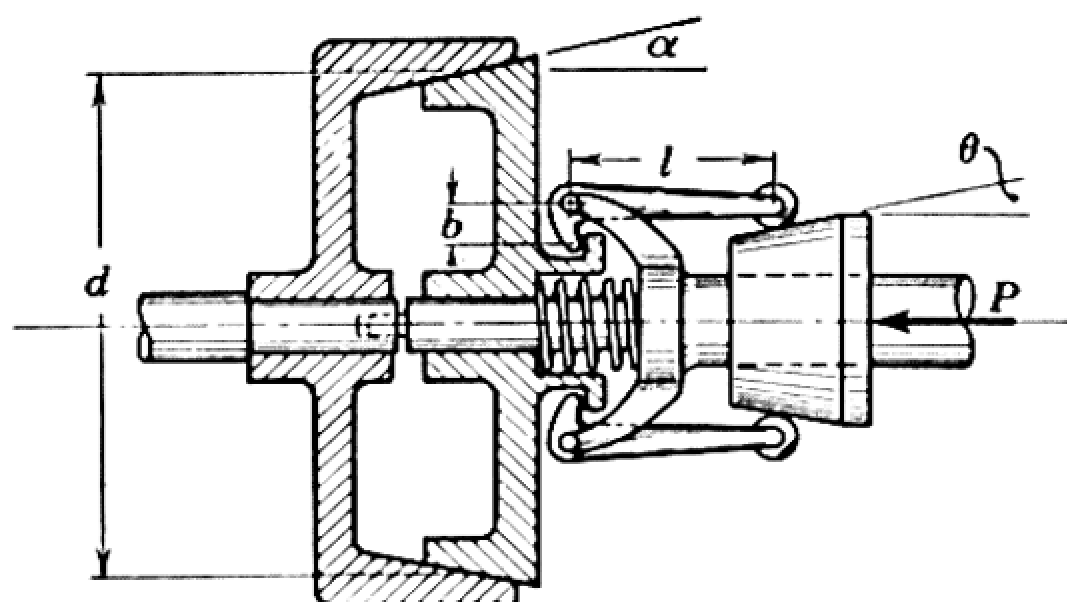
512. For the friction tongs shown determine the minimum coefficient of friction f which can exist between the block and the tongs without the block slipping.



Prob. 512

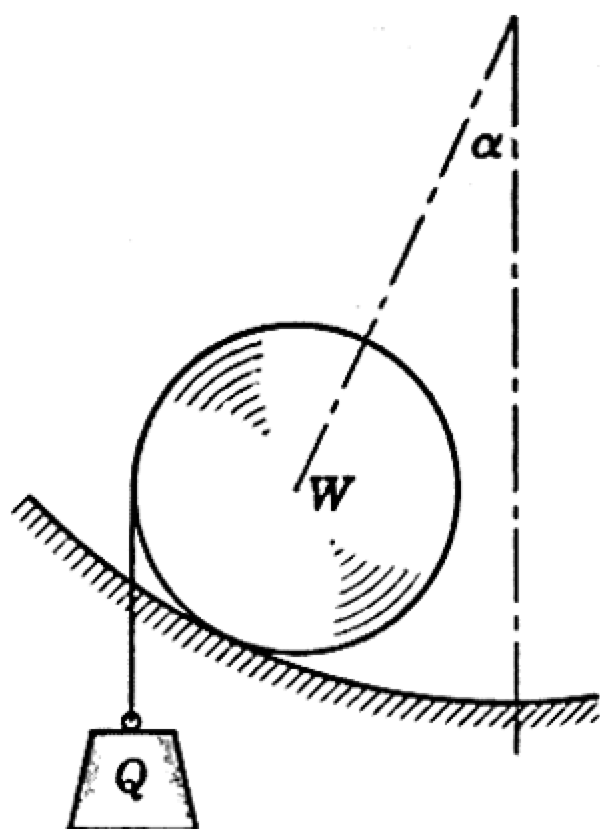
513. Determine the torque M which the cone clutch shown can transmit when the engaging spring exerts a compressive force R and the force P on the disengaging collar is equal to zero. Assume that the forces between the clutch surfaces act at the mean diameter d and that after engagement any friction force in the direction of a cone element is negligible. The coefficient of friction between the cone surfaces is f.

Ans. $M = \dfrac{fRd}{2 \sin \alpha}$

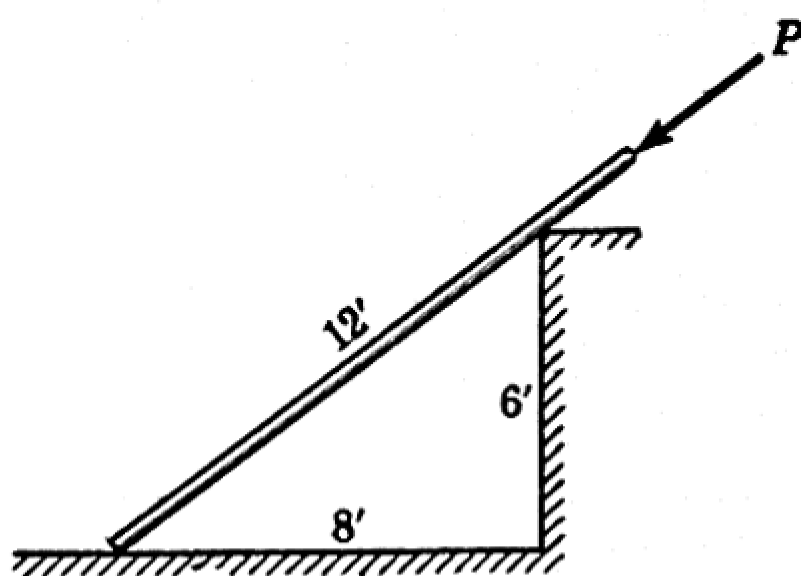


Prob. 513

514. A wheel whose weight is W carries a load Q by means of a cord wrapped around its periphery as shown. It rolls up the circular track without slipping and comes to rest in the equilibrium position shown. Find angle α and the minimum coefficient of friction f such that the wheel does not slip.



PROB. 514

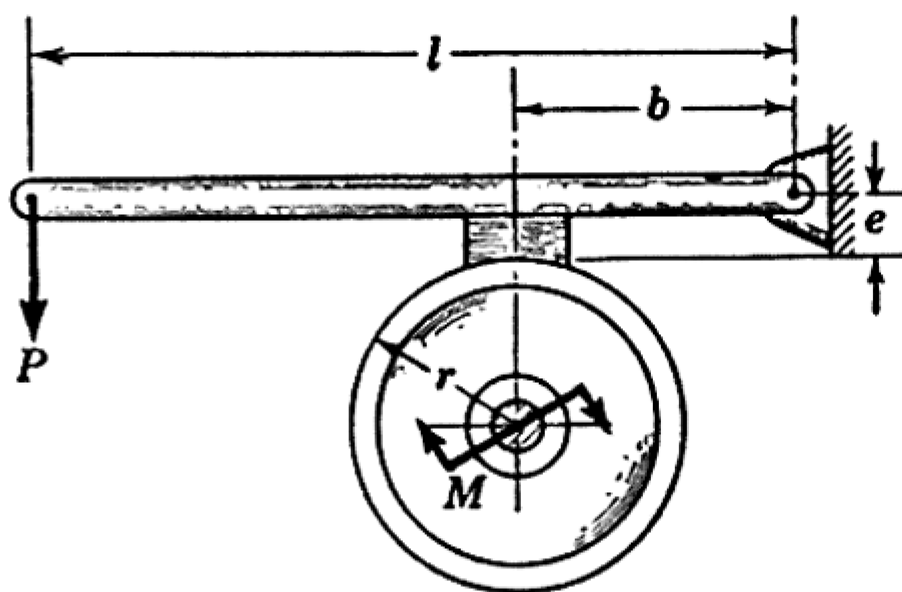


PROB. 515

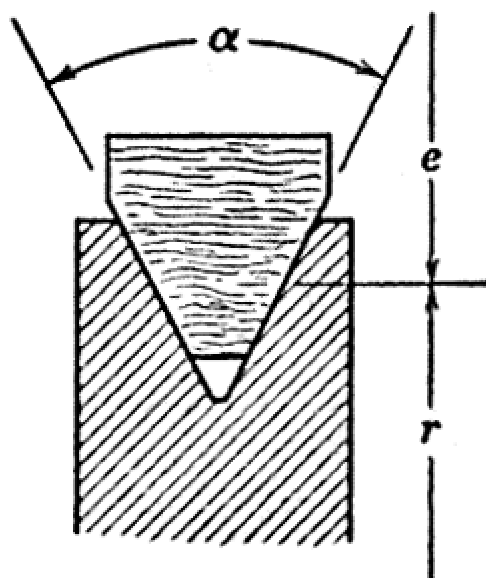
515. Determine the force P required to slip the uniform 50 lb. bar. The coefficient of friction at both contacts is 0.40.

516. The single-lever block brake shown prevents rotation of the flywheel under an applied clockwise torque M. (a) Find the required force P if the coefficient of friction is f. (b) How much force is required if $b = fe$?

Ans. (a) $P = \frac{M}{r}\frac{b - fe}{fl}$, (b) zero



PROB. 516



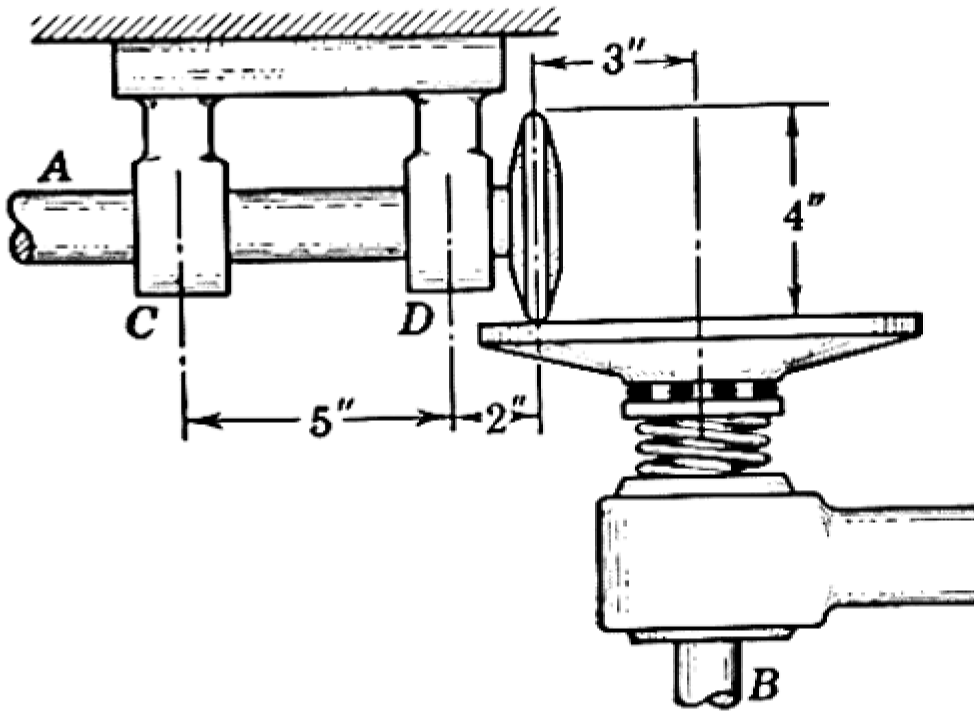
PROB. 517

517. The brake block and flywheel of Prob. 516 are replaced by a V-grooved pulley and matching V-shaped brake block of angle α as shown in the figure. The distance e is measured to the mean radius r of the pulley groove. (a) Determine the force P necessary to brake the applied clockwise torque M. (b) For what angle α will the brake be self-locking for given values of b, f, and e?

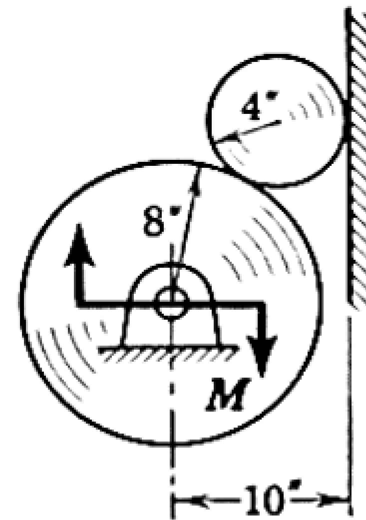
Ans. (a) $P = \frac{M}{r}\frac{b \sin\frac{\alpha}{2} - fe}{fl}$, (b) $\alpha = 2\sin^{-1}\frac{fe}{b}$, $b > fe$

518. Shaft A drives shaft B at constant speed by means of the attached wheel and friction disk. The center lines of the two shafts intersect at right angles. Constant pressure between the friction surfaces is maintained by the coil spring which bears against the underside of the disk. In assembly the spring is compressed 3/8 in. If the coefficient of friction is 0.6 and a torque of 10 lb. ft. is to be maintained on shaft A, determine the necessary spring constant k to prevent slipping and the total forces on the bearings C and D.

Ans. k = 267 lb./in., C = 46.6 lb., D = 163 lb.



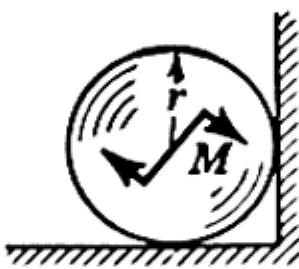
Prob. 518



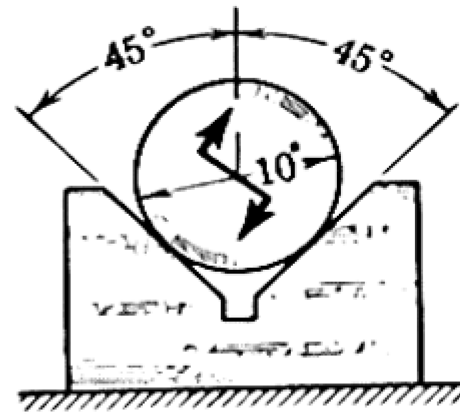
Prob. 519

519. Calculate the torque M required to turn the cylinder against the action of the 50 lb. roller above it. The coefficient of friction for both pairs of contacting surfaces is 0.3.

520. Determine the torque M required to spin the wheel of weight W in its position against the wall as shown. The coefficient of friction is f.



Prob. 520



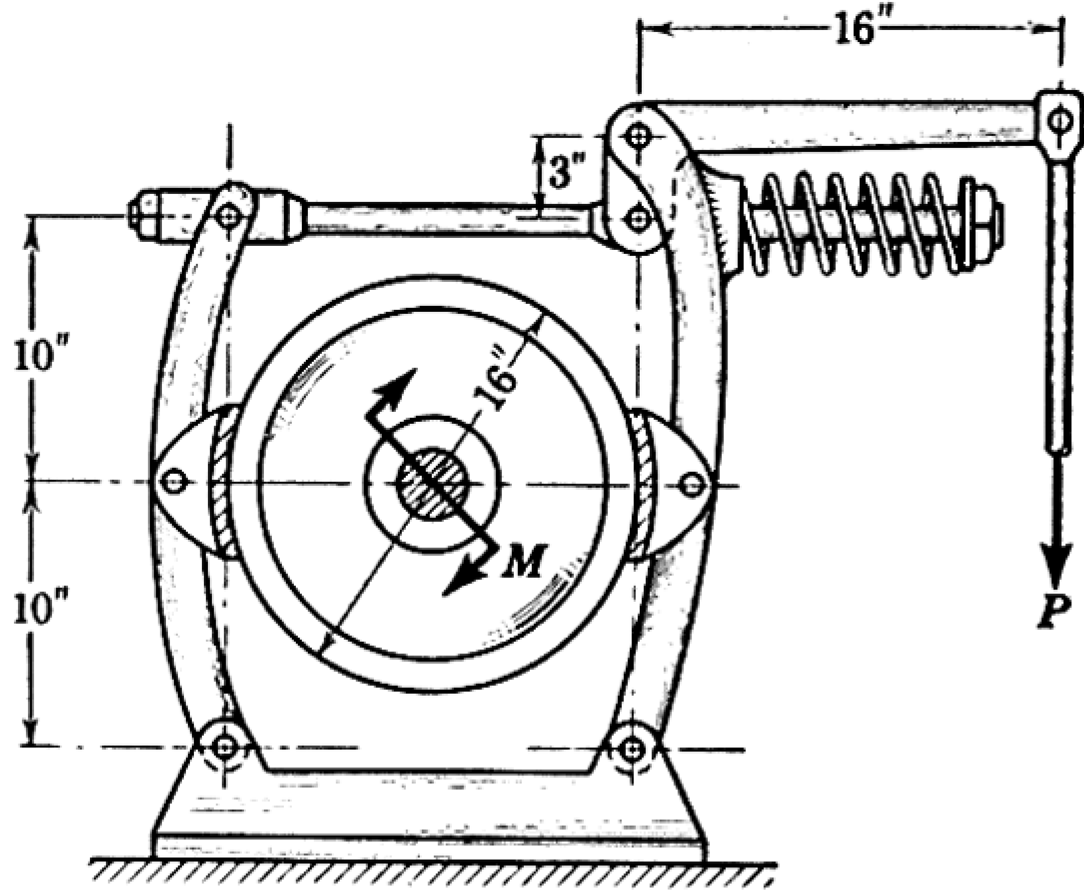
Prob. 521

* **521.** A torque of 9 lb. ft. is required to turn the 80 lb. shaft in the V-block shown. Determine the coefficient of friction between the shaft and the block.

Ans. f = 0.20

* **522.** The double block brake shown is applied to the flywheel by means of the action of the spring. To release the brake a force P is applied to the control lever. In the operating position with P = 0 the spring is compressed 1 in. Select a spring with an appropriate constant k which will provide sufficient

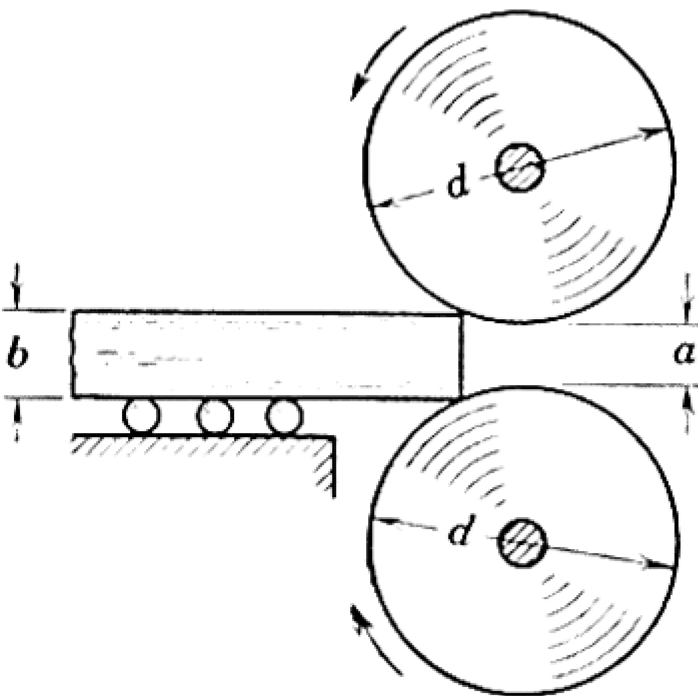
force to brake the flywheel under the torque $M = 60$ lb. ft. if the coefficient of friction for both brake shoes is $f = 0.2$. Neglect the dimensions of the brake shoes.



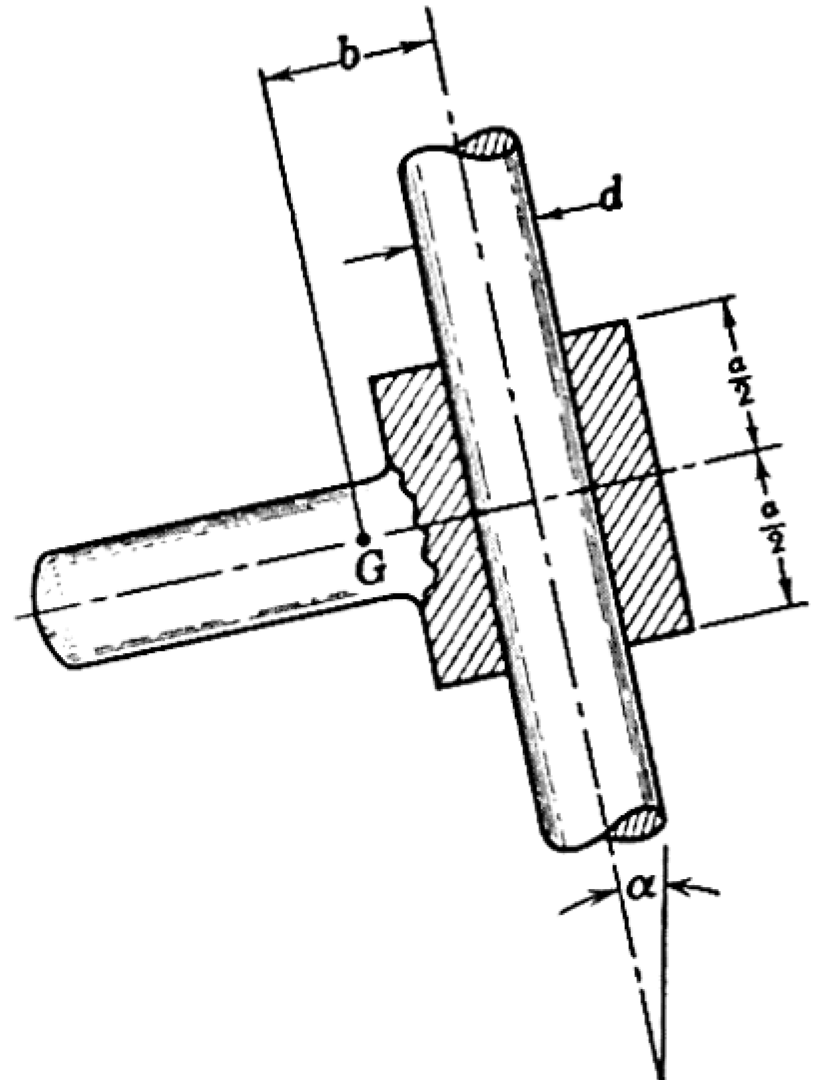
PROB. 522

* **523.** In the figure are shown the elements of a rolling mill. Determine the maximum thickness b which the slab may have and still enter the rolls by means of the friction between the slab and the rolls. Assume that the coefficient of friction is f and that $b - a$ is small compared with d.

Ans. $b = a + \dfrac{f^2 d}{2}$



PROB. 523

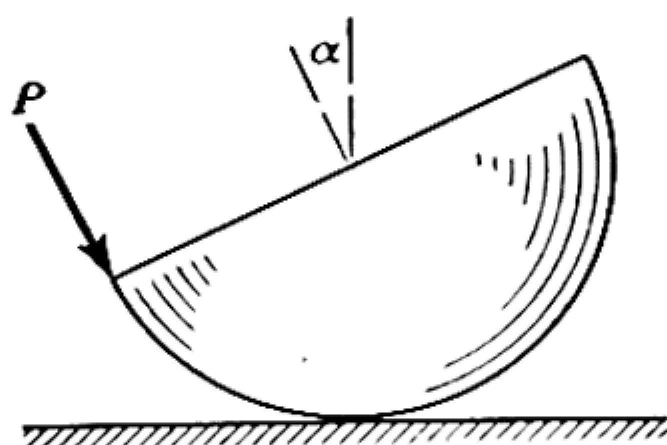


PROB. 524

* **524.** The collar with center of gravity at G fits loosely on the shaft. As the angle α is decreased, the collar will begin to slide down the shaft. Find the value of α at which slipping begins for a given coefficient of friction f.

Ans. $\alpha = \tan^{-1} \dfrac{a - 2bf}{f^2 d}$ provided $b < \dfrac{a}{2f}$

* **525.** A hemispherical body of uniform density is subjected to a force P at the rim as shown. If P is increased slowly and is always kept normal to the rim surface, determine the angle of tilt α at which slipping occurs. The coefficient of friction is 0.3. *Ans.* $\alpha = 41°$ (graphical solution)



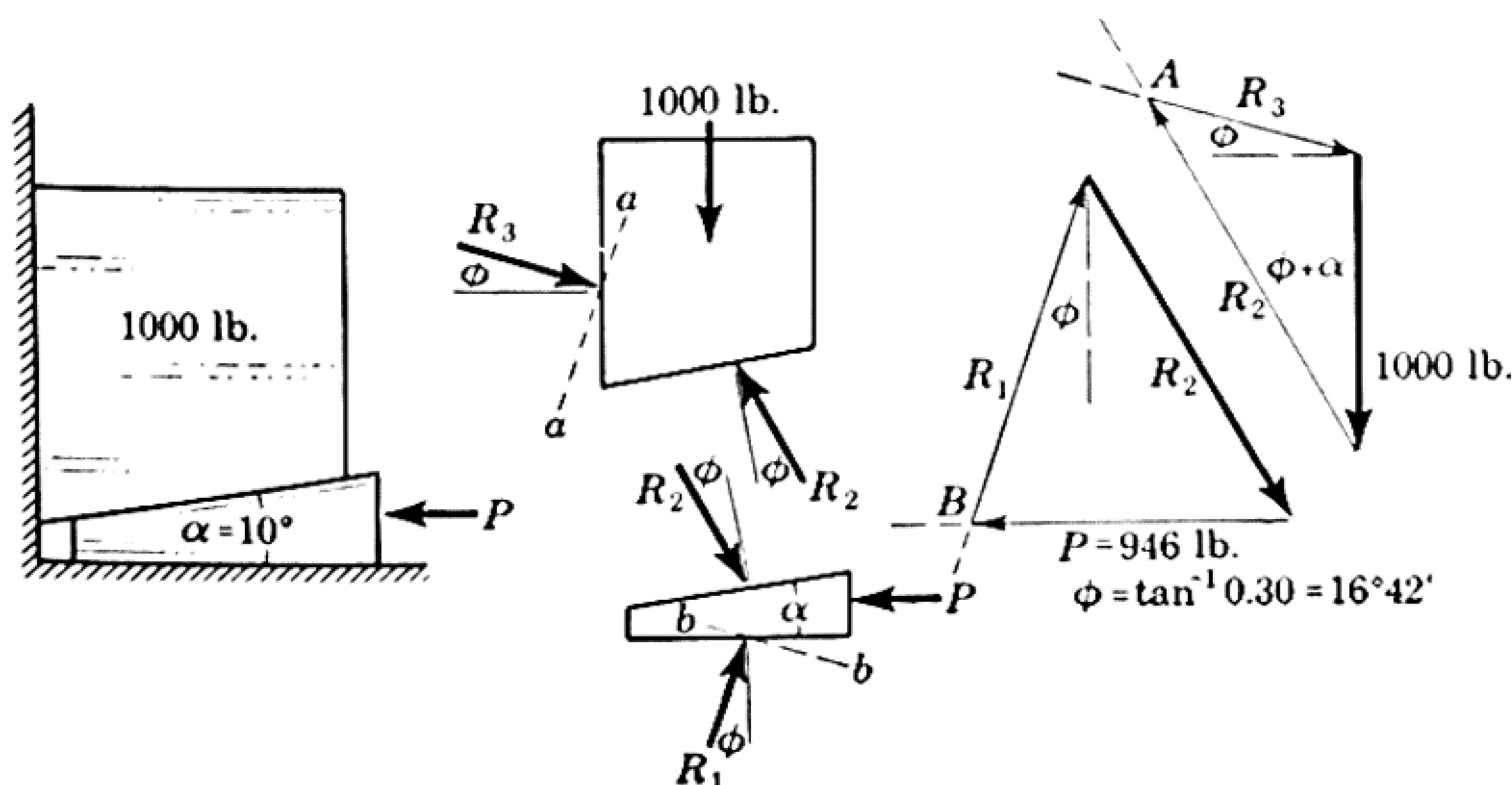
PROB. 525

52. Wedges.

A wedge is used as a means of producing small adjustments in position of a body or as a means of applying large forces. Wedges depend on friction for their action. The solution of wedge problems is often simplified by using graphical methods, and errors inherent in graphical construction are well within those due to the uncertainty of the friction coefficients.

SAMPLE PROBLEM

526. Determine the force P applied to the 10 deg. wedge necessary to start the 1000 lb. block upward. The coefficient of static friction between all surfaces is 0.30, and the weight of the wedge is negligible.



PROB. 526

Graphical Solution: The free-body diagrams of the block and wedge are shown in the figure. The reactions R_1, R_2, and R_3 are all inclined with their normals an amount equal to the friction angle ϕ, since slipping occurs simultaneously at both surfaces, and lean in a direction to oppose the impending motion.

The equilibrium of the block is established by first laying off the 1000 lb. force to scale, as shown in the upper right part of the figure. Next the known directions of the reactions R_2 and R_3 are used to determine point A, which then establishes the magnitudes of R_2 and R_3. A similar equilibrium triangle for the wedge is constructed by taking the reaction R_2 which is now known and constructing the two lines parallel to the known directions of P and R_1, as shown in the second diagram. The intersection of these two lines determines point B, and thus the magnitude

$$P = 946 \text{ lb.} \qquad \textit{Ans.}$$

is scaled off the drawing.

Algebraic Solution: The simplest choice of axes for the necessary force summations is, in the case of the block, in the direction a–a normal to R_3, and for the wedge in the direction b–b normal to R_1. In this way a simultaneous solution of two equations is avoided. The angle between R_2 and the a-direction amounts to $2\phi + \alpha = 43° 24'$. Thus for the block

$$[\Sigma F_a = 0] \qquad 1000 \cos (16° 42') - R_2 \cos (43° 24') = 0,$$

$$R_2 = 1320 \text{ lb.}$$

For the wedge the angle between R_2 and the b-direction is $\pi/2 - (2\phi + \alpha) = 46° 36'$. Equilibrium then requires that

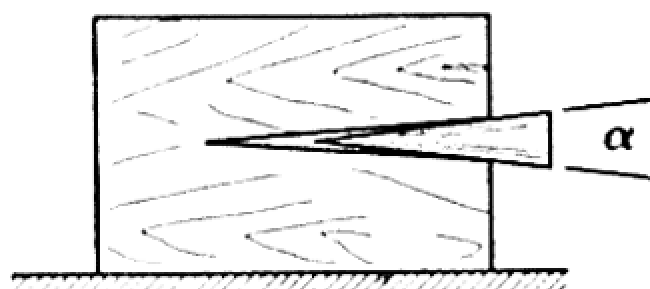
$$[\Sigma F_b = 0] \qquad 1320 \cos (46° 36') - P \cos (16° 42') = 0,$$

$$P = 946 \text{ lb.} \qquad \textit{Ans.}$$

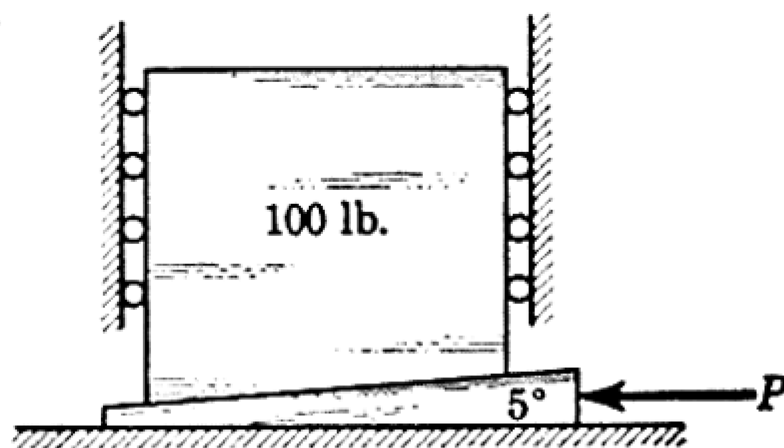
If the given coefficients of friction between mating surfaces were not all alike, the same procedure would be employed and the modified friction angles would be used.

PROBLEMS

527. A wedge of angle α is used to split wood as shown. If the coefficient of friction between the wedge and the wood is f, determine the maximum angle α for which the wedge will be self-locking.



PROB. 527

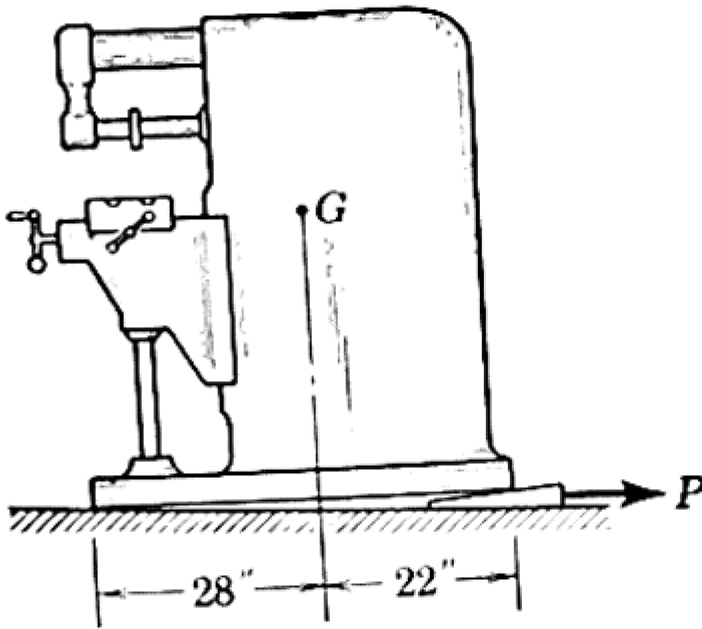


PROB. 528

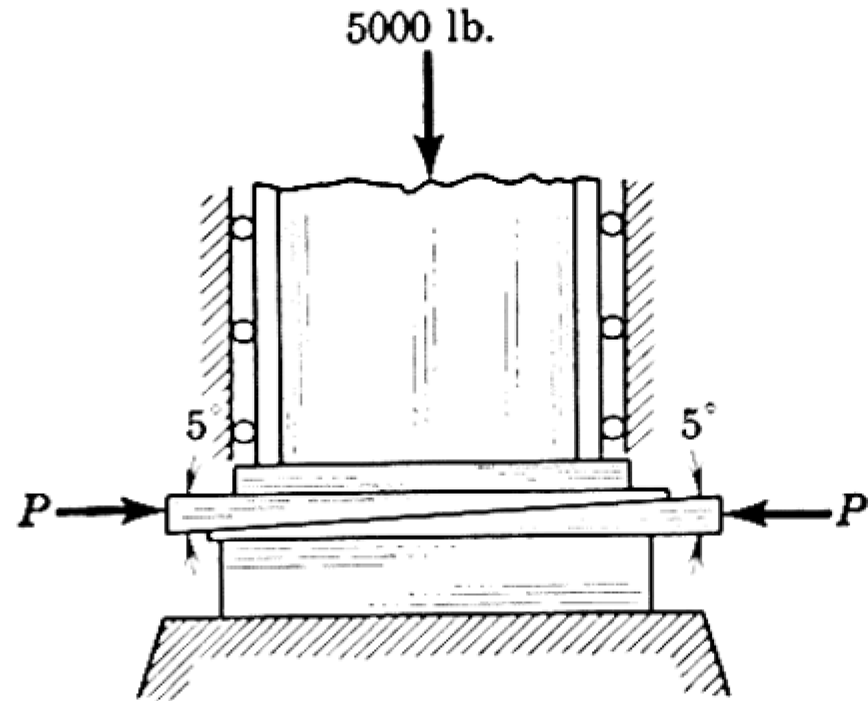
528. By neglecting friction in the rollers, find (a) the force P necessary to start the wedge moving under the weight and (b) the horizontal force P' required to pull the wedge from under the block. The coefficient of friction between the wedge and the weight is 0.25, and that between the wedge and the horizontal surface is 0.20.

529. A 5 deg. wedge is used to position the 3500 lb. milling machine shown. If the center of gravity of the machine is at G and the coefficient of friction of both wedge surfaces is 0.30, find the force P required to withd.aw the wedge. Neglect the small angle through which the machine is tipped.

Ans. $P = 994$ lb.



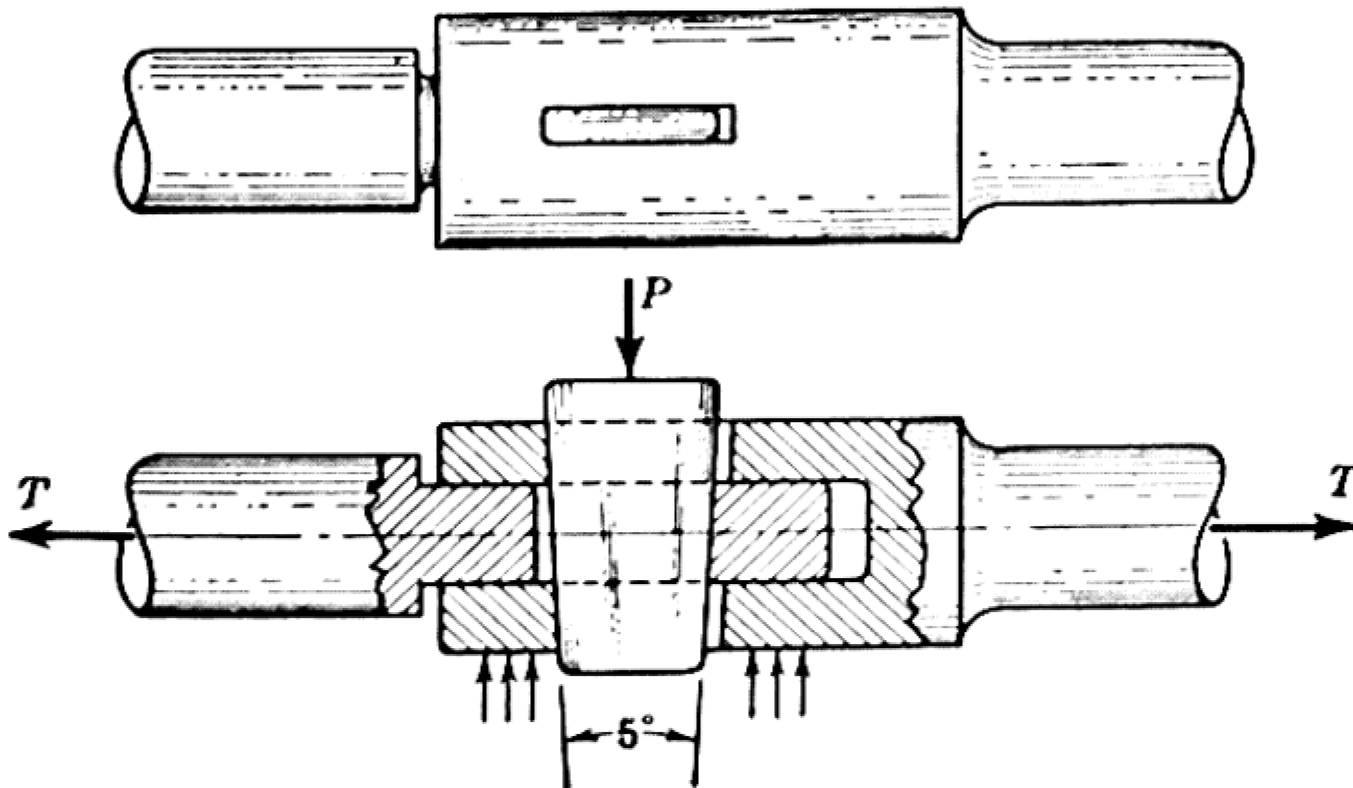
Prob. 529



Prob. 530

530. In adjusting the position of a vertical column under a load of 5000 lb., two 5 deg. wedges are used as shown. Determine the force P necessary to raise the load if the coefficient of friction for all surfaces is 0.4. *Ans.* $P = 4520$ lb.

531. Two shafts connected by a flat 5 deg. tapered cotter, as shown by the two views in the figure, are under a constant tension T of 200 lb. Find the force P required to move the cotter and take up any slack in the joint. The coefficient of friction is 0.20. Neglect horizontal friction between the shafts.

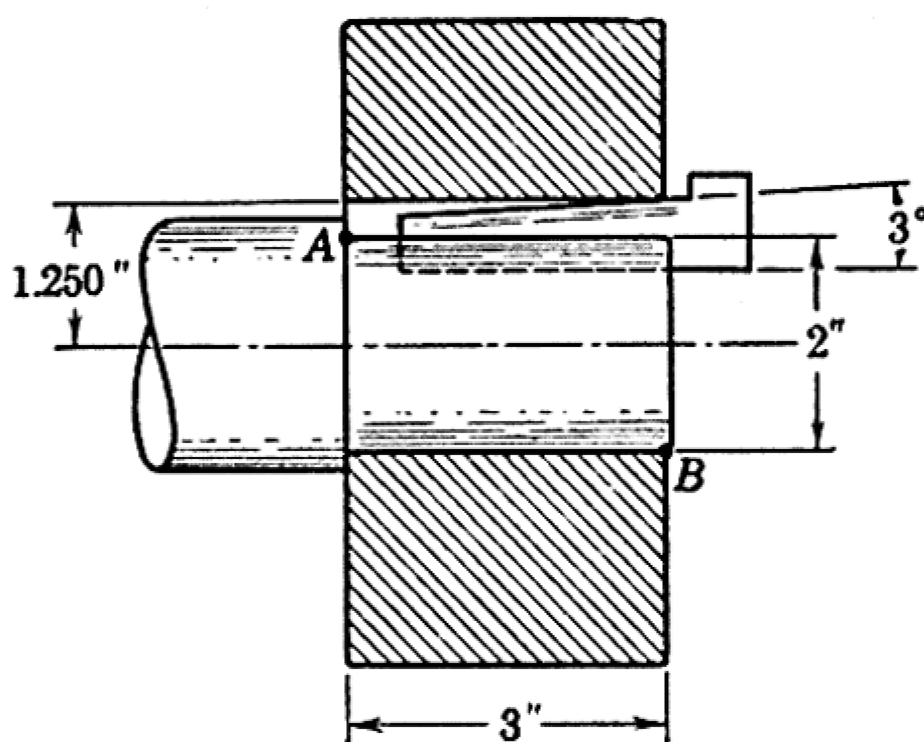


Prob. 531

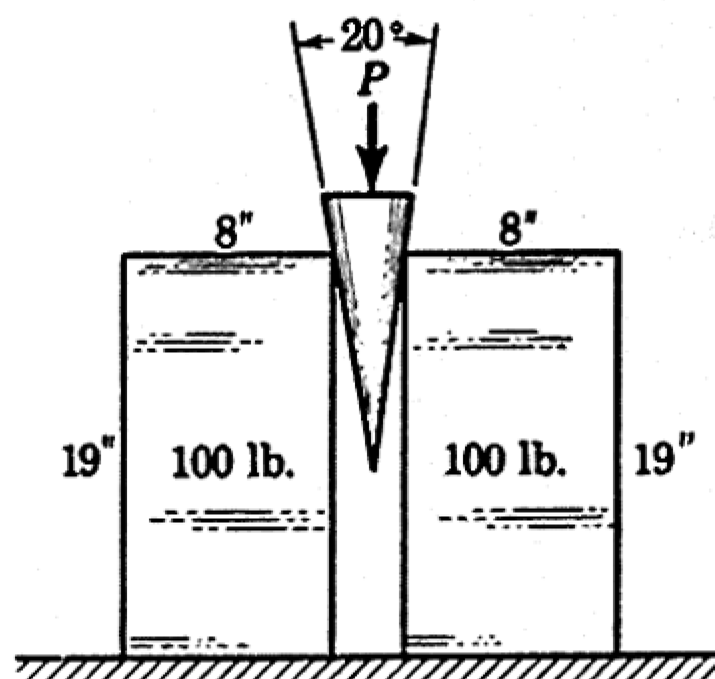
532. A collar which fits freely on a 2 in. shaft is fastened as shown by a key with 3 deg. of taper. With the key in position, contact between the collar and shaft occurs at points A and B. All resistance to horizontal movement of the collar is provided by the shoulder at A so that any friction at B is negligible.

If the key is seated by a horizontal force P of 400 lb. and the coefficient of friction is 0.3, determine the total reaction R at A while P is applied.

Ans. $R = 218$ lb.



PROB. 532



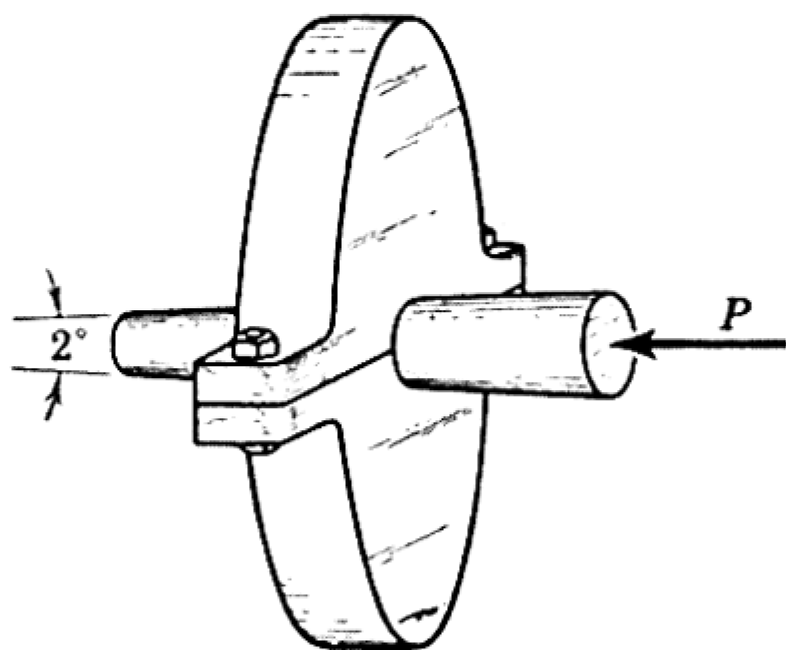
PROB. 533

533. The 20 deg. wedge shown is used to move the two 100 lb. blocks. If the coefficient of friction for all surfaces is 0.2, determine whether the blocks tip or slide and the force P necessary to cause them to tip or slide, whichever is the case.

* **534.** For the cone clutch of Prob. 513 determine the force P on the sliding collar required to disengage the clutch if (*a*) $f < \tan \alpha$ and (*b*) $f > \tan \alpha$. Assume that no torque is being transmitted so that all friction forces are in the direction of cone elements. (*Hint:* When $f > \tan \alpha$ the clutch surfaces are wedged together and a force P greater than that necessary to counterbalance R will be required. It may be assumed that the pressure normal to the cone surface is the same as when the clutch was initially wedged under the action of R alone.)

Ans. (*a*) $P = \frac{Rb}{l}\tan\theta$, (*b*) $P = \frac{Rb}{l}\left[1 + \frac{\sin(\phi - \alpha)}{\sin(\phi + \alpha)}\right]\tan\theta,\ \phi = \tan^{-1} f$

* **535.** The split disk shown has a hole with 2 deg. of taper and is held together by two bolts each of which has a negligible initial tension. A pin with mating taper is pressed into the hole with a force P of 1000 lb. If the coefficient of friction is 0.20, determine the tension T in each bolt (*a*) while P is acting and (*b*) after P has been removed. Assume that the pressure is constant over the surface of the hole and that its component normal to the surface remains unchanged upon removal of the load.



PROB. 535

Ans. (*a*) $T = 728$ lb., (*b*) $T = 732$ lb.

* **536.** A taper pin 2 in. long and 0.620 in. in diameter at the large end has a taper of 2.500 in. per foot of length. A force of 200 lb. is used to press the pin

into a matching tapered hole 2 in. deep, and ½ in. of the pin is left projecting out of the hole. If the coefficient of friction is 0.20 and the normal pressure between the pin and the hole is constant over the contact area of the pin, determine the torque M necessary to twist the pin in the hole after the 200 lb. force has been removed. (*Hint:* The pressure normal to the pin surface will remain essentially unchanged upon removal of the 200 lb. force. Also, the limiting friction force at any point will be the vector sum of the residual friction force along the cone element and the friction force tangent to the cross-section circle of the pin at that point.) *Ans.* $M = 21.5$ lb. in.

53. Screws. Screws are used for fastenings and for transmitting power or motion, and in each case the friction developed in the threads is important. For transmitting power or motion the square thread is more efficient than the V-thread; the analysis which follows will be confined to the square thread.

Consider the square-threaded jack, Fig. 70, under the action of the axial load W and the applied moment Pa. The force R exerted by the

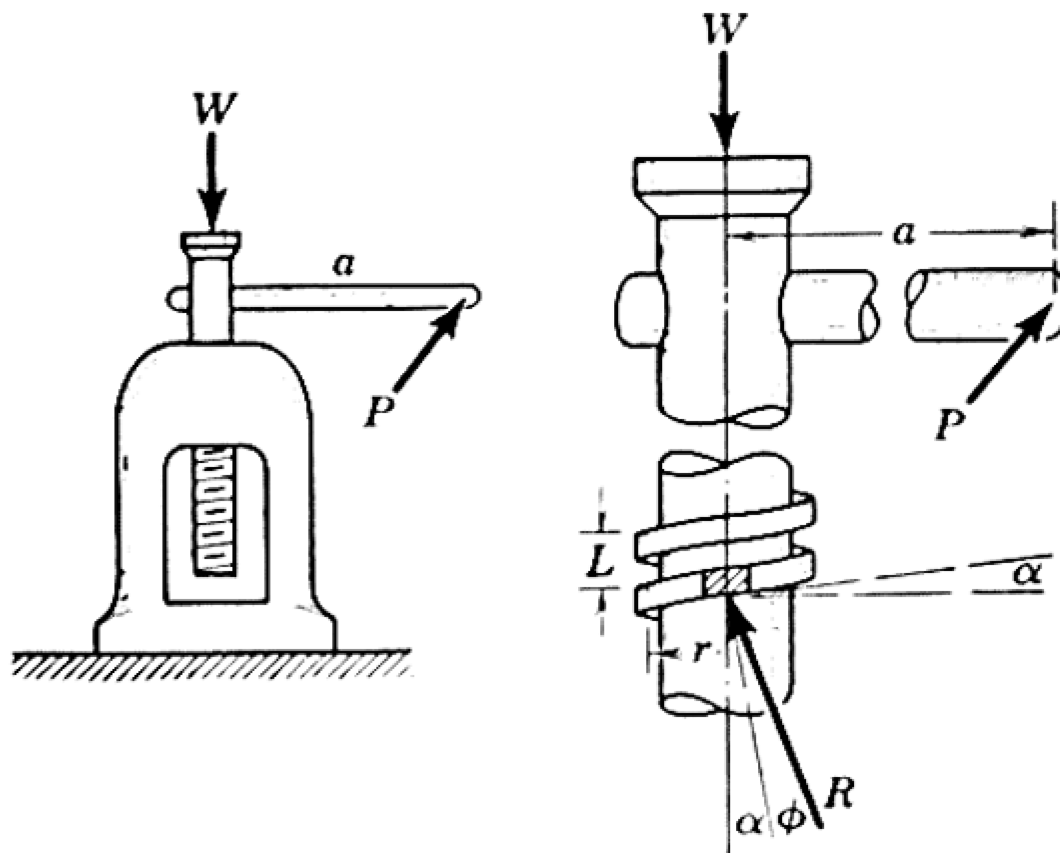


Fig. 70

thread of the jack frame on a small portion of the thread of the screw is shown on the free-body diagram of the screw. If P is just sufficient to turn the screw, the thread of the screw will slide around and up on the fixed thread of the frame. The angle ϕ made by R with the normal to the thread will be the angle of friction, so that $\tan \phi = f$. The moment of R about the vertical axis of the screw is $Rr \sin (\alpha + \phi)$, and the total moment due to all reactions on the screw threads is $\Sigma Rr \sin (\alpha + \phi)$. Since $r \sin (\alpha + \phi)$ appears in each term, it may be factored out. The moment equilibrium equation for the screw becomes

$$Pa = [r \sin (\alpha + \phi)]\Sigma R.$$

Equilibrium of forces in the axial direction further requires

$$W = \Sigma R \cos (\alpha + \phi) = [\cos (\alpha + \phi)]\Sigma R.$$

Dividing Pa by W gives

$$Pa = Wr \tan (\alpha + \phi), \tag{38}$$

where $\phi = \tan^{-1} f$. The angle α is determined from the lead L or advancement per revolution of the screw. Thus $\alpha = \tan^{-1} (L/2\pi r)$. This relation is easily seen by unwrapping the thread for one turn of the screw and forming the right triangle whose base is the mean circumference $2\pi r$ and whose altitude is the lead L.

Equation (38) gives the moment required to start the screw upward or to maintain upward movement, depending on whether the static or kinetic coefficient of friction is used. If the force P is removed, the friction force changes direction so that ϕ is measured to the other side of the normal. Thus the screw will remain in place and be self-locking providing $\phi > \alpha$ and will be on the verge of unwinding if $\phi = \alpha$. A force P must be applied in the direction opposite to that shown in Fig. 70 to lower the screw when $\phi > \alpha$ and is given by

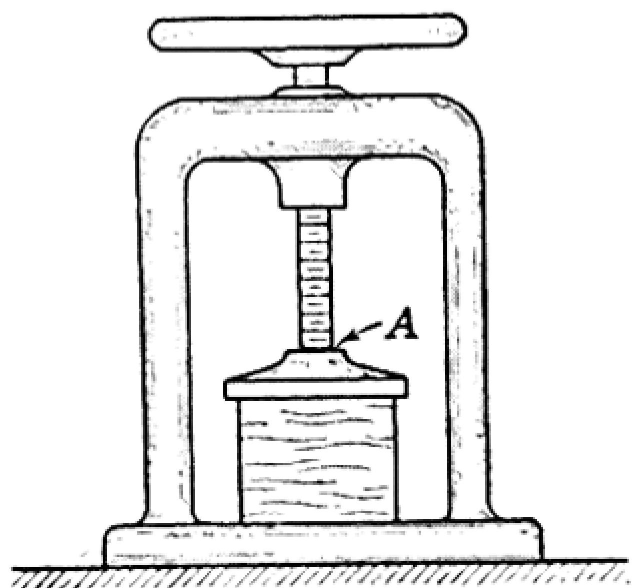
$$Pa = Wr \tan (\phi - \alpha).$$

If $\phi < \alpha$ the screw will unwind by itself and would require a moment to keep it from unwinding equal to

$$Pa = Wr \tan (\alpha - \phi).$$

PROBLEMS

537. A torque of 40 lb. ft. is applied to the handwheel of the small press shown. The screw has a mean diameter of 2 in. and has a double square thread with a lead of 1 in. If the coefficient of friction in the threads is 0.25, how much force W does the press exert? Neglect friction between the end of the screw and the clamp A.



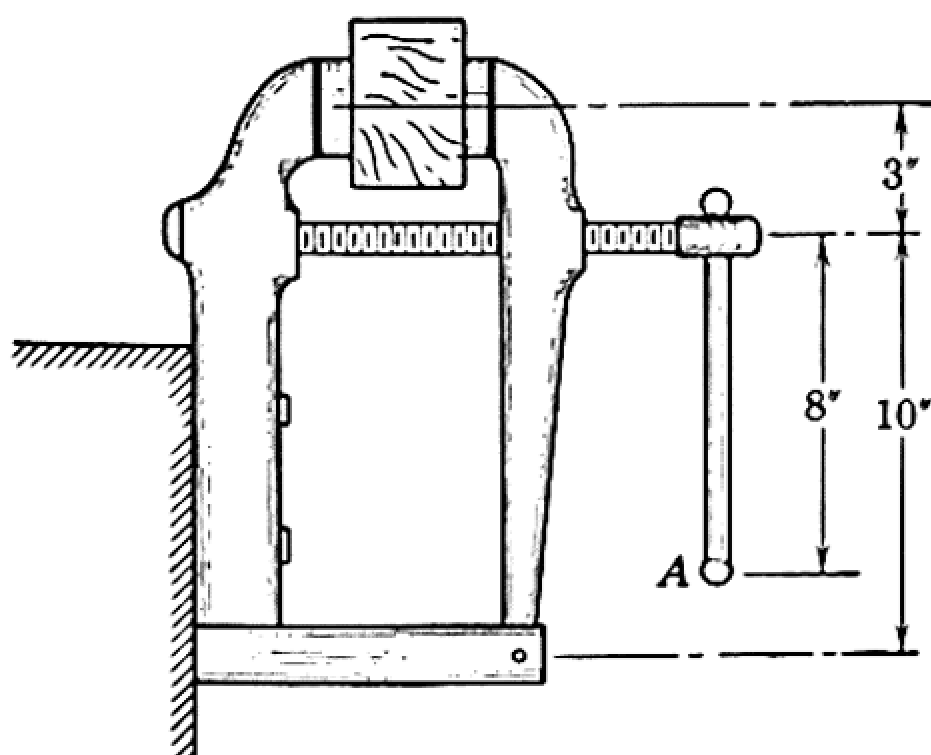
Prob. 537

538. A screw jack with square threads having a mean radius of 1 in. supports a load of 1000 lb. The lever arm of the jack handle is 12 in. If the coefficient of friction is 0.3, what is the greatest lead L of the screw for which the screw will not unwind by itself? For this condition what force P on the lever will raise the load?

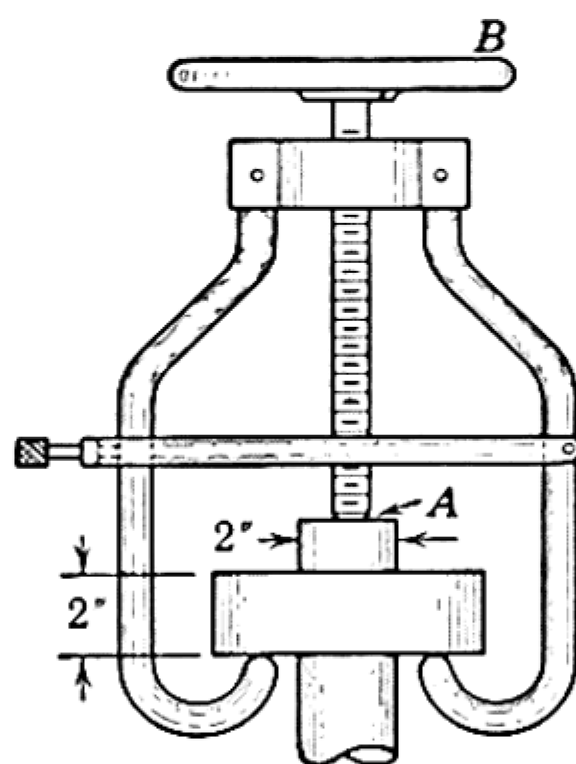
Ans. $L = 1.88$ in., $P = 55.0$ lb.

539. The single-threaded screw of the vise has a mean diameter of 1 in. and has 5 square threads per inch. Find the clamping force C if a 30 lb. pull is ap-

plied normal to the handle at A. The coefficient of friction may be taken to be 0.20.



PROB. 539

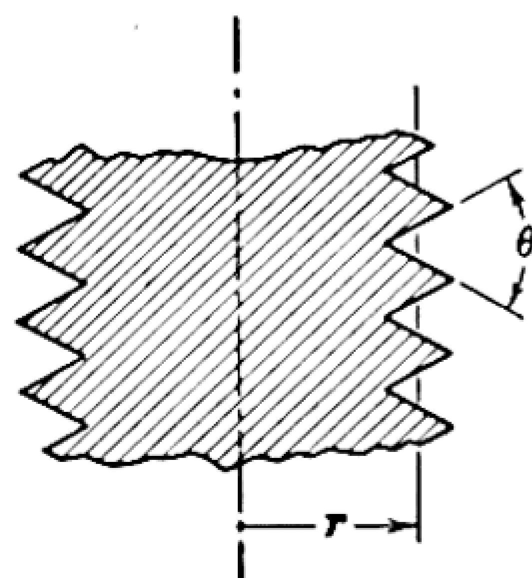


PROB. 540

540. A device for removing wheels and collars from shafts is shown in the figure. The screw is single-threaded with a mean diameter of 1 in. and has 4 square threads per inch. If a torque of 20 lb. ft. is required on wheel B to slip the collar off the shaft, determine the average pressure p between the shaft and the collar. The coefficient of friction for the screw is 0.20, and that for the collar and shaft is 0.30. Neglect any friction at A.

541. A turnbuckle with 10 square threads per inch on each of its two screws supports a cable tension of 500 lb. The mean diameter of the screws is ½ in. While both screws are prevented from rotating, a force P of 6.7 lb. applied 10 in. from the axis of the screws to the handle of a wrench is required to tighten the turnbuckle. If the same wrench is used, determine the force P' required to loosen the turnbuckle.

Ans. $P' = 3.4$ lb.



PROB. 542

* **542.** Replace the square thread of the screw jack in Fig. 70 by a V-thread as indicated in the figure accompanying this problem and determine the force P required to raise the load W.

$$\textit{Ans.}\quad P = \frac{Wr}{a}\,\frac{\tan\alpha + f\sqrt{1+\tan^2\frac{\theta}{2}\cos^2\alpha}}{1 - f\tan\alpha\sqrt{1+\tan^2\frac{\theta}{2}\cos^2\alpha}},\quad \tan\alpha = \frac{L}{2\pi r}$$

54. Journal Bearings. A journal bearing is a bearing which gives lateral support to a shaft in contrast to axial or thrust support. For many partially lubricated bearings analysis by the principles of dry friction gives a good approximation. For fully lubricated bearings the frictional resistance involves the clearance in the bearing, the speed of

rotation, and the viscosity of the lubricant, and is considerably more complicated than the approximation of partially lubricated bearings based on dry friction. A partially lubricated journal bearing with contact or near contact between the shaft and the bearing is shown in Fig. 71, where the clearance is greatly exaggerated. As the shaft begins to turn in the direction shown, it rolls up on the bearing until slippage occurs. Here it remains in more or less of a fixed position during rotation. The torque M required to maintain rotation and the radial load L on the shaft will cause a reaction R at the contact point. For equilibrium in the vertical direction R must equal L but will not be collinear with it. The force R will be tangent to a small circle of radius r_f called the *friction circle*. The angle between R and its normal component N is the friction angle ϕ. Equating the sum of the moments about A to zero gives

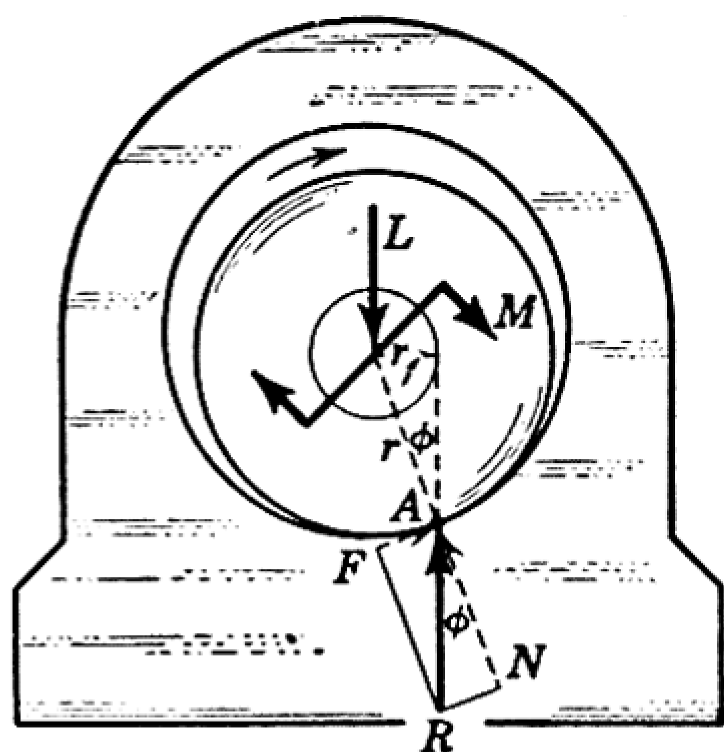


FIG. 71

$$M = Lr_f = Lr \sin \phi.$$

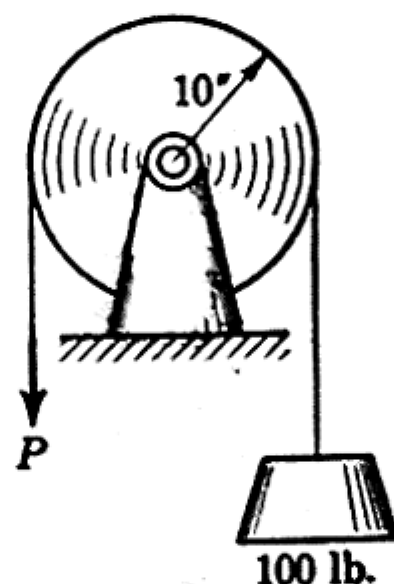
For a small coefficient of friction the angle ϕ is small, and the sine and tangent may be interchanged with only small error. Since $f = \tan \phi$, a good approximation to the torque is

$$M = fLr. \tag{39}$$

Equation (39) gives the amount of torque or moment applied to the shaft which is necessary to overcome friction for a partially lubricated journal bearing.

PROBLEMS

543. A force $P = 102$ lb. is required to rotate the 50 lb. drum at a constant counterclockwise speed. Find the coefficient of friction f for the bearing if the diameter of the shaft is 2 in.

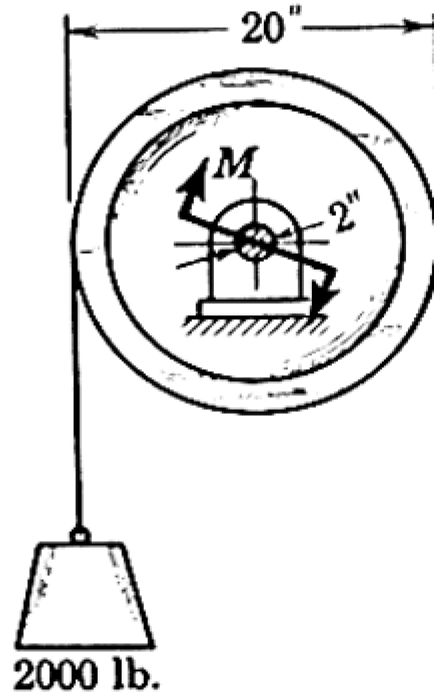


PROB. 543

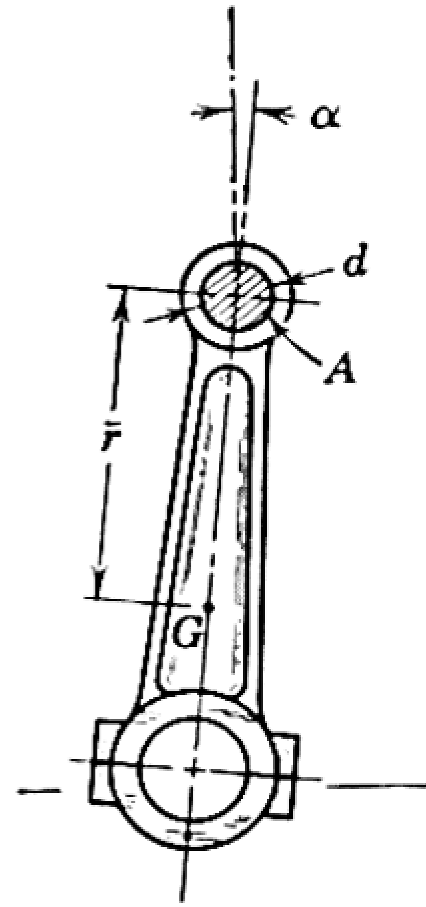
544. A freight car with 33 in. diameter wheels and 6 in. diameter journals is on the verge of rolling when on a track with a 1 per cent grade. With the assumption that all the resistance to rolling is due to journal friction and that the weight of the wheels and axles may be neglected compared with the total weight of the car, determine the coefficient of static friction f for the bearings.

Ans. $f = 0.055$

545. The shaft of the hoisting drum shown is mounted in two bearings symmetrically spaced on either side of the drum and the 2000 lb. weight. If the drum and shaft together weigh 200 lb. and the coefficient of friction in the bearings is 0.15, determine the torque M required to raise the load.



PROB. 545



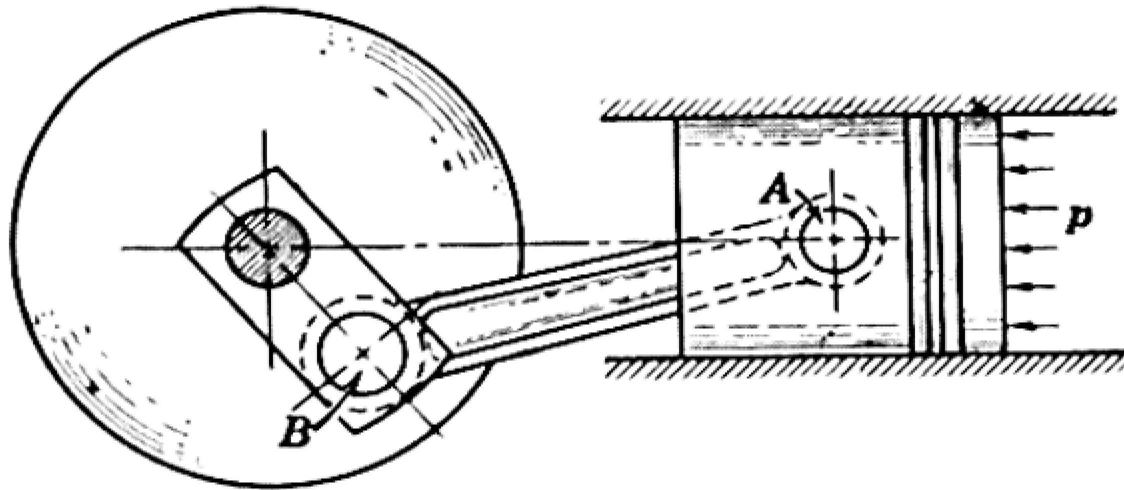
PROB. 546

546. The shaft A fits loosely in the wrist pin bearing of the connecting rod with center of gravity at G as shown. With the rod initially in the vertical position the shaft is rotated slowly until the rod slips at the angle α. Determine the coefficient of friction, assuming α to be small.

Ans. $f = \dfrac{2\bar{r}\alpha}{d}$

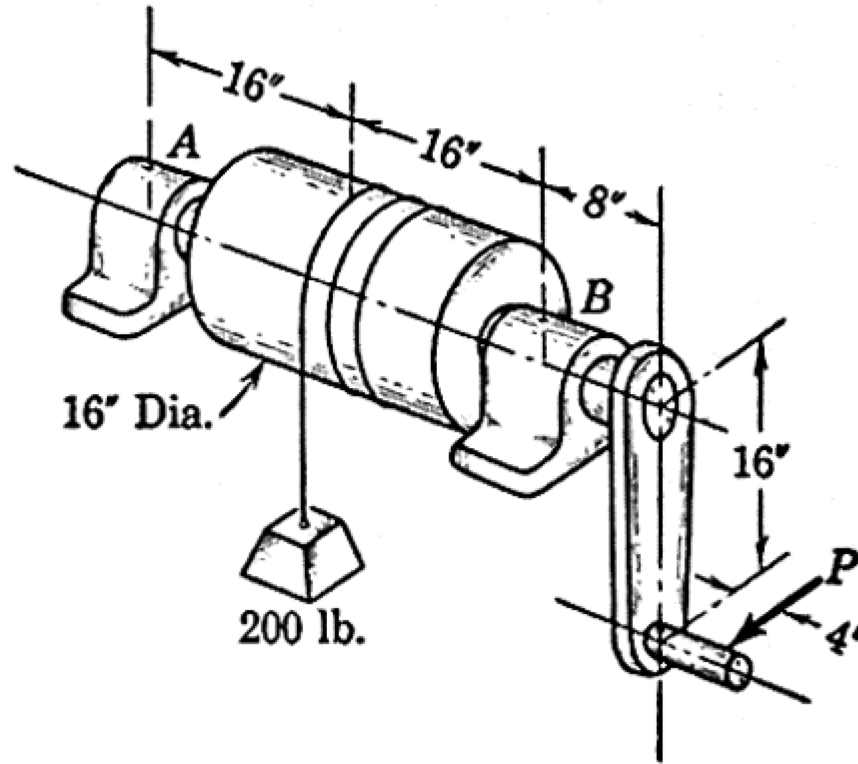
547. A single-cylinder reciprocating engine starts from rest in the position shown when the cylinder pressure p reaches a certain value. Determine the angle α which the resultant compressive force in the rod makes with the axis of the rod if the coefficient of friction is 0.05 for both rod bearings. The diameters of the wrist pin A and crank pin B are 2 in. and 3 in., respectively, and the rod is 24 in. between bearing centers. Neglect the weight of the rod.

Ans. $\alpha = 0° 18'$



PROB. 547

548. For the hoisting rig shown a force of $P = 104$ lb. is required to overcome friction in the bearings A and B and lift the 200 lb. load. If the shaft is 2 in. in diameter, determine the coefficient of friction for the bearings. *Ans.* $f = 0.227$



PROB. 548

* **549.** For the connecting rod of Prob. 546 determine (*a*) the exact expression for the angle α at which the rod slips on the loosely fitting shaft and (*b*) the maximum possible value of α with the corresponding coefficient of friction f.

Ans. (*a*) $\alpha = \sin^{-1}\left(\frac{d}{2\bar{r}}\frac{f}{\sqrt{1+f^2}}\right)$, (*b*) $\alpha = \sin^{-1}\frac{d}{2\bar{r}}, f \rightarrow \infty$

55. Disk Friction. The friction between circular surfaces under normal pressure is encountered in pivot bearings, clutch plates, and disk brakes. Consider the two circular disks of Fig. 72 whose shafts are mounted in bearings (not shown) such that they can be brought in contact under the axial force P. The maximum torque that this clutch can transmit will be equal to the torque M required to slip one disk against the other. If p is the normal pressure at any location between the plates, the frictional force acting on an elemental area is $fp\,dA$, where f is the friction coefficient and dA is the area $r\,dr\,d\theta$ of the element. The moment of this elemental friction force about the shaft axis is $fpr\,dA$, and the total moment is

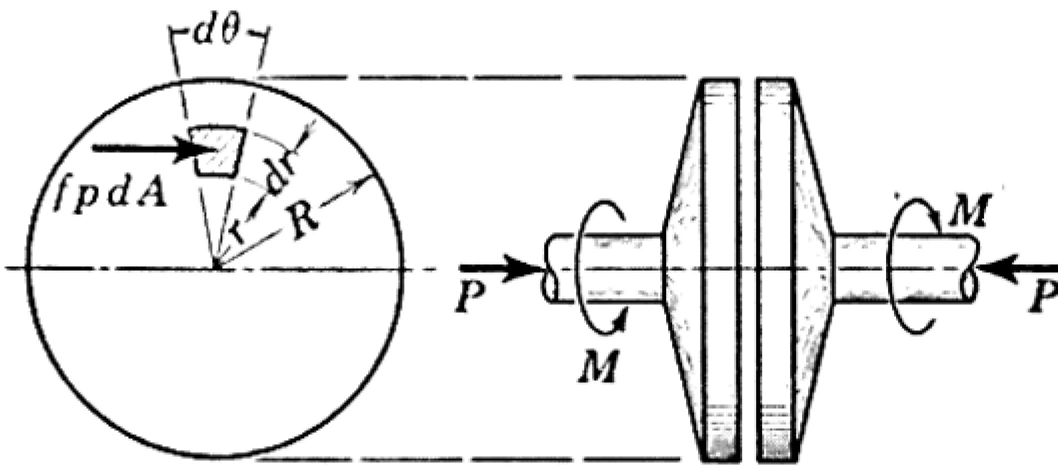


FIG. 72

$$M = \int fpr\,dA,$$

where the integral is evaluated over the area of the disk. If the plates

are new and well supported, it is reasonable to assume that the pressure is uniformly distributed so that $\pi R^2 p = P$. Substituting this constant value of p in the expression for M gives

$$M = \frac{fP}{\pi R^2} \int_0^{2\pi} \int_0^R r^2 \, dr \, d\theta = \frac{2}{3} fPR. \tag{40}$$

This result may be interpreted as being the moment due to a friction force fP acting at a distance $2R/3$ from the center of the shaft.

If the friction disks are rings, the limits of integration are the inside and outside radii R_i and R_o, respectively, and the frictional torque is

$$M = \frac{2}{3} fP \frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}. \tag{40a}$$

After some wear of the surfaces has taken place, it is found that the frictional moment decreases somewhat. When the wearing-in period is over, the surfaces retain their new relative shape and further wear is therefore constant over the surface. This wear depends on the circumferential distance traveled and the pressure p. Since the distance traveled is proportional to r, the expression $rp = K$ may be written, where K is a constant. The value of K is determined by equating the axial forces to zero, or

$$P = \int p \, dA = K \int_0^{2\pi} \int_0^R dr \, d\theta = 2\pi KR.$$

The expression for M may now be integrated, and it becomes

$$M = \int f \frac{P}{2\pi R} \, dA = \frac{fP}{2\pi R} \int_0^{2\pi} \int_0^R r \, dr \, d\theta,$$

or

$$M = \tfrac{1}{2} fPR. \tag{41}$$

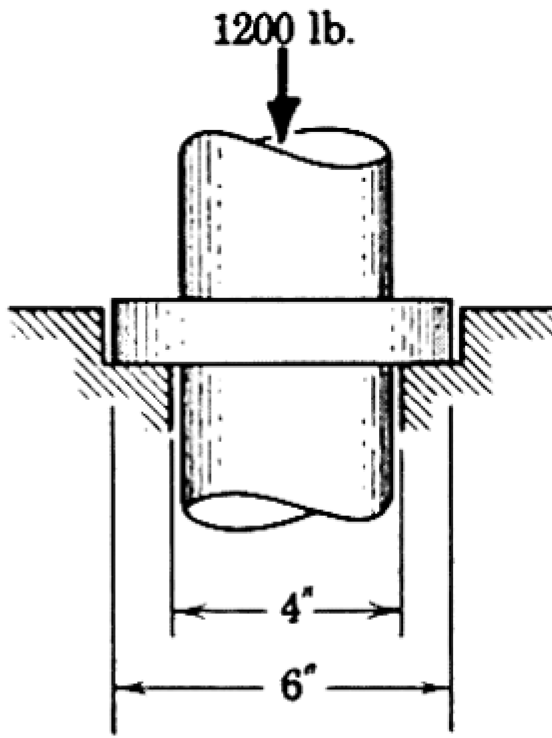
The frictional moment for worn-in plates is, therefore, only $(\frac{1}{2})/(\frac{2}{3})$, or $\frac{3}{4}$ as much as for new surfaces.

If the friction disks are rings of inside radius R_i and outside radius R_o, substitution of these limits in the integrations shows that the frictional torque for worn-in surfaces is

$$M = \tfrac{1}{2} fP(R_o + R_i). \tag{41a}$$

PROBLEMS

550. The collar bearing shown supports a shaft thrust of 1200 lb. If a torque of 45 lb. ft. is required to overcome friction in the bearing, what is the coefficient of friction f? Assume the bearing to be worn-in.



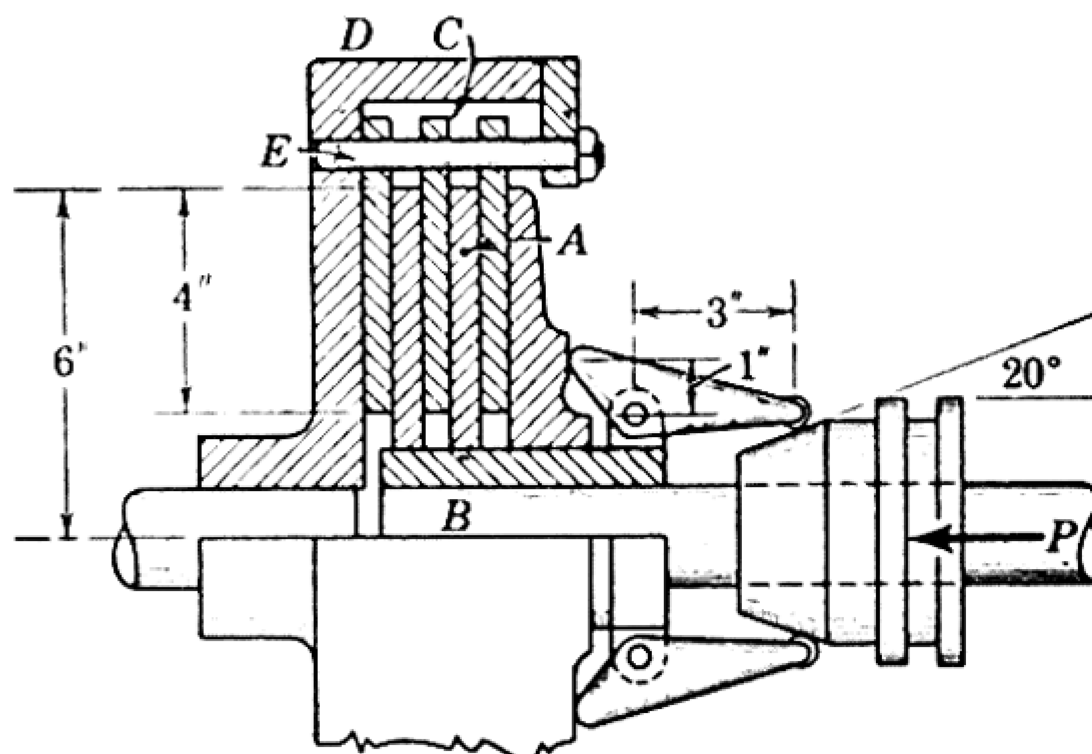
PROB. 550

551. Plot the radial distribution of pressure for the worn-in bearing of Prob. 550. What is the maximum pressure p? *Ans.* $p = 95.5$ lb./in.^{2}

552. A disk brake used on the landing wheels of an airplane contains three floating metal rings keyed to the wheel and three metal rings keyed to a flange which is attached to the axle. There are five pairs of friction surfaces, and the action is identical with that of the multi-plate clutch shown in the figure for Prob. 554 which follows. The contact surface of each ring has outside and inside diameters of 12 in. and 10 in., respectively. If the friction coefficient is 0.15 and the maximum normal pressure between rings is 80 lb./in.^{2}, what is the frictional braking torque M for each wheel? Is it necessary in this case to use either of Eqs. (40a) or (41a)?

553. The friction disk of a single-plate clutch has outside and inside diameters of 10 in. and 4 in., respectively. When new the single pair of surfaces transmits up to 250 lb. in. of torque before slipping when the clutch surfaces are under a normal engaging force of 400 lb. Determine the coefficient of friction f for this clutch. Assume that this friction coefficient remains the same and find the maximum torque M which the clutch transmits under the same engaging force after the surfaces are worn-in. *Ans.* $f = 0.168$, $M = 235$ lb. in.

554. In the figure is shown a multiple-disk clutch for marine use. The driving disks A are splined to the driving shaft B so that they are free to slip along the shaft but must rotate with it. The disks C drive the housing D by means of the bolts E along which they are free to slide. In the clutch shown there are

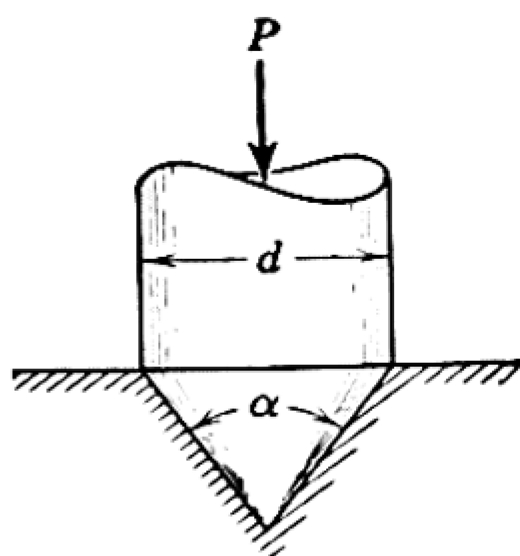


PROB. 554

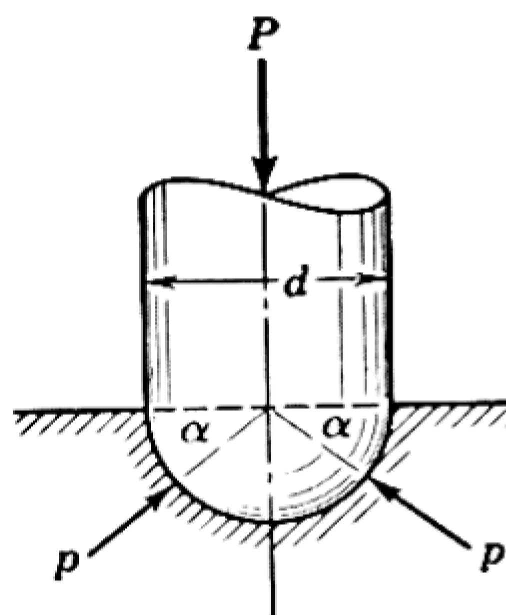
5 pairs of friction surfaces. Assuming the pressure to be uniformly distributed over the area of the disks, determine the maximum torque M which can be transmitted if the coefficient of friction is 0.15 and $P = 100$ lb.

*** 555.** The conical pivot bearing shown supports a shaft thrust P. Derive an expression for the torque M required on the shaft to overcome friction, assuming (*a*) new bearing surfaces with uniform pressure distribution and (*b*) worn-in surfaces where the constant wear is proportional to the product of the normal pressure and the radial distance from the shaft axis.

Ans. (*a*) $M = \dfrac{fPd}{3 \sin \frac{\alpha}{2}}$, (*b*) $M = \dfrac{fPd}{4 \sin \frac{\alpha}{2}}$



PROB. 555



PROB. 556

*** 556.** The spherical thrust bearing on the end of the shaft supports an axial load P. Determine the expression for the moment M required to turn the shaft against friction in the bearing. Assume that the pressure p is directly proportional to the angle α and that the coefficient of friction is f.

Ans. $M = \left(\dfrac{\pi}{4} - \dfrac{1}{\pi}\right) f\,Pd$

56. Belt Friction. The impending slippage of belts and ropes over sheaves and drums is of importance in the design of belt drives of all types, band brakes, and hoisting rigs. In Fig. 73 is shown a drum subjected to the two belt tensions T_1 and T_2 and the moment M necessary to prevent rotation. With M in the direction shown T_2 is greater than T_1. The free-body diagram of an element of the belt of length $r\,d\theta$ is also shown in the figure. The tension increases

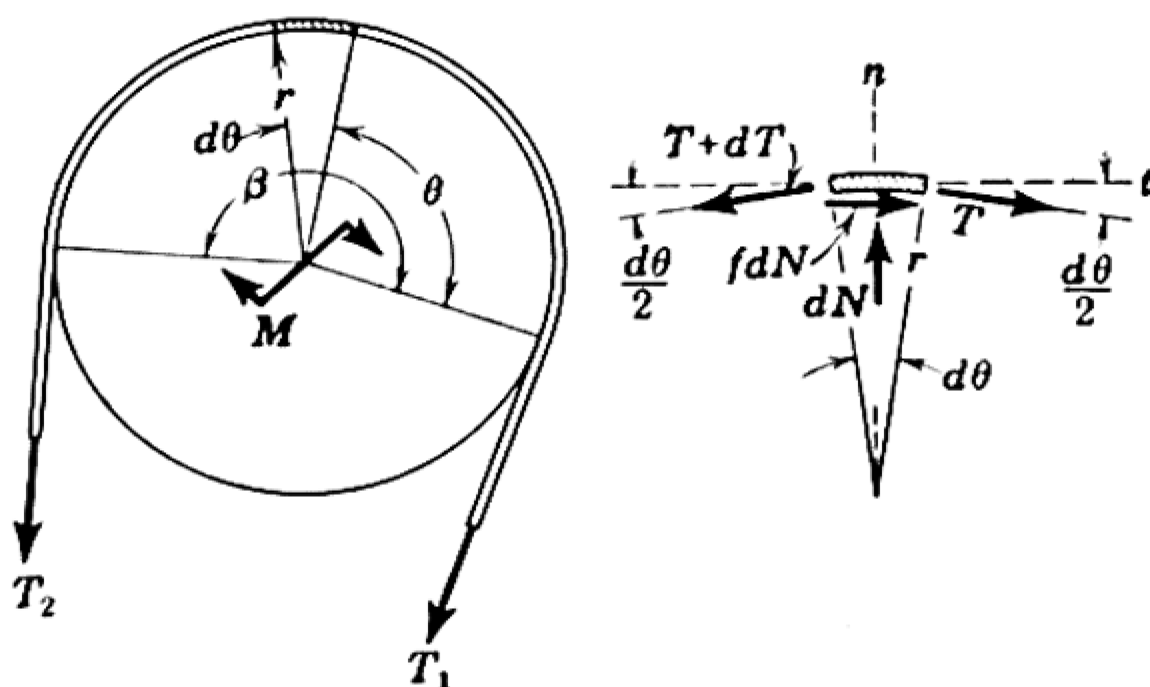


FIG. 73

from T at the angle θ to $T + dT$ at the angle $\theta + d\theta$. The normal force is a differential dN since it acts on a differential element of area. Likewise the friction force, which must act on the belt in a direction to oppose slipping, is a differential and is $f\,dN$ for impending motion. Equilibrium in the t-direction gives

$$T \cos \frac{d\theta}{2} + f\,dN = (T + dT) \cos \frac{d\theta}{2},$$

or

$$f\,dN = dT,$$

since the cosine of a differential quantity is unity. Equilibrium in the n-direction requires that

$$dN = (T + dT) \sin \frac{d\theta}{2} + T \sin \frac{d\theta}{2},$$

or

$$dN = T\,d\theta.$$

In this reduction it must be remembered that the sine of a differential angle equals the angle and that the product of two differentials must be neglected in the limit compared with the first-order differentials remaining. Combining the two equilibrium relations gives

$$\frac{dT}{T} = f\,d\theta.$$

Integrating between corresponding limits yields

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\beta f\,d\theta,$$

or

$$\log \frac{T_2}{T_1} = f\beta,$$

where the $\log (T_2/T_1)$ is a natural logarithm to the base e. Solving for T_2 gives

$$T_2 = T_1 e^{f\beta}. \tag{42}$$

It should be noted that β is the total angle of belt contact and is expressed in radians. If a rope were wrapped around a drum n times, the angle β would be $2\pi n$ radians.

The relation expressed by Eq. (42) also applies to belt drives where both the belt and the pulley are rotating at constant speed. In this case the equation describes the ratio of belt tensions for slipping or impending slipping. When the speed of rotation becomes large, there is a tendency for the belt to leave the rim so that Eq. (42) will involve some error.

SAMPLE PROBLEM

557. Determine the torque M required to maintain rotation of the pulley if $P = 50$ lb. for the simple band brake shown. The band is steel and the drum is cast iron.

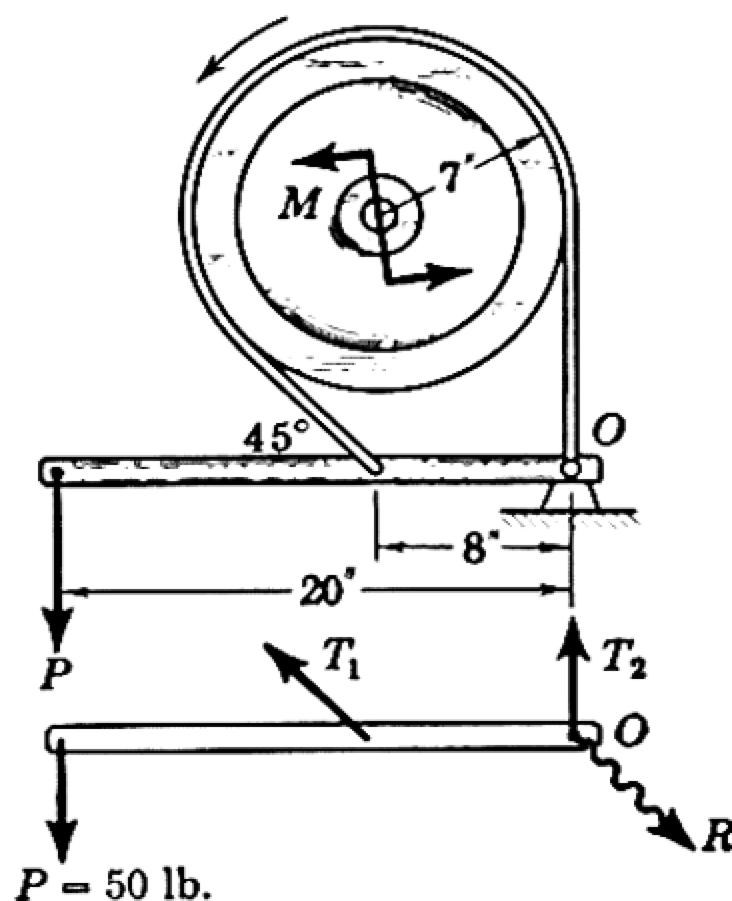
Solution: From the free-body diagram of the brake lever the moment equation gives

$$[\Sigma M_O = 0] \quad 50 \times 20 - 8T_1 \cos 45° = 0,$$

$$T_1 = 177 \text{ lb.}$$

For counterclockwise rotation the tension T_2 will be greater than T_1. The angle of belt contact is $\beta = \frac{225}{360} \times 2\pi = 3.93$ radians, and a suitable coefficient of kinetic friction for steel on cast iron is $f = 0.10$, assuming a dry surface. Substituting these values into Eq. (42) gives

$$[T_2 = T_1 e^{f\beta}] \quad T_2 = 177e^{0.393}, \quad T_2 = 262 \text{ lb.}$$



PROB. 557

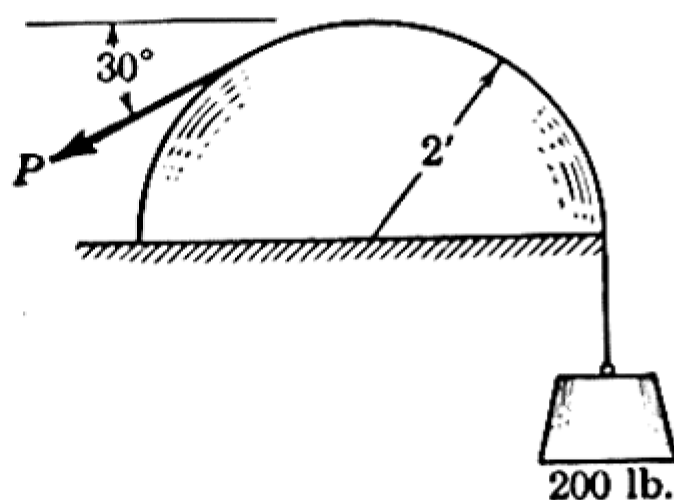
The equilibrium of moments about the axle of the pulley requires

$$[\Sigma M = 0] \qquad (262 - 177) \times 7 - M = 0, \qquad M = 595 \text{ lb. in.} \qquad \textit{Ans.}$$

PROBLEMS

558. Find the force P necessary (a) to raise the load and (b) to prevent the load from slipping down. The coefficient of friction is 0.3.

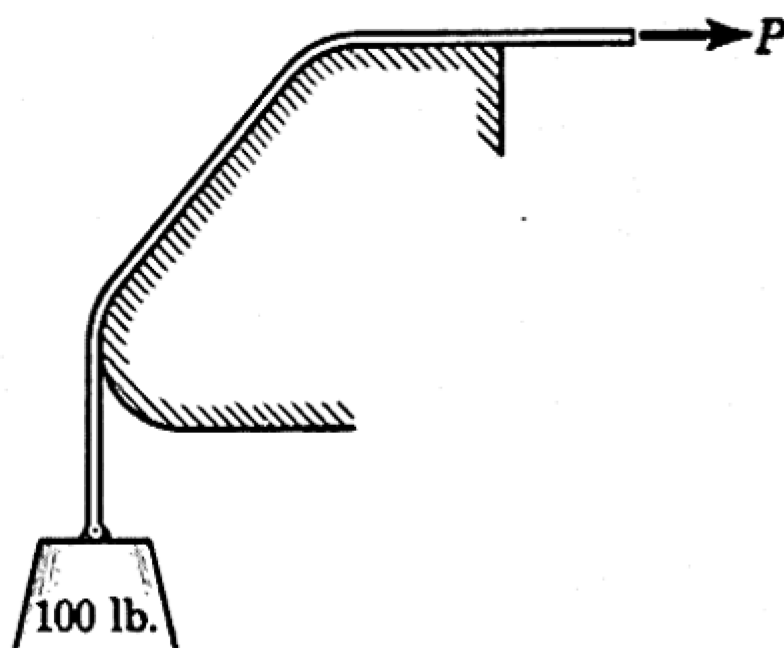
Ans. (a) $P = 375$ lb., (b) $P = 107$ lb.



PROB. 558

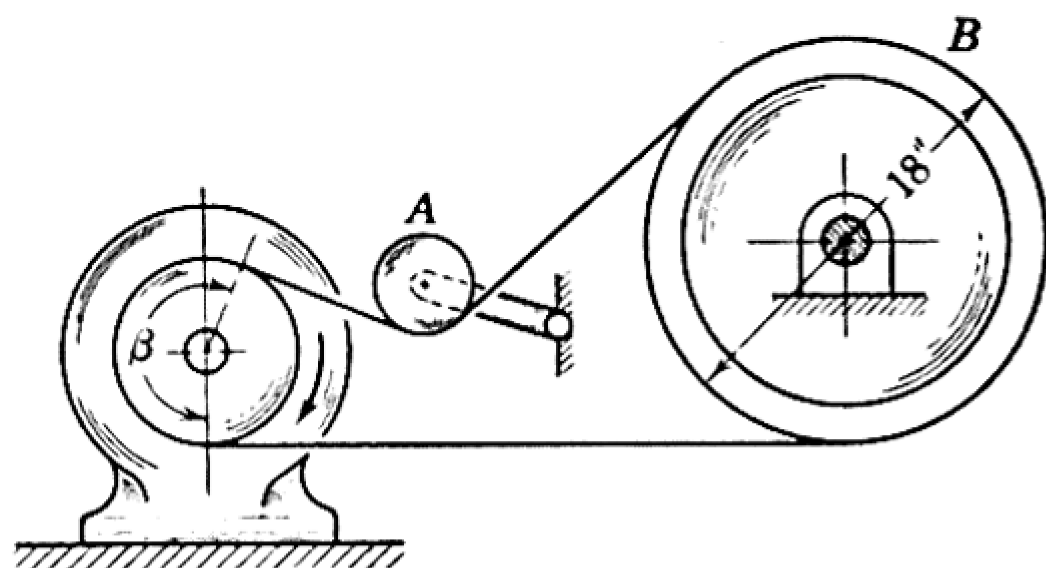
559. A dock hand secures a bow line from a tug by wrapping it three complete turns around a capstan and exerting a force of 25 lb. on the free end. The tug is just able to slip the rope by running its engine full speed in reverse. Determine the thrust T developed by the tug's propeller. Take the coefficient of friction between the rope and the capstan to be 0.3.

560. What force P is required to lower the 100 lb. load at constant speed? The coefficient of friction is 0.30.



Prob. 560

561. By adjusting the position of the idler pulley A so as to increase the angular contact β of the flat belt shown it is possible to transmit more torque from the motor to the pulley B. If the safe tension in the belt is limited to 120 lb. and the pulley B must supply a torque of 60 lb. ft. to its shaft, determine the minimum angular contact β. The coefficient of friction is 0.3 for both pulleys, and the motor rotates clockwise as shown.

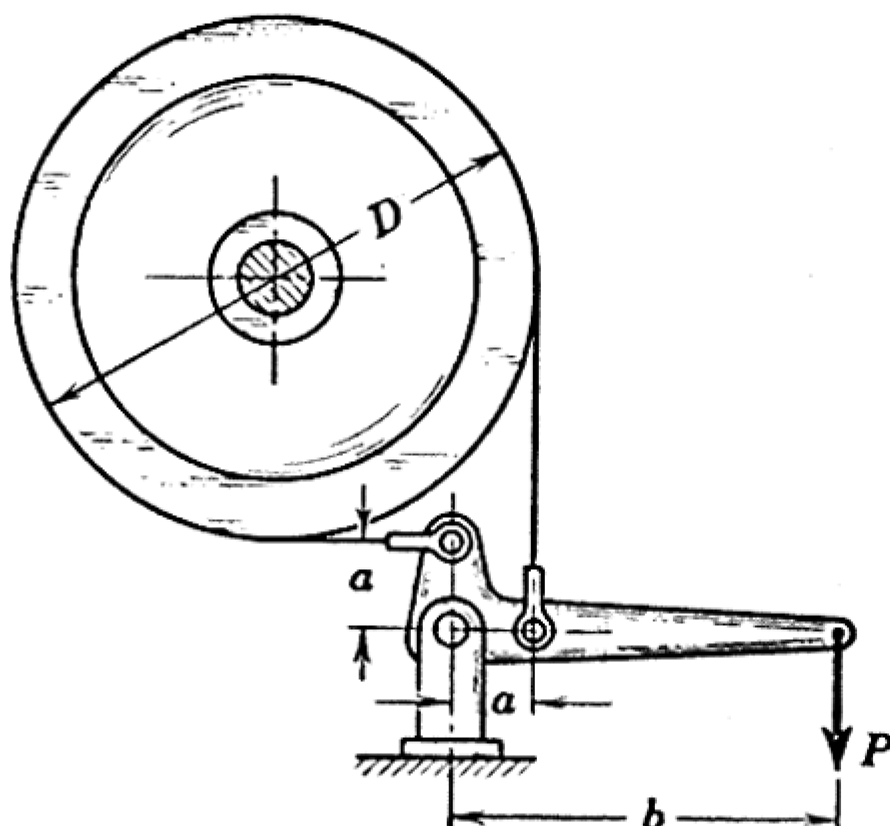


Prob. 561

Ans. $\beta = 210°$

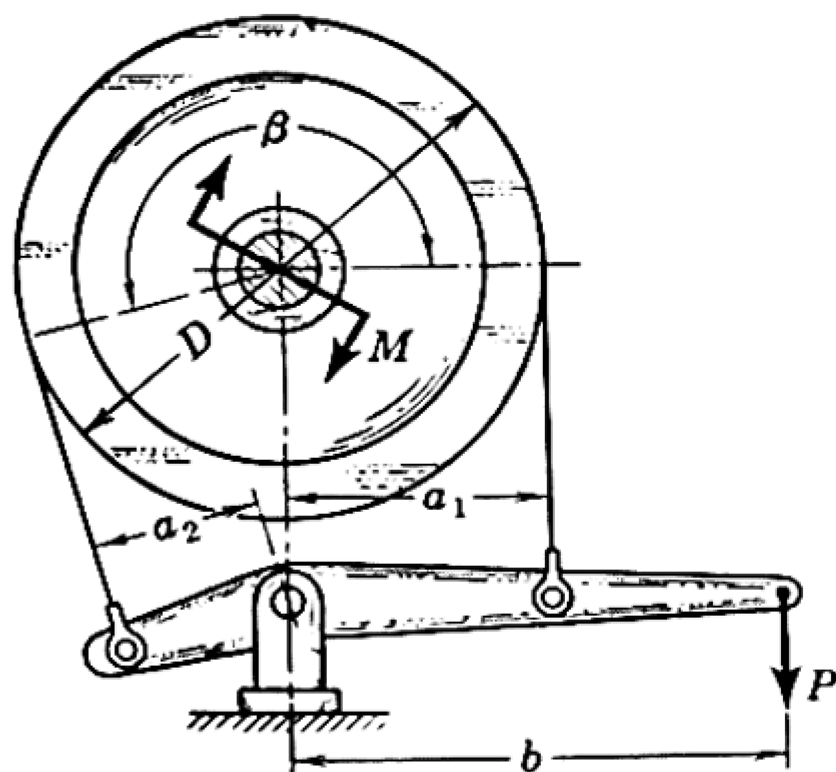
562. A band brake for use with rotation in either direction is shown in the figure. If the coefficient of friction is f and the braking force is P, determine the torque M required to turn the wheel.

$$Ans. \quad M = \frac{PbD}{2a}\,\frac{e^{3\pi f/2} - 1}{e^{3\pi f/2} + 1}$$

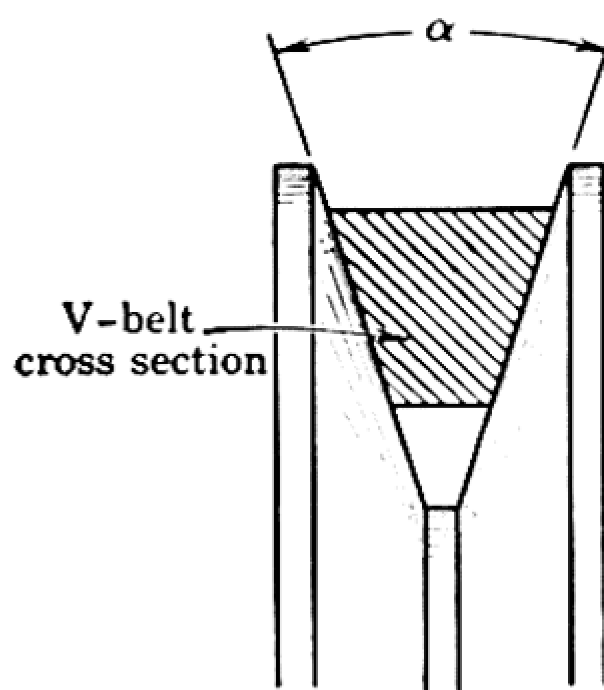


Prob. 562

563. In the figure is shown a differential band brake. Determine the force P required to brake the wheel under a clockwise torque M if the coefficient of friction is f. What happens if $a_1/a_2 \leq e^{f\beta}$?



Prob. 563



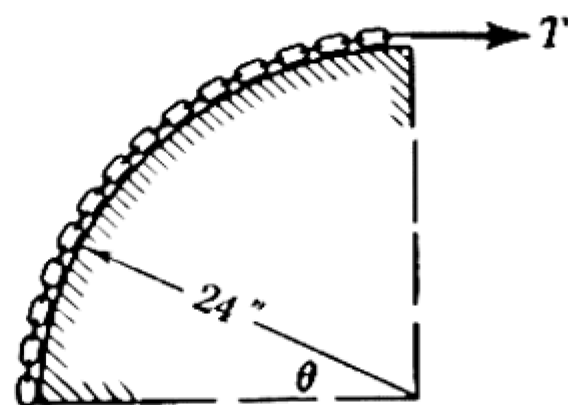
Prob. 564

* **564.** Replace the flat belt and pulley of Fig. 73 by a V-belt and matching grooved pulley as indicated by the cross-sectional view accompanying this problem. Derive the relation between the belt tensions, the angular contact, and the coefficient of friction for the V-belt when slipping impends.

Ans. $T_2 = T_1 e^{\frac{f\beta}{\sin(\alpha/2)}}$

* **565.** The V-belt angle most commonly used is $\alpha = 35$ deg. (see the figure for Prob. 564). With the results of Prob. 564 show that using a V-belt of this angle in place of a flat belt is equivalent to increasing the coefficient of friction for the flat belt by a factor of 3.33. By what factor F is the limiting ratio of belt tensions increased?

Ans. $F = \left(\frac{T_2}{T_1}\right)^{2.33}_{\text{Flat belt}}$



Prob. 566

* **566.** A chain weighing 12 lb./ft. of length is to be pulled over the circular form shown. The coefficient of friction is 0.30. Determine the tension T which must be applied to start the chain moving. (*Hint*: Approximate the solution of the equation by a step-by-step numerical solution, taking 10 deg. increments for θ.)

Ans. $T = 41$ lb.

57. Rolling Resistance. Another type of resistance to motion is encountered in a rolling object. The wheel shown in Fig. 74 carries a load L on the axle, and a force P is applied to produce rolling. The deformation of the wheel and supporting surface as shown is greatly exaggerated. The distribution of pressure p over the area of contact is similar to that indicated, and the resultant R of this distribution will act at some point A and will pass through the center of the wheel for equilibrium. The

force P necessary to produce rolling may be found by equating the moments of all forces about A to zero. This gives

$$P = \frac{a}{r} L = f_r L, \tag{43}$$

where the moment arm of P is taken to be r, and $f_r = a/r$ is called the coefficient of rolling friction. The coefficient f_r is the ratio of resisting force to normal load and in this respect is similar to the coefficients of static and kinetic friction. On the other hand there is no slipping involved in the interpretation of f_r.

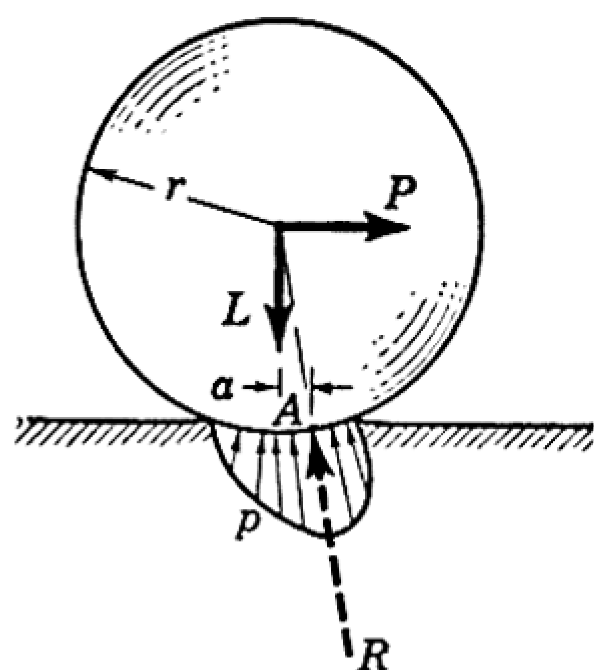


Fig. 74

The quantity a in Eq. (43) depends on many factors which are not completely understood, so that a comprehensive theory of rolling resistance is not available. This distance a is a function of the elastic (and in some cases plastic) properties of the mating materials, the radius of the wheel, the speed of travel, and the roughness of the surfaces. Some tests indicate only a small variation with wheel radius, and a is often taken to be independent of the rolling radius. The quantity a is frequently referred to as the "coefficient of rolling resistance." It is not a dimensionless coefficient since it has the dimension of length and is usually expressed in inches.

Two typical values of f_r are listed in Table B2, Appendix B. These values are only approximate and may be used merely for rough calculations.

CHAPTER VIII

Virtual Work

58. Work. In this chapter a method involving the concept of work will be developed for solving the equilibrium problem. For certain types of equilibrium problems this method will be found greatly superior to the method based on the summation of forces and moments. The term *work* is used in an exact sense as contrasted to its common nontechnical usage. The work done by a force F during a displacement ds of its point of application O, Fig. 75, is

$$dU = F\,ds \cos \alpha,$$

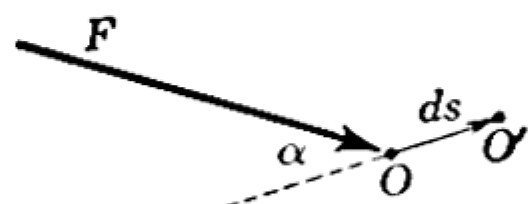


FIG. 75

where α is the angle between F and ds. This expression may be interpreted either as the displacement multiplied by the force component $F \cos \alpha$ in the direction of the displacement or as the force multiplied by the displacement component $ds \cos \alpha$ in the direction of the force. With this definition it should be noted that the component $F \sin \alpha$ normal to the displacement does no work. Work is positive if the working component $F \cos \alpha$ is in the direction of the displacement and negative if in the opposite direction. Work is a scalar quantity with the dimensions of (distance) × (force) and is usually expressed in foot pound units. Dimensionally, work and moment are the same. In order to distinguish between the two quantities it is recommended that work be expressed as foot pounds (ft. lb.) and moment as pound feet (lb. ft.). It should be noted that work involves the product of a force and a distance, both measured along the same line, whereas moment is the product of force and distance measured at right angles to the force.

During a finite displacement s of the point of application of a force the force does an amount of work equal to

$$U = \int F \cos \alpha\, ds.$$

In order to carry out this integration, it is necessary to know the relationships between F and s and between $\cos \alpha$ and s. The path s need not be restricted to a straight line. The work done by the force of gravity

equals the weight multiplied by the vertical displacement of the center of gravity of the body. If this movement is down, the weight does positive work. If the movement is up, the weight does negative work. The force of gravity does no work during a horizontal movement of the center of gravity.

In the case of concurrent forces acting on a body the work done by their resultant equals the total work done by the several forces. This may be seen from the fact that the component of the resultant in the direction of the displacement equals the sum of the components of the several forces in the same direction.

The work done by a couple M acting on a body during a rotation $d\theta$ of the body in the plane of the couple is $dU = M\,d\theta$, and the total work done during a finite angular displacement is

$$U = \int M\,d\theta.$$

This result may be seen from Fig. 76, where the couple consists of the two forces F separated a distance b. Consider any displacement of the body which moves line AB to $A''B''$ in the plane of the paper. This movement may be represented as taking place in two parts. The first is a linear movement to a parallel position $A'B'$, during which time the work done by one force F is equal and opposite to that done by the other force F. Second, the line rotates an amount $d\theta$ to the final position $A''B''$, during which time each force does an amount of work equal to $Fb\,d\theta/2$ so that the total work done is $Fb\,d\theta$ or $M\,d\theta$. The angle θ is expressed in radian measure. Positive work is done when the rotation is in the direction of the couple, and negative work is done when the rotation is in the direction opposite to the couple.

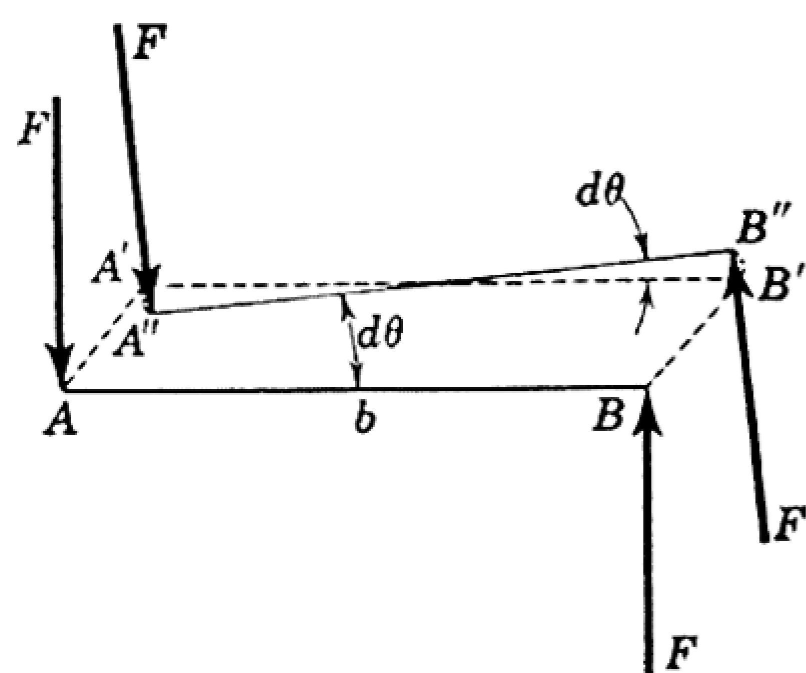


Fig. 76

59. Virtual Work. Consider a particle whose position is determined by the forces which act upon it. Any assumed and arbitrary small displacement δs away from this natural position is called a *virtual displacement*. The term *virtual* is used to indicate that the displacement does not exist in reality but is only assumed. The virtual movement is used in order that various possible equilibrium positions may be compared in the process of selecting the correct one. The work done by any force F acting

on the particle during the virtual displacement is called *virtual work* and is

$$\delta U = F\ \delta s \cos \alpha,$$

where α is the angle between F and δs. The difference between ds and δs is that ds refers to an infinitesimal change in an actual movement, whereas δs refers to an infinitesimal virtual or assumed movement. Mathematically both quantities are differentials of the first order.

A virtual displacement may also be measured by a rotation. The angular rotation $\delta\theta$ of a line could be a virtual angular displacement. The virtual work done by a couple M during a virtual angular displacement $\delta\theta$ is, then, $\delta U = M\ \delta\theta$.

The magnitude of the force F or couple M may be considered as remaining constant during any corresponding infinitesimal virtual displacement δs or $\delta\theta$. If account of the change in F or M during the movement is made, higher-order terms will result which disappear in the limit. This consideration is identical with that which permits the writing of an element of area under the curve $y = f(x)$ as $dA = y\ dx$.

60. Equilibrium of a Particle. If a particle is in equilibrium, the virtual work done by all forces acting on it during any arbitrary virtual displacement δs away from the equilibrium position is

$$\begin{aligned} \delta U &= (F_1 \cos \alpha_1)\delta s + (F_2 \cos \alpha_2)\delta s + (F_3 \cos \alpha_3)\delta s + \cdots, \\ &= \delta s \Sigma(F \cos \alpha) = 0, \end{aligned}$$

where $\Sigma(F \cos \alpha)$ represents the sum of the components of all forces in the δs direction. This sum is zero since the component of the resultant force in that same direction is zero for equilibrium. The equation $\delta U = 0$ is, then, an alternate way of stating the equilibrium conditions for a particle. This condition is both necessary and sufficient since it may be applied to movements, one at a time, in all three mutually perpendicular directions and is therefore equivalent to the equilibrium requirements $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$.

The principle of zero virtual work for the equilibrium of a single particle usually does not simplify this already simple problem since δs appears in every term and hence cancels. The concept of virtual work for a particle is introduced so that it may be applied to systems of particles in the discussion which follows.

61. Equilibrium of a Rigid Body. For a "rigid" body the distances between particles remain constant, and no part of the work done by forces external to the body can be absorbed by internal friction or by elastic elongations or compressions which are assumed absent in a rigid body. Since the virtual work done on each particle of the body in equilibrium is

zero, it follows that the virtual work done on the entire rigid body is zero. Only the virtual work done by external forces appears in the evaluation of $\delta U = 0$ for the entire body, since the internal forces occur in pairs of equal and opposite forces and the net work done by these forces during any movement is necessarily zero.

Again, as in the case of a particle, the principle of virtual work offers no real advantage to the solution for a single rigid body in equilibrium. Any assumed virtual displacement defined by a linear or angular movement will appear in each term in $\delta U = 0$ and when canceled will leave the same expression as would have been obtained by using one of the force or moment equations of equilibrium directly. This condition is illustrated in Fig. 77, where it is required to determine the reaction R under the roller for the hinged plate of negligible weight under the action of a given force P. A small assumed rotation $\delta\theta$ of the plate about O is consistent with the constraining hinge at O and is taken as the virtual displacement. The work done by P is $-Pa\,\delta\theta$, and the work done by R is $+Rb\,\delta\theta$. Therefore the principle $\delta U = 0$ gives

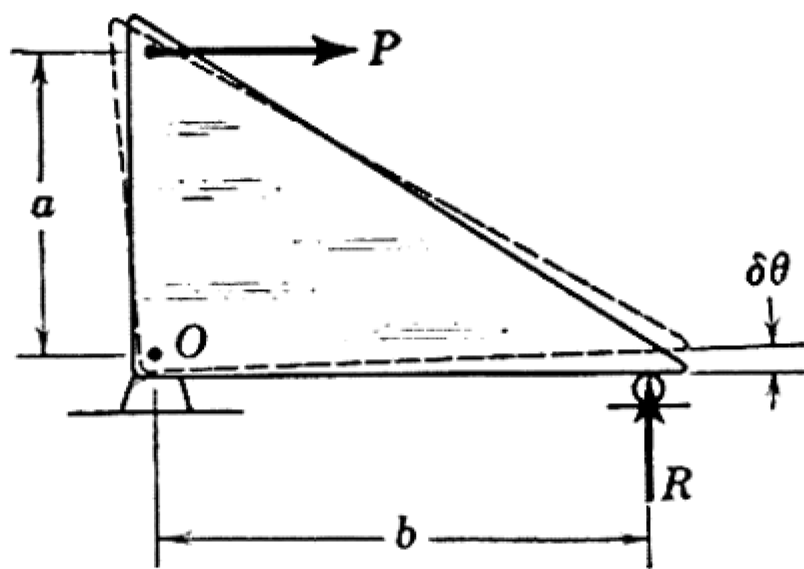


FIG. 77

$$-Pa\,\delta\theta + Rb\,\delta\theta = 0.$$

Canceling out $\delta\theta$ leaves

$$Pa - Rb = 0,$$

which is simply the moment equilibrium equation about O. Therefore nothing is gained by the use of the virtual work principle for a single rigid body. Use of the principle will be a decided advantage for interconnected bodies as described in the next two articles.

62. Ideal Systems. Two or more rigid bodies fastened together by mechanical connections which are frictionless and which cannot absorb energy by elongation or compression constitute an ideal mechanical system. Examples of ideal systems are shown in Fig. 78. In each system there is assumed to be no friction in the joints. It is permissible to have friction acting on the sides of the piston in Fig. 78*c* since such friction forces are *external* to the system composed of the piston, connecting rod, and crankshaft. The rope and pulley arrangements in Figs. 78*d* and *e* are ideal systems as long as the stretch in the rope is negligible and as long as the rope does not slip on the pulleys, thereby inducing kinetic friction and heat loss. By isolating an ideal system with a free-body di-

agram showing all forces external to the system, such as indicated in Fig. 78*b*, it is seen that during any and all possible movements of the system or its parts the net work done by the internal forces at the connections is zero. This is so because the internal forces exist in pairs of equal and opposite forces, as illustrated for joint B in Fig. 78b, and any reaction will do work equal and opposite to that done by its corresponding action.

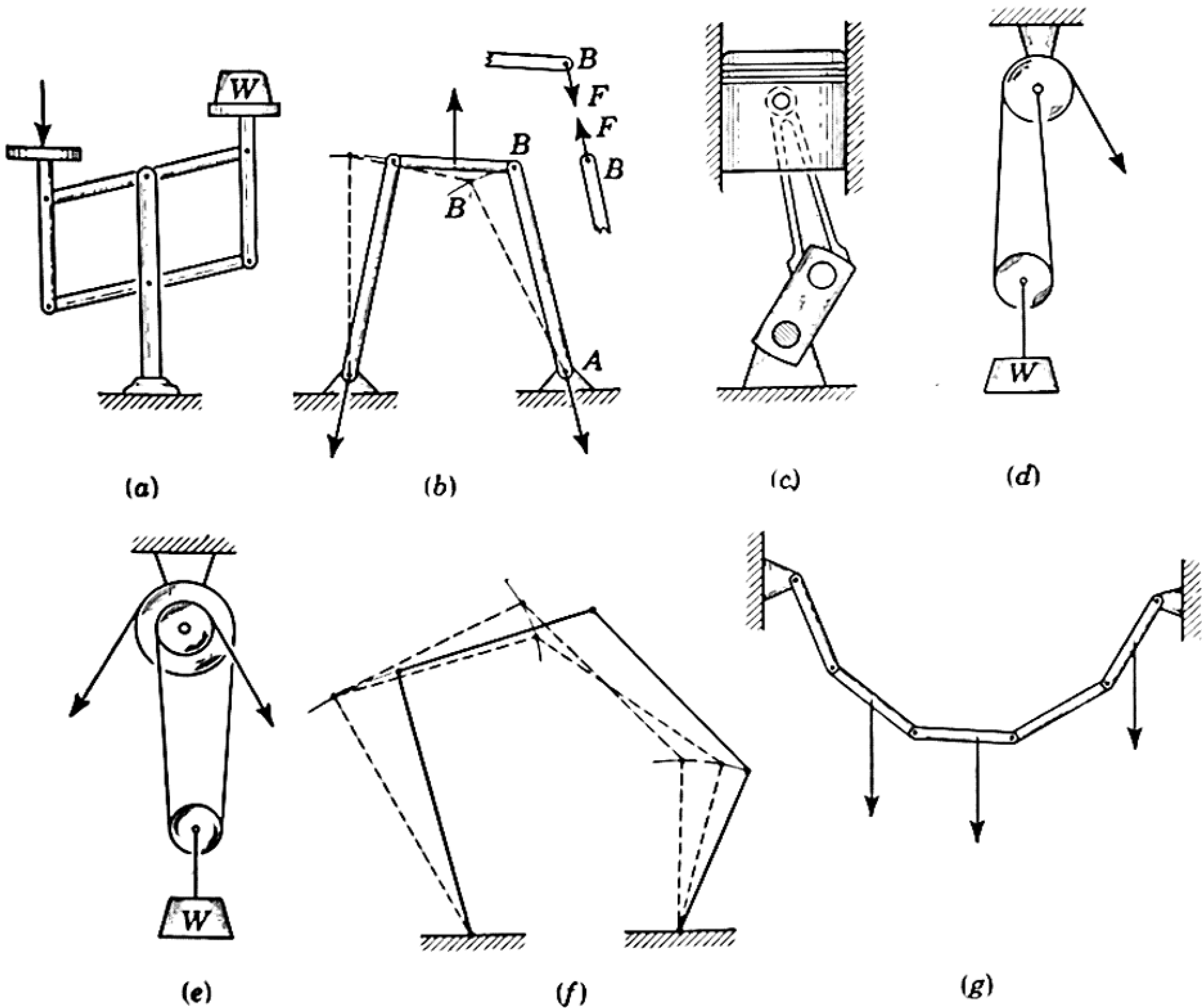


Fig. 78

Hence the principle of virtual work applies to such a connected system. The principle may now be restated as follows: *For any ideally connected system in equilibrium the net work done by all forces external to the system is zero for any and all possible virtual displacements consistent with the connections.*

For the slider-crank mechanism shown in Fig. 78c, where the piston is confined to move in the vertical cylinder and the crankshaft is restrained in its bearing, only one coordinate is needed to specify the virtual displacements of the three parts. Thus a small movement δs of the piston will uniquely define the corresponding consistent movements of the con-

necting rod and crankshaft. With the existing constraints provided by the cylinder sleeve and crankshaft bearing this system is said to have *one degree of freedom.* The pulley device of Fig. 78*d* is also a system of one degree of freedom since the position of each of its parts is determined uniquely by the movement of the free end of the rope. In the four-link mechanism in the *b*-part of the figure (one link is the ground) a small rotation of one of the members such as AB to AB' completely determines the positions of the other movable ones. Thus it has a single degree of freedom, as does the balance in Fig. 78*a*. In the pulley device in Fig. 78*e* the two independent movements of the free ends of the rope must be known to determine the position of W. Therefore this device has two degrees of freedom. The five-link mechanism in Fig. 78*f* will require two separate specified rotations before the configuration of each part is uniquely determined, and hence it has two degrees of freedom. The connection of links in Fig. 78*g* has three degrees of freedom. A rigid body free to move in space has six degrees of freedom, three components of translation and three components of rotation. The *number of degrees of freedom,* then, equals the *number of independent coordinates* required to determine the unique configuration of the system. The number of possible virtual displacements of a system will equal the number of degrees of freedom.

All external forces at fixed reactions do no work during any virtual displacement consistent with the constraints at these reactions. Such external forces are called *reactive forces* as distinguished from the externally applied or *active forces.* Weight or any other body force must be considered an active force.

The principle of virtual work is now restated in a restricted form for ideal systems with constraints. *The virtual work done by all external active forces on an ideal system in equilibrium is zero for any and all virtual displacements consistent with the constraints.* It is in this form that the principle is most useful for ideal systems. This principle may be stated symbolically by the equation

$$\delta U = 0, \tag{44}$$

where δU stands for the total work done by all external active forces during a virtual displacement.

The real advantages of the method of virtual work can only now be seen. There are essentially two. First, it is not necessary to dismember ideal systems to establish the relations between the active forces, as is usually the case with the equilibrium method based on force and moment summations. Second, the relations between the active forces may be determined directly without reference to the reactive forces. These ad-

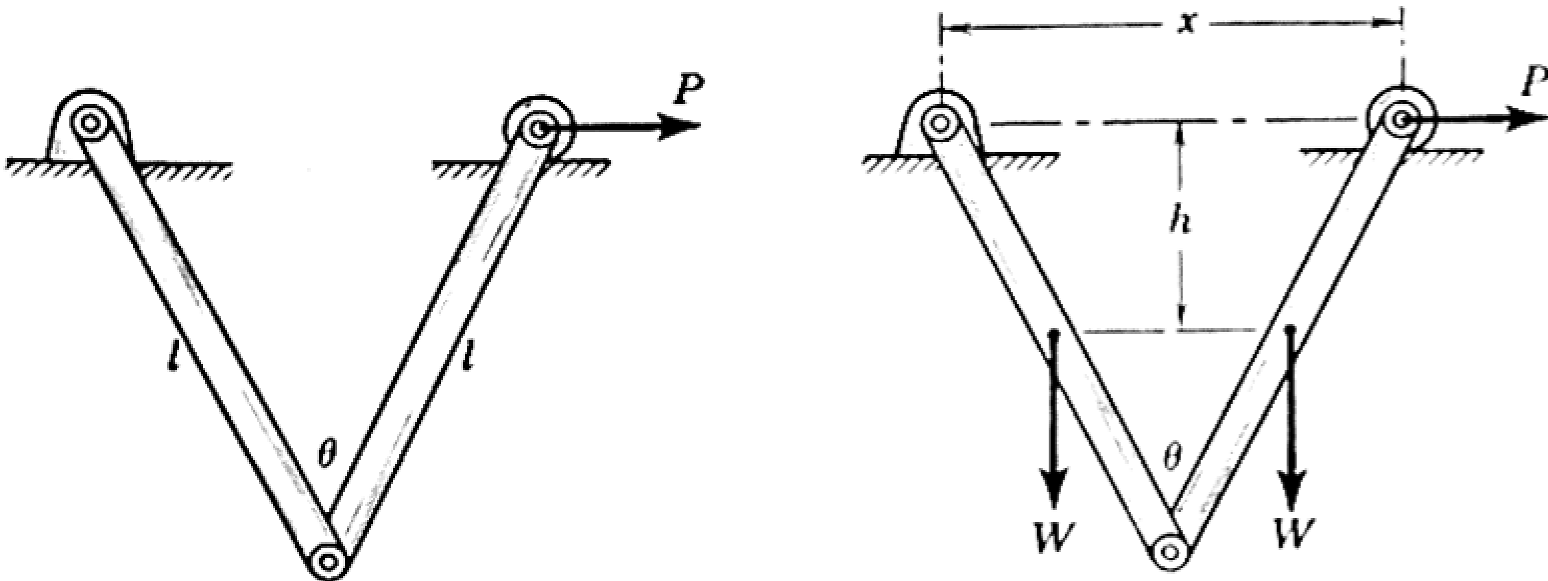
vantages make the method of virtual work particularly useful in determining the position of equilibrium of a system under known loads. This type of problem is in contrast with the problem of determining the forces acting on a body whose equilibrium position is fixed.

In the method of virtual work a diagram which isolates the system under consideration must be drawn. Unlike the free-body diagram, where all forces are shown, the diagram for the method of work need show only the *active* forces since the reactive forces do not enter into the application of $\delta U = 0$ in the restricted sense for constrained systems. Such a drawing may be termed an *active-force diagram*.

Most problems involve a single degree of freedom, and, since the number of possible virtual displacements equals the number of degrees of freedom, such problems will require one application of Eq. (44) for the one virtual displacement. For systems of n degrees of freedom it is necessary to solve n equations, each expressing zero virtual work of all active forces due to each of the n possible virtual displacements considered separately while the remaining ones are held zero.

SAMPLE PROBLEMS

567. Each of the two uniform hinged bars has a weight W and a length l, and is supported and loaded as shown. For a given force P determine the angle θ for equilibrium.



PROB. 567

Solution: The active-force diagram for the system composed of the two members is shown separately and includes the two weights in addition to the force P. All other forces acting externally on the system are reactive forces which do no work during a virtual movement $\delta\theta$ and are not shown.

The work done by each weight is evaluated from the change in h due to the virtual change in θ. Thus

$$h = \frac{l}{2}\cos\frac{\theta}{2} \qquad \text{and} \qquad \delta h = -\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta.$$

The minus sign comes from differentiating the cosine and indicates the evident fact that for a positive change in θ the change in h is negative. The work done by both weights during a positive $\delta\theta$ is, then,

$$2W\,\delta h = -\frac{Wl}{2}\sin\frac{\theta}{2}\,\delta\theta.$$

The work done by P during this same virtual movement is

$$P\delta x = P\delta\left(2l\sin\frac{\theta}{2}\right) = Pl\cos\frac{\theta}{2}\,\delta\theta.$$

The principle of virtual work requires that the sum total of the work of all external active forces is zero, or

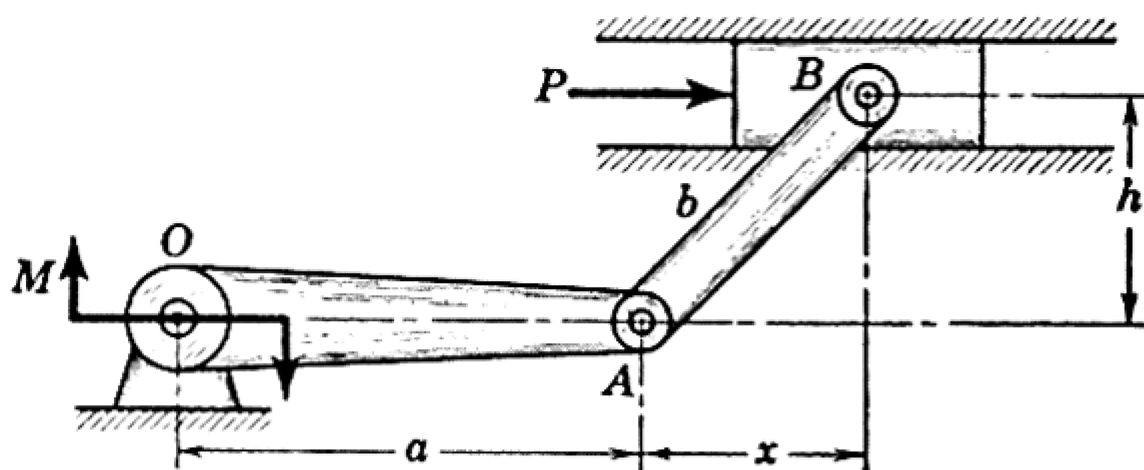
$$[\delta U = 0] \qquad P\,\delta x + 2W\,\delta h = 0,$$

$$Pl\cos\frac{\theta}{2}\,\delta\theta - \frac{Wl}{2}\sin\frac{\theta}{2}\,\delta\theta = 0,$$

$$\tan\frac{\theta}{2} = \frac{2P}{W}, \qquad \theta = 2\tan^{-1}\frac{2P}{W}. \qquad \textit{Ans.}$$

In order to obtain this result by the principles of force and moment summation, it would be necessary to dismember the frame and take into account all forces acting on each member. Solution by the method of virtual work involves a much simpler operation.

568. A couple M is applied to the shaft fixed to the lever OA. Determine the force P on the smooth slider block required to keep OA in the horizontal position. The length of the link AB is b.



PROB. 568

Solution: The active-force diagram includes only the force P and the couple M as shown. A small rotation of the arm OA is accompanied by a small horizontal movement of the slider block at B. The total work done by P and M during these consistent movements is zero. The relation between these movements is obtained from the right triangle of which b is the hypotenuse. Differentiating the relation $x^2 + h^2 = b^2$ and noting that the change δb in the length of the link is zero give

$$2x\,\delta x + 2h\,\delta h = 2b\,\delta b = 0, \qquad x\,\delta x = -h\,\delta h.$$

If δh is arbitrarily taken as positive, δx is negative, and the rotation $\delta\theta$ of the arm OA is clockwise with the value $\delta\theta = \delta h/a$.

The work done by P during the virtual displacement is

$$P\,\delta x = -P\frac{h}{x}\,\delta h,$$

and the work done by the couple M is

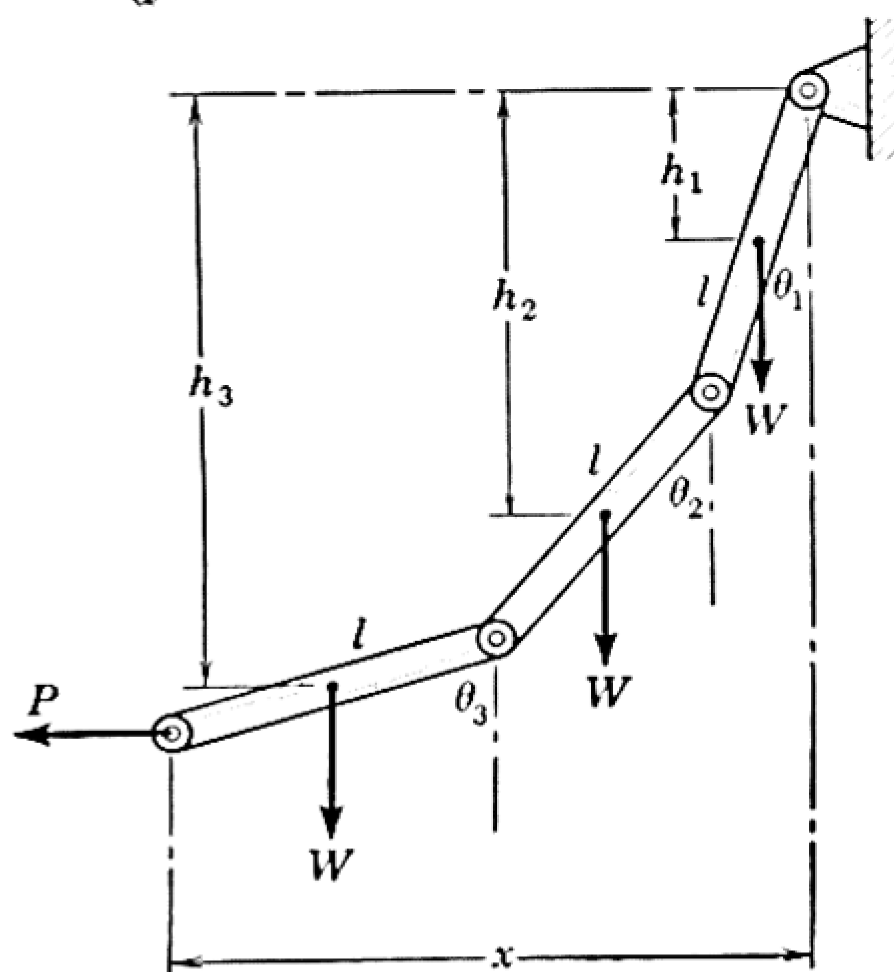
$$M\,\delta\theta = M\frac{\delta h}{a}.$$

Applying the principle of virtual work requires

$$[\delta U = 0] \qquad P\,\delta x + M\,\delta\theta = 0, \qquad -P\frac{h}{x}\,\delta h + M\frac{\delta h}{a} = 0,$$

$$P = \frac{x}{h}\frac{M}{a} = \frac{\sqrt{b^2 - h^2}}{h}\frac{M}{a}. \qquad \textit{Ans.}$$

569. A horizontal force P is applied to the end of one of three identical hinged links each of which has a weight W and a length l. Determine the equilibrium configuration.



PROB. 569

Solution: The equilibrium configuration may be specified by determining the angles θ_1, θ_2, and θ_3. Thus three independent coordinates or measurements are required to describe uniquely the positions of the bars. The problem, consequently, is one of *three degrees of freedom*. It will be necessary to apply the principle $\delta U = 0$ three times, where each application is made by varying *one* of the coordinates at a time with the remaining two held constant.

The active-force diagram shows the three weights and the applied load P. The geometry of the figure gives

$$h_1 = \frac{l}{2}\cos\theta_1, \qquad h_2 = l\,(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2),$$

$$h_3 = l(\cos\theta_1 + \cos\theta_2 + \tfrac{1}{2}\cos\theta_3),$$

$$x = l(\sin\theta_1 + \sin\theta_2 + \sin\theta_3).$$

If θ_3 is allowed to vary first and θ_1 and θ_2 are held constant, the principle of virtual work requires

$$[\delta U = 0]_{\theta_3} \qquad P\,\delta x + W\,\delta h_3 = 0, \qquad Pl\cos\theta_3\,\delta\theta_3 - W\frac{l}{2}\sin\theta_3\,\delta\theta_3 = 0,$$

$$\tan\theta_3 = \frac{2P}{W}. \qquad \textit{Ans.}$$

Next, θ_2 is allowed to vary and θ_1 and θ_3 are held constant, so that

$$[\delta U = 0]_{\theta_2} \qquad P\,\delta x + W\,\delta h_3 + W\,\delta h_2 = 0,$$

$$Pl\ \mathrm{eos}\,\theta_2\,\delta\theta_2 - Wl\sin\theta_2\,\delta\theta_2 - W\frac{l}{2}\sin\theta_2\,\delta\theta_2 = 0,$$

$$\tan\theta_2 = \frac{2P}{3W}. \qquad \textit{Ans.}$$

Lastly, θ_1 is allowed to vary while θ_2 and θ_3 are held constant. Thus

$$[\delta U = 0]_{\theta_1} \quad P\,\delta x + W\,\delta h_3 + W\,\delta h_2 + W\,\delta h_1 = 0,$$

$$Pl\cos\theta_1\,\delta\theta_1 - Wl\sin\theta_1\,\delta\theta_1 - Wl\sin\theta_1\,\delta\theta_1 - W\frac{l}{2}\sin\theta_1\,\delta\theta_1 = 0,$$

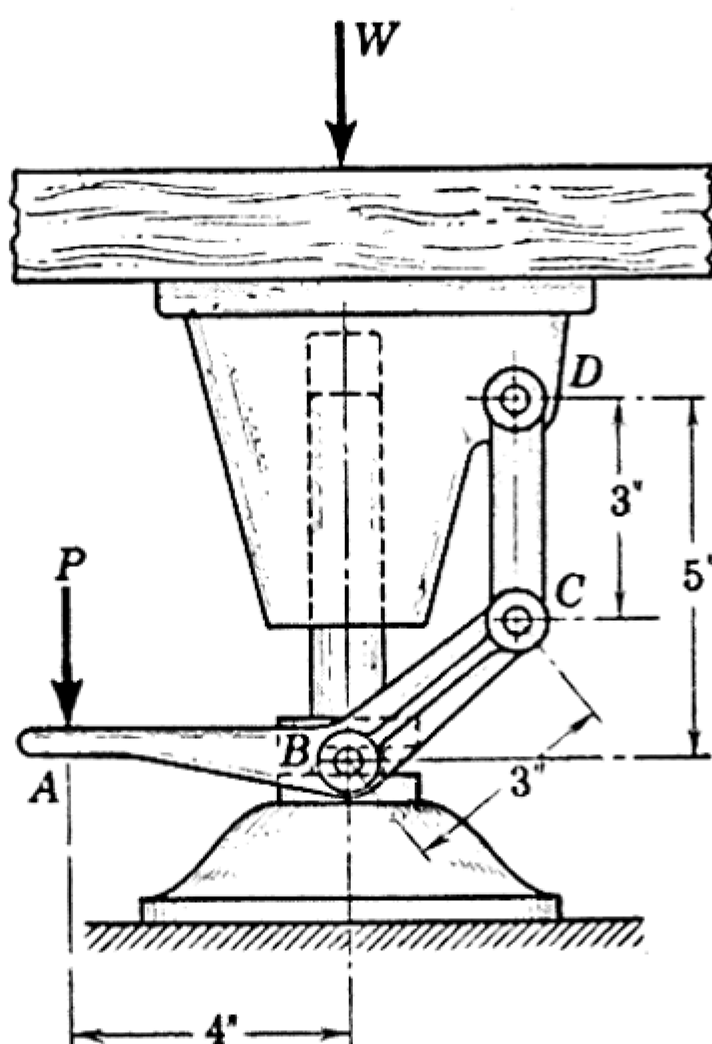
$$\tan\theta_1 = \frac{2P}{5W}. \qquad \textit{Ans.}$$

The order in which the coordinates are allowed to vary makes no difference. In this problem the three angles were independently determined by each of the three applications of $\delta U = 0$. In other problems of multiple degrees of freedom it may be necessary to effect a simultaneous solution of the equations arising from each application of $\delta U = 0$ in order to separate the unknowns.

PROBLEMS

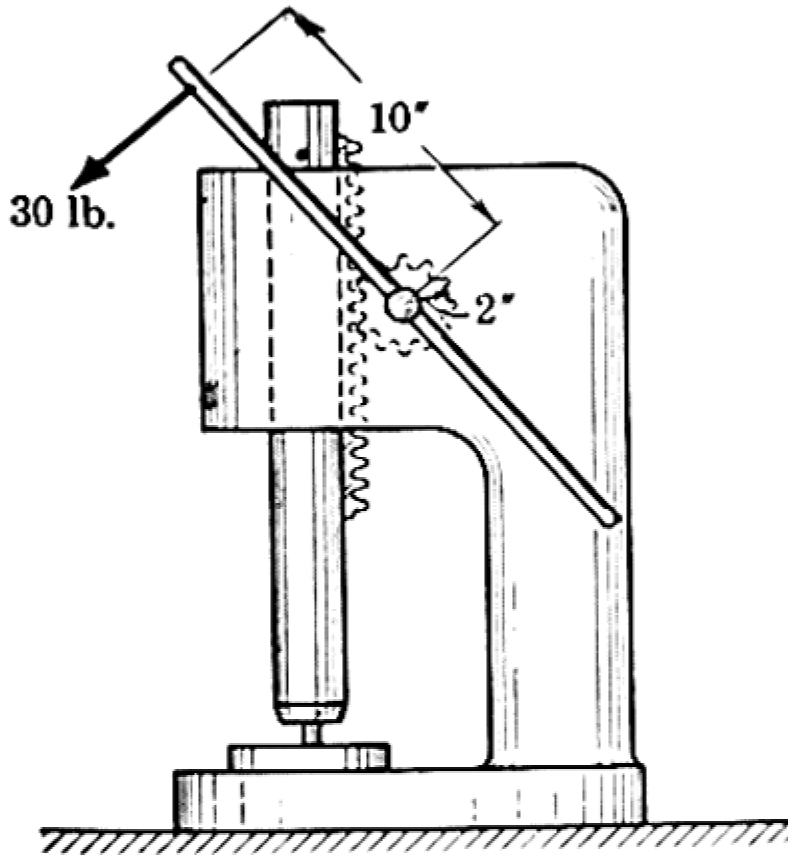
Solve the following problems by the method of virtual work.

570. The foot-operated truck lock shown is used to jack up a small hand truck a short distance above the floor. Stepping on the lever at A with a force P causes the link CD to raise the upper frame. Find the weight W which can be lifted from the position shown by a force P of 100 lb. *Ans.* $W = 179$ lb.

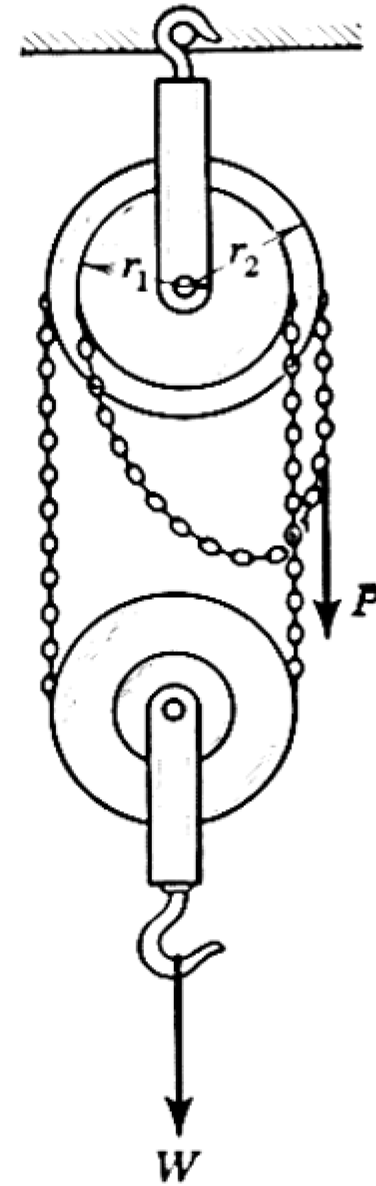


PROB. 570

571. Find the force P exerted by the arbor press for the 30 lb. pull on the lever. Neglect the weight of the spindle.



Prob. 571



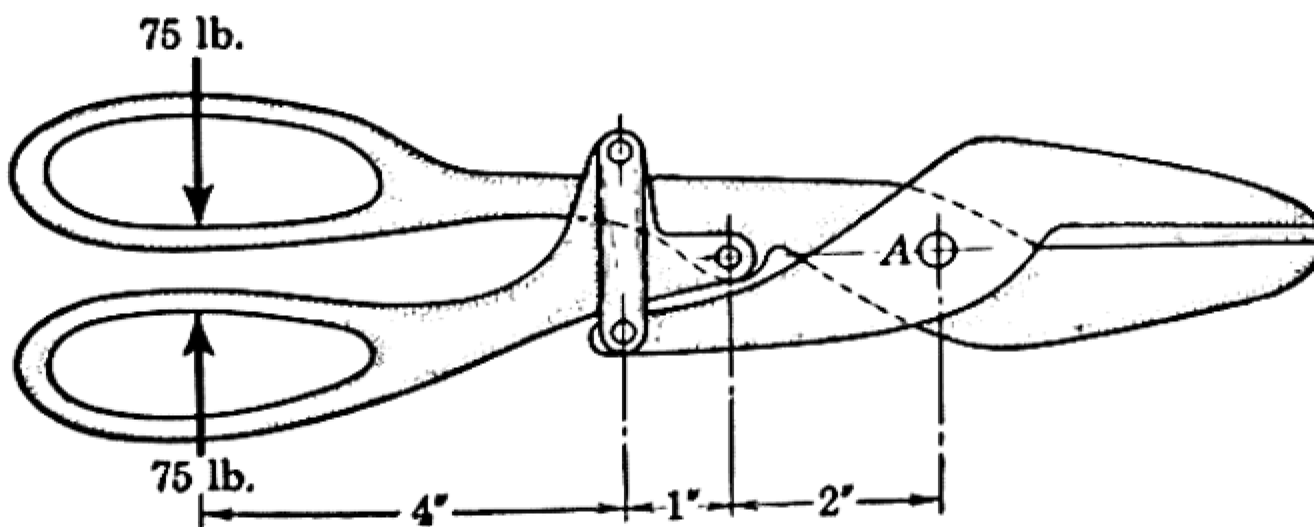
Prob. 572

572. Find the force P required to lift the load W for the differential chain hoist of Prob. 163 shown again here.

573. For a certain car it takes one turn of the steering wheel to turn the front wheels through 20 deg. When the car is standing still, determine the frictional moment between each front wheel and the road if a moment of 60 lb. ft. is applied to the steering wheel. Neglect small frictional loss in the linkage.

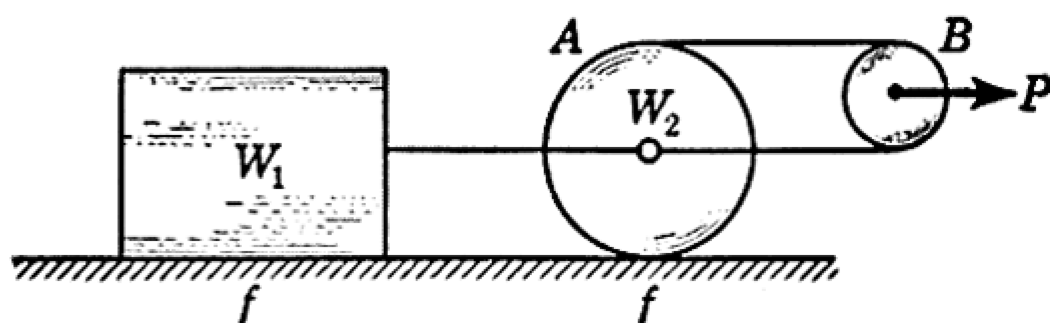
Ans. 540 lb. ft.

574. The compound snips of Prob. 301 are shown again here. Find the cutting force P at a distance of 1 in. along the blade from A.



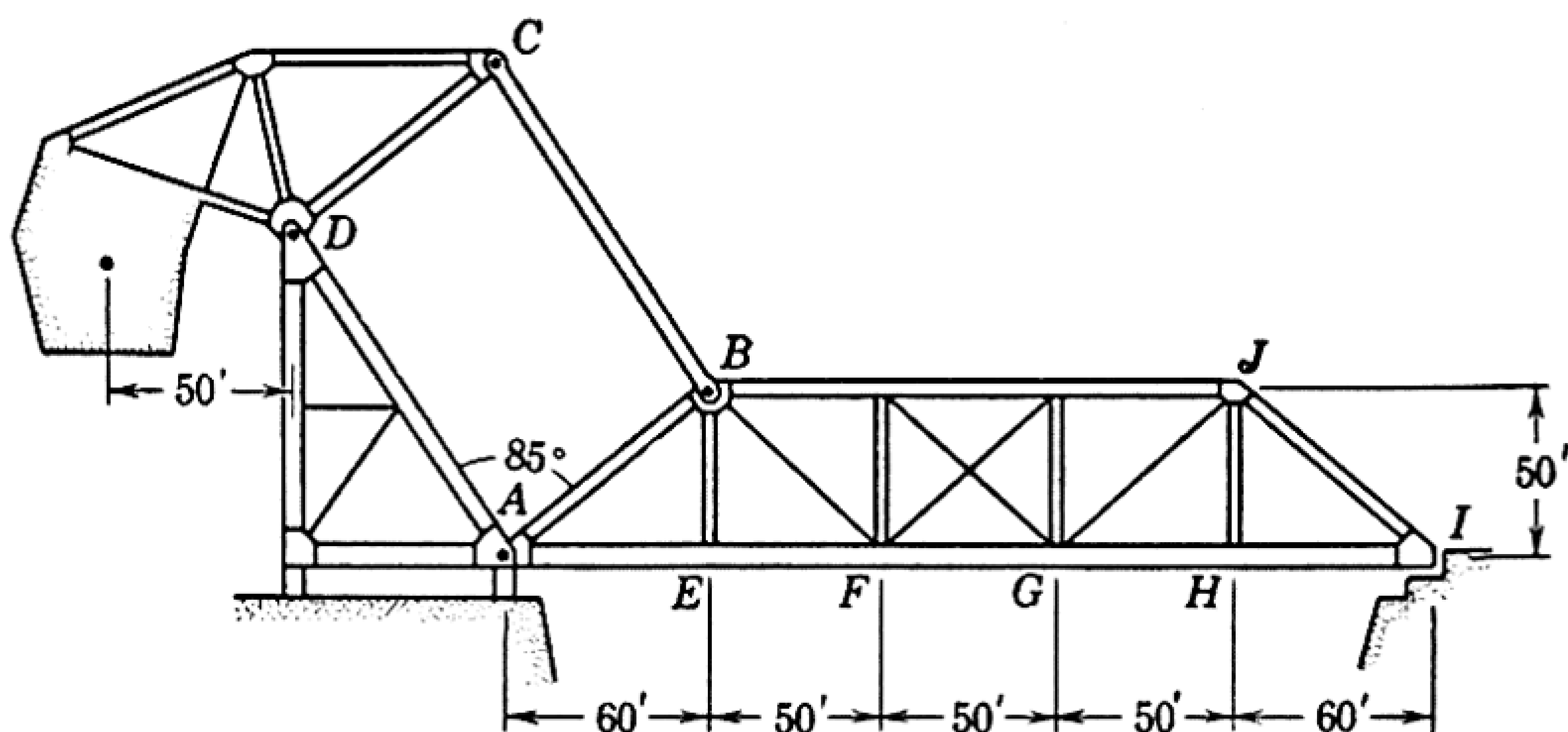
Prob. 574

575. Determine the force P required to move the system of Prob. 504 repeated here. Assume wheel A does not slip.



PROB. 575

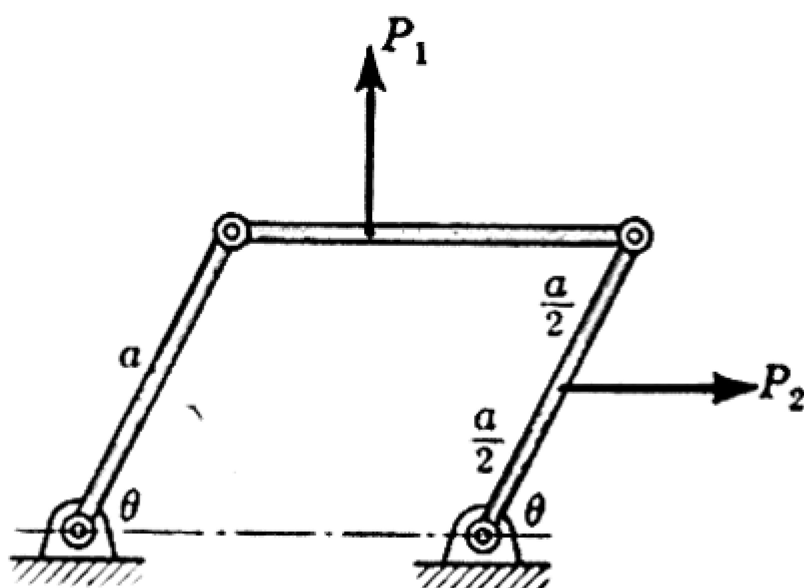
576. Determine the weight W of the uniform roadway of the balanced bascule bridge of Prob. 294 repeated here. *Ans.* $W = 667$ tons



PROB. 576

577. For the parallel linkage shown determine the angle θ for equilibrium under the action of the vertical load P_1 and the horizontal load P_2.

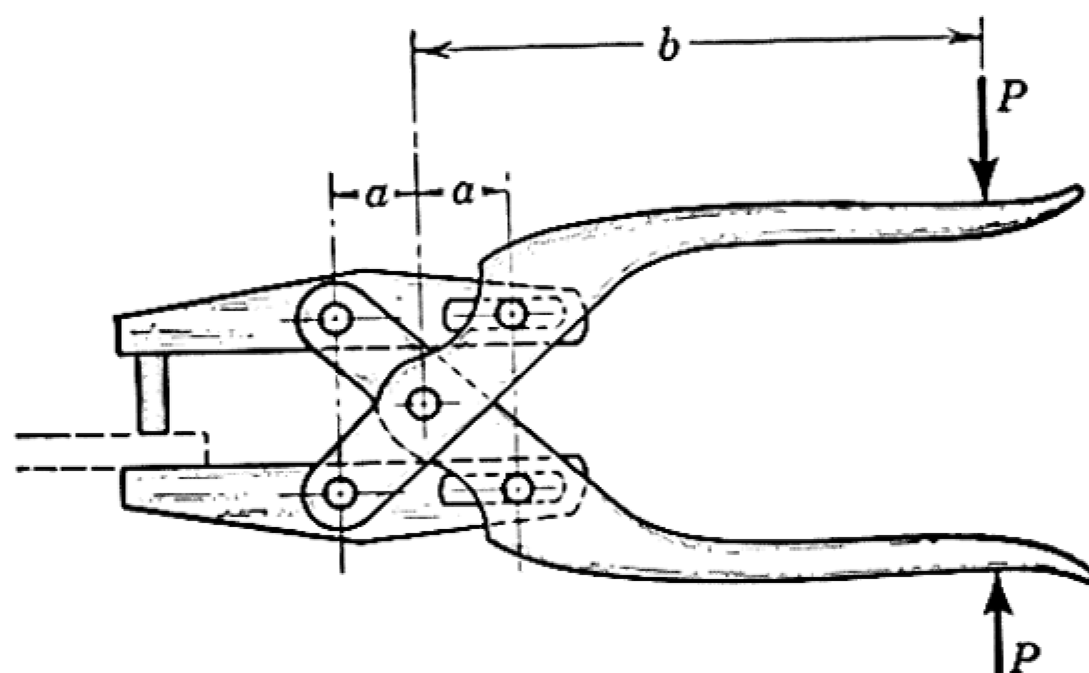
Ans. $\theta = \tan^{-1} \dfrac{2P_1}{P_2}$



PROB. 577

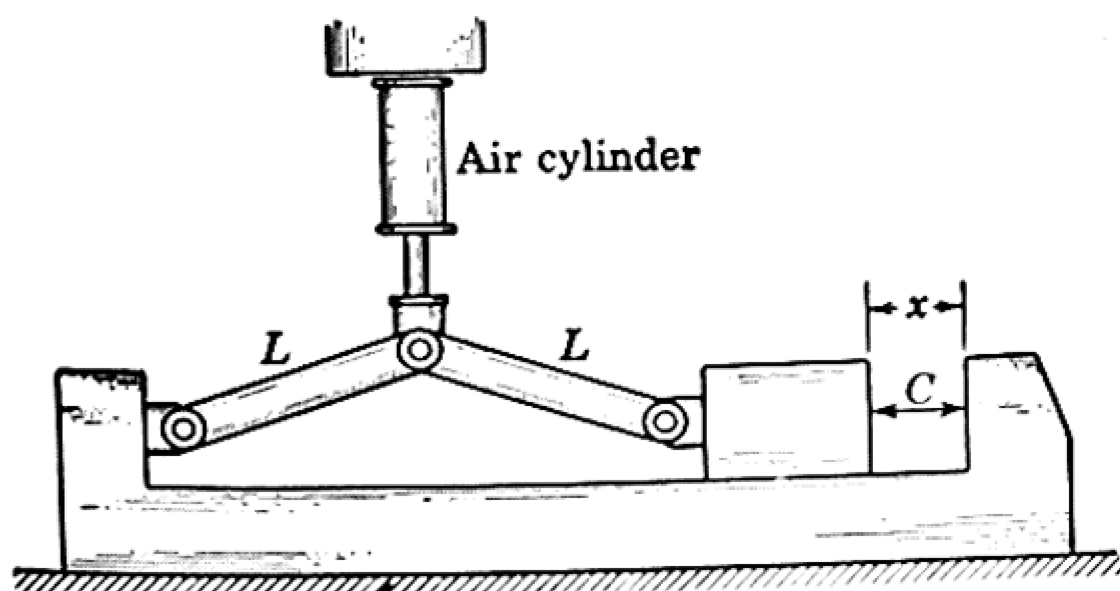
578. Replace the force P acting on the right-hand member in Prob. 567 by a clockwise couple M applied to the same member at any point and determine the angle θ for equilibrium.

579. Find the shear Q for the paper punch of Prob. 311 repeated here.

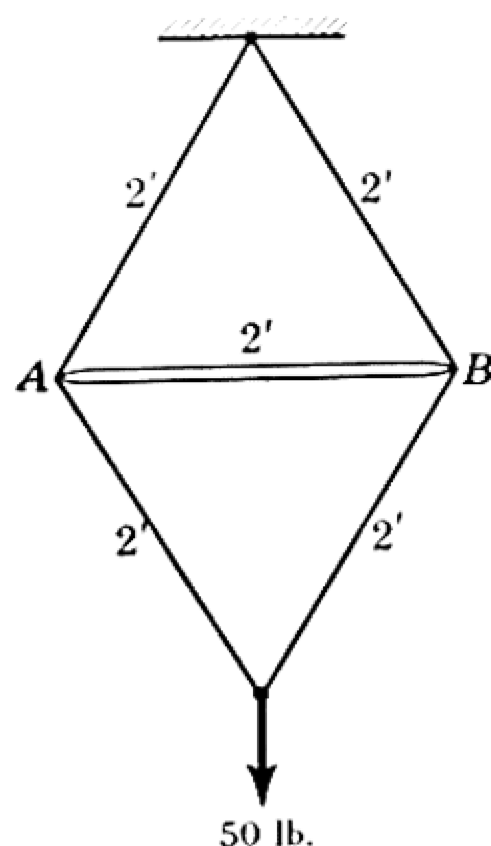


PROB. 579

580. Determine the clamping force C as a function of the jaw opening x for the quick-acting vise of Prob. 302 shown again here. The piston area is A, the air pressure is p, and the jaws are just closed when the toggle links are horizontal. The air cylinder remains in a vertical position.



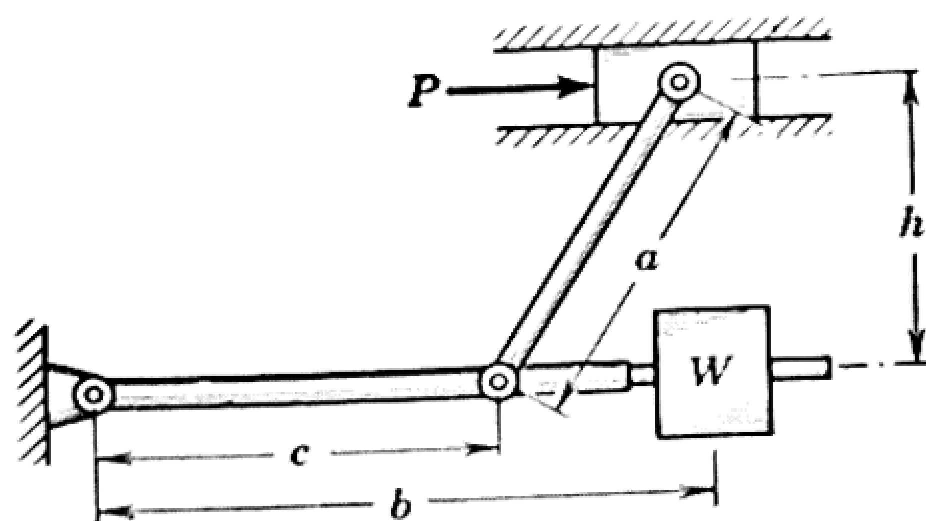
PROB. 580



PROB. 581

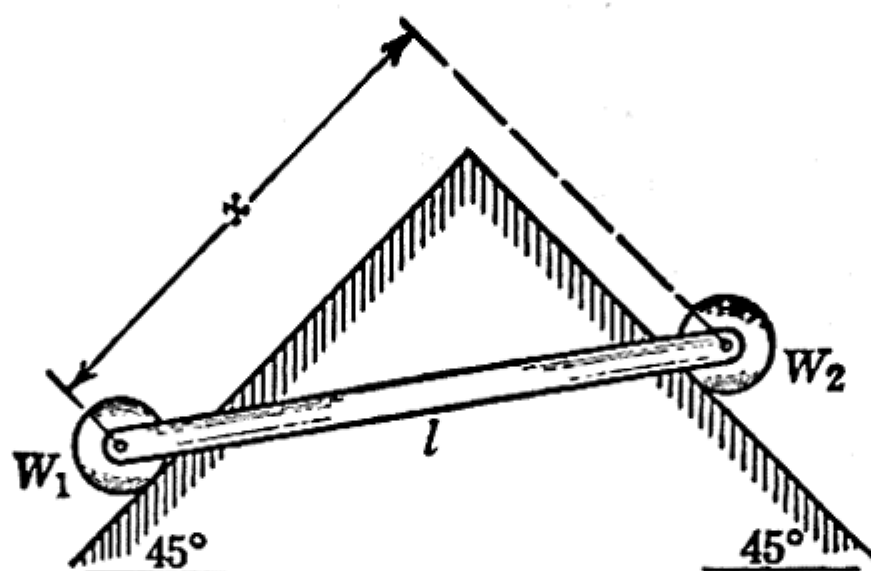
581. Determine the compression C in the strut AB. (*Hint:* Replace the strut by the forces it exerts on the cables.)

582. Determine the force P for equilibrium of the links in the position shown. Neglect friction in the guide. *Ans.* $P = \frac{Wb}{ch}\sqrt{a^2 - h^2}$



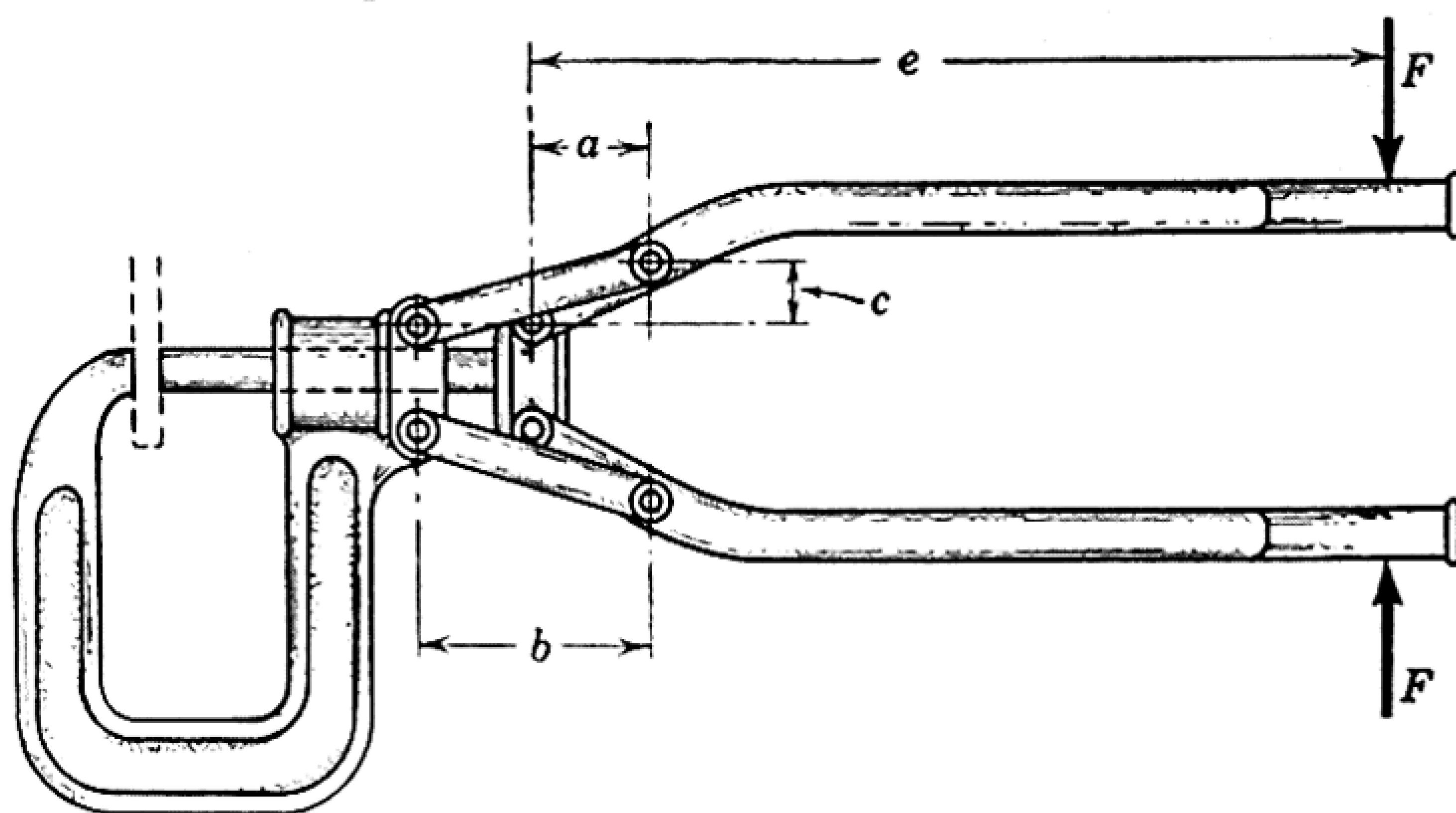
PROB. 582

583. Find the distance x for equilibrium of the bar and wheels of Prob. 160 reproduced here. The weight of the bar is negligible.



PROB. 583

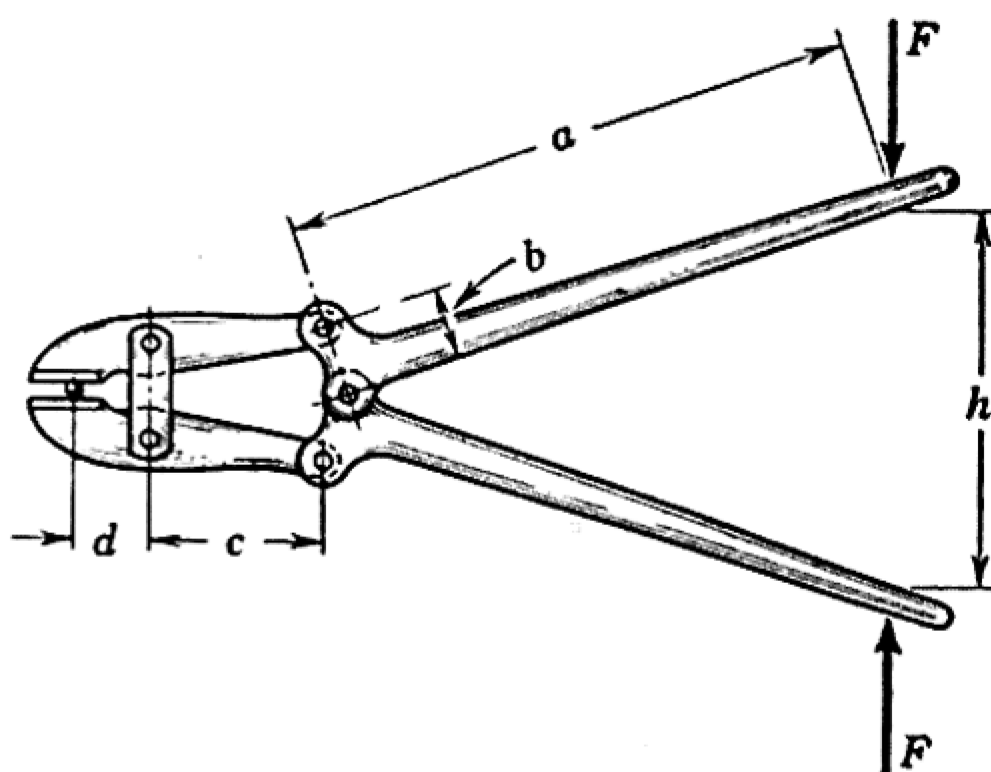
584. The rivet squeezer of Prob. 317 is shown here with symbol dimensions. Determine the punching force P.



PROB. 584

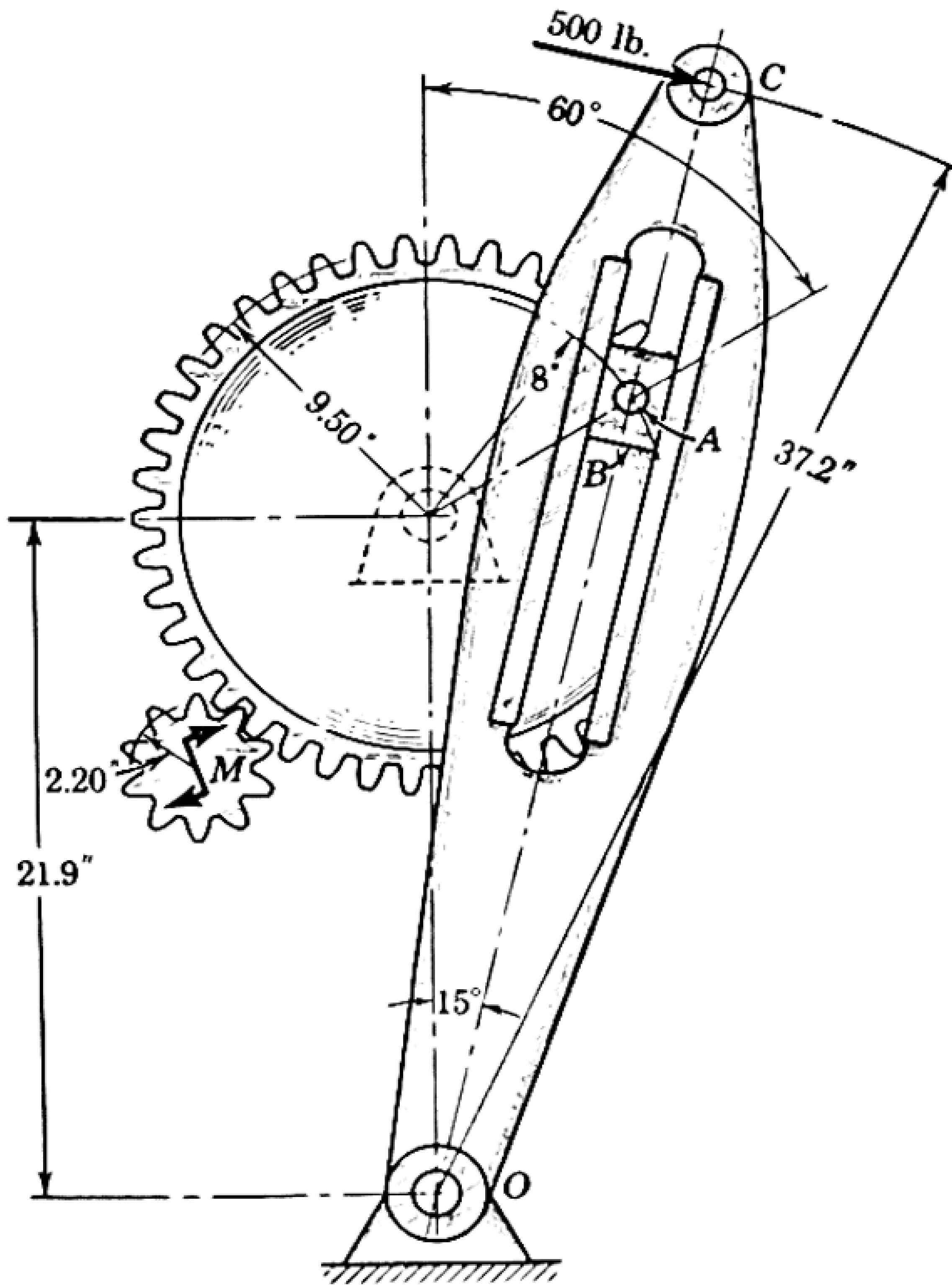
585. Solve for the shearing force Q exerted by the bolt cutters in Prob. 312 redrawn here with symbolic dimensions.

$$Ans.\quad Q = \frac{acF}{bd}\sqrt{\left(\frac{2a}{h}\right)^2 - 1}$$



PROB. 585

* **586.** The elements of a shaper quick-return mechanism are shown in the figure. The pin A is fastened to the drive wheel, and the slider block B is free



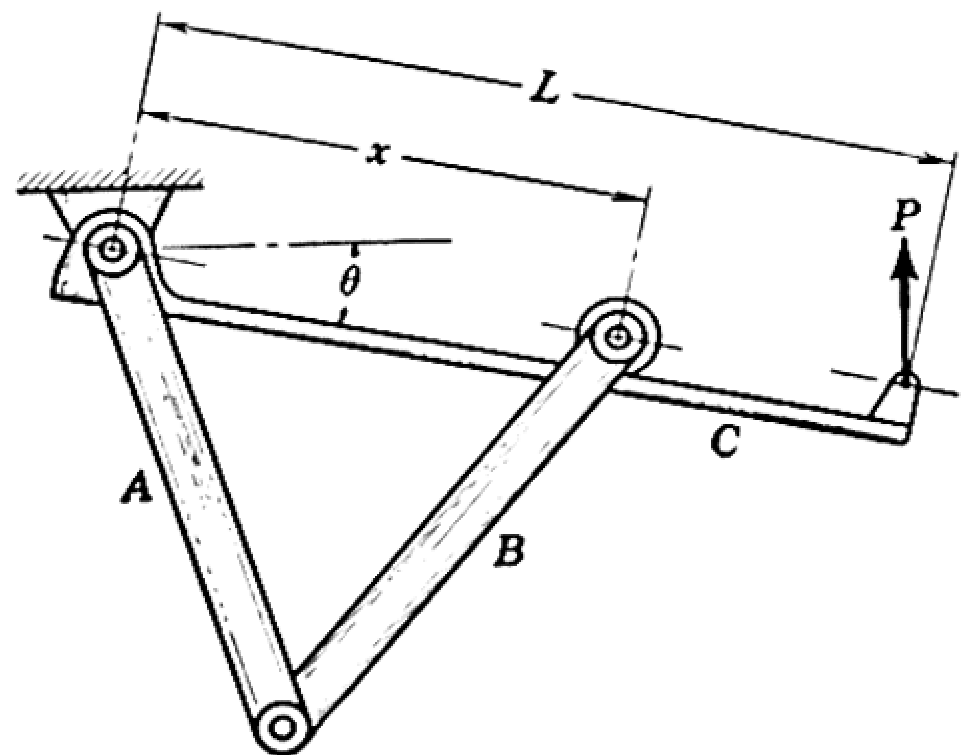
Prob. 586

to turn on the pin. Determine the torque M which must be applied to the driving pinion to balance the 500 lb. force applied to the end of the oscillating arm for the position shown. *Ans.* $M = 908$ lb. in.

* **587.** Each of the uniform hinged bars A and B has a weight W and a length l. The weight of the supporting bar C is negligible. For any given value P for which $0 < x < L$ determine the equilibrium configuration by specifying x and θ.

Ans. $x = \dfrac{4PL}{3W}$,

$$\theta = \tan^{-1} \frac{PL}{\sqrt{(3Wl)^2 - (2PL)^2}}$$



Prob. 587

63. Real Systems. When a system of bodies involves elastic members which stretch or compress under applied loads or involves internal connections which are not free from friction, some of the work done on the system by the active forces is absorbed in the elastic members by their compressions or extensions and some is dissipated in the form of heat generated by the friction forces. Such systems are called ***real systems.*** The principle of virtual work as stated for ideal mechanical systems does not apply to real systems. Before modifying the virtual work principle to apply to real systems, the work of deformation of an elastic body by a force and the work done by a friction force will be discussed.

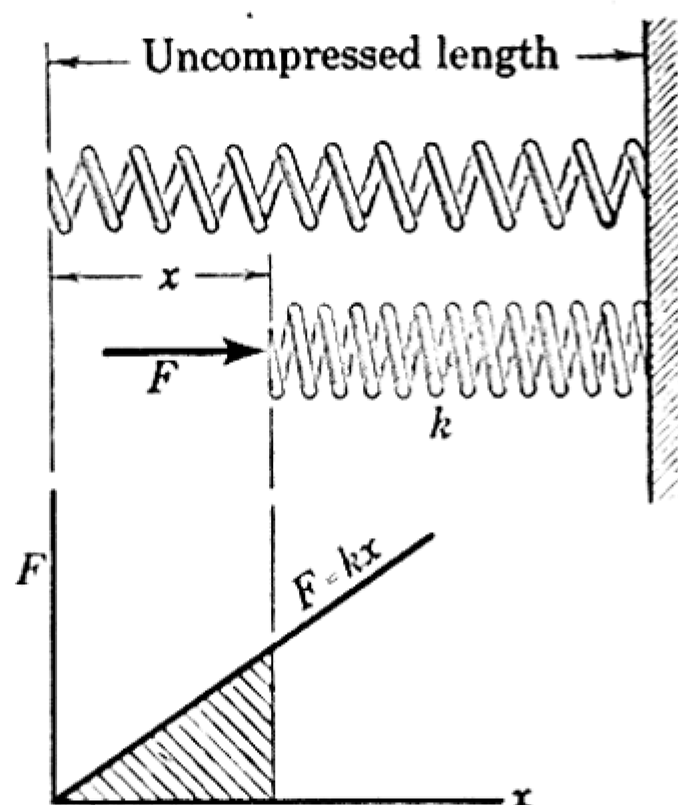


FIG. 79

The work done on an elastic member is stored in the member and is called ***elastic potential energy*** V_e. This energy is potentially usable and may be recovered by allowing the member to do work on some body during the relief of its compression or extension. Consider a spring, Fig. 79, which is being compressed by a force F. The spring is assumed to be elastic or linear, i.e., the force F is found to be directly proportional to the deflection x. This relation is written as $F = kx$, where k is the ***spring constant*** and is a measure of the stiffness of the spring. The work done by F during a movement dx is $dU = F\,dx$ so that the elastic potential energy of the spring for a compression x is

$$V_e = \int_0^x F\,dx = \tfrac{1}{2}kx^2.$$

Thus the potential energy of the spring equals the triangular area on the diagram of F versus x. The virtual work done by F during a displacement δx is the virtual change in potential energy of the spring and is

$$\delta V_e = F\,\delta x = kx\,\delta x.$$

If the spring is stretched instead of compressed, then a tensile force is required which also does positive work on the spring since the movement would likewise be in the direction of the force. Any body which exhibits a linear relation between applied force and resulting deformation is said to be elastic and may be analyzed in this same manner.

When a kinetic friction force is present, it will move through a definite distance and will do negative work on the body. The work is negative

since the direction of the force is always opposite to the direction of the movement of the body on which it acts. Thus the friction force fW for the sliding block in Fig. 80a does work on the block during the displacement x by an amount equal to $-fWx$. During a virtual displacement δx the friction force does work equal to $-fW\,\delta x$. The friction force acting on the rolling wheel in Fig. 80b, on the other hand, does no work

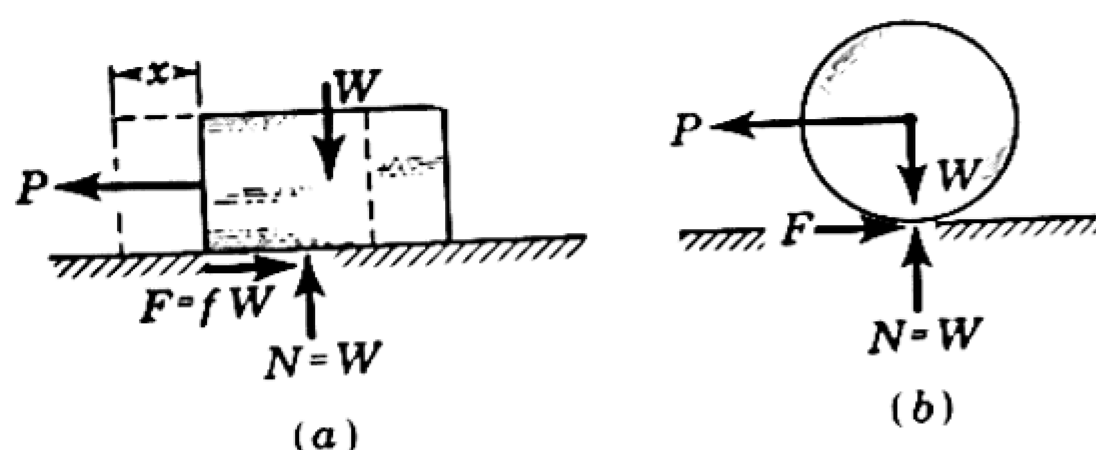


Fig. 80

if the wheel does not slip as it rolls. The negative work done by kinetic friction forces cannot be regained, as all of it is dissipated in the form of heat.

By including the two kinds of energy just described, then, *for a real system of bodies in equilibrium the net work done by all active forces external to the system during any virtual displacement consistent with the constraints equals the elastic energy stored in the system plus the energy dissipated by friction*. This principle may be stated symbolically by the equation

$$\delta U = \delta V_e + \delta Q, \tag{45}$$

where δU is the total virtual work of all external active forces including the weights of the members, δV_e is the recoverable energy stored in elastic compressions or extensions, and δQ is the nonrecoverable energy lost by the friction process, all evaluated for a given virtual displacement. In a great many problems friction is negligible, so that the virtual work principle is merely

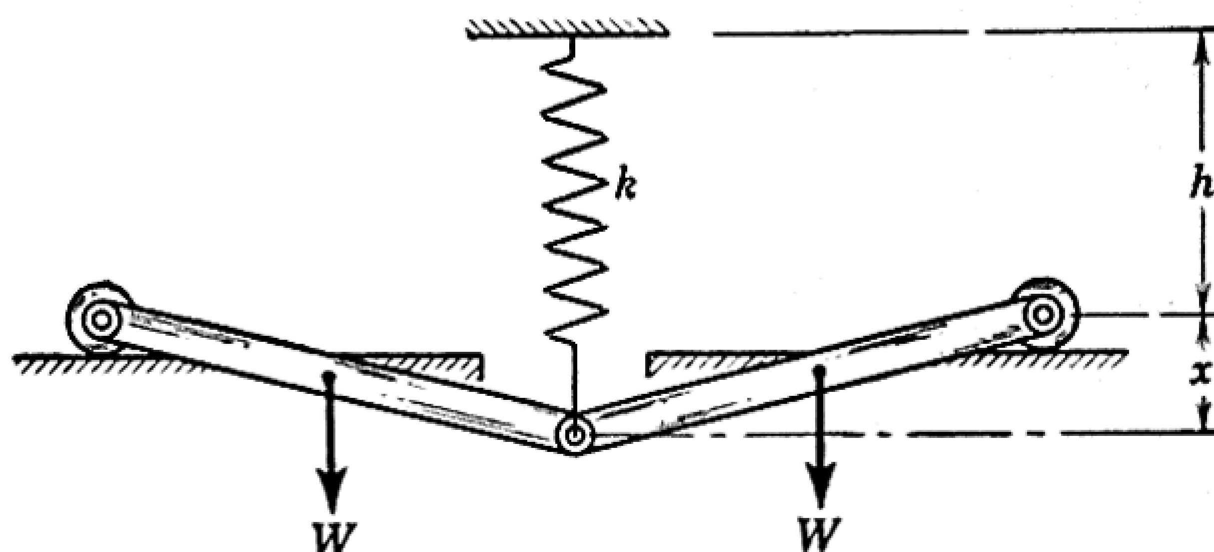
$$\delta U = \delta V_e. \tag{46}$$

For problems with a single degree of freedom Eq. (45) or (46) need be applied only once. If there are n degrees of freedom, it will be necessary to apply the principle n times. For each application the movements of the system due to one virtual displacement at a time are considered while the remaining $n - 1$ coordinates are held constant.

The principle of virtual work is very useful for real systems having internal springs but is often of little use for systems with internal friction. When the magnitudes of the internal friction forces must be known, it is often necessary to dismember the system so that one advantage of the virtual work method is lost.

SAMPLE PROBLEMS

588. The unstretched length of the spring is h and its stiffness is k. Find the deflection x for equilibrium if each of the identical uniform bars has a weight W.



Prob. 588

Solution: From the active-force diagram it is seen that the work done by the weights of the two bars during a virtual movement δx is

$$\delta U = 2W\frac{\delta x}{2},$$

and the change in the elastic energy of the spring during the same movement is

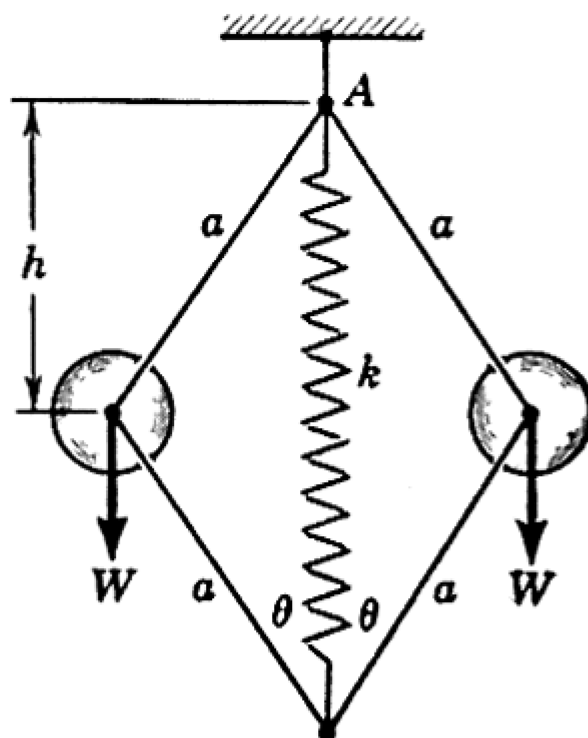
$$\delta V_e = \delta(\tfrac{1}{2}kx^2) = kx\,\delta x.$$

Equilibrium is specified by

$[\delta U = \delta V_e]$ $\qquad W\,\delta x = kx\,\delta x, \qquad x = \frac{W}{k}.$ *Ans.*

The method illustrated in this simple problem is the same as used with more complex problems. The results here, of course, could have been obtained by inspection.

589. When θ is 45 deg. the force in the spring is zero. Determine the angle θ for equilibrium. The spring constant is k, there is negligible friction in the joints, and the weights of the arms and spring are negligible.



Prob. 589

Solution: The initial length of the spring is $2a/\sqrt{2}$ and the length in the equilibrium position is $2a\cos\theta$. The spring extension becomes $x = 2a\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$. The variation in elastic potential energy due to a small increase in θ is

$$\delta V_e = kx\,\delta x = 2ka\left(\cos\theta - \frac{1}{\sqrt{2}}\right)(-2a\sin\theta\,\delta\theta).$$

The work done by the weights is

$$\delta U = 2W\,\delta h = 2W\,\delta(a\cos\theta) = -2Wa\sin\theta\,\delta\theta.$$

The equilibrium position requires

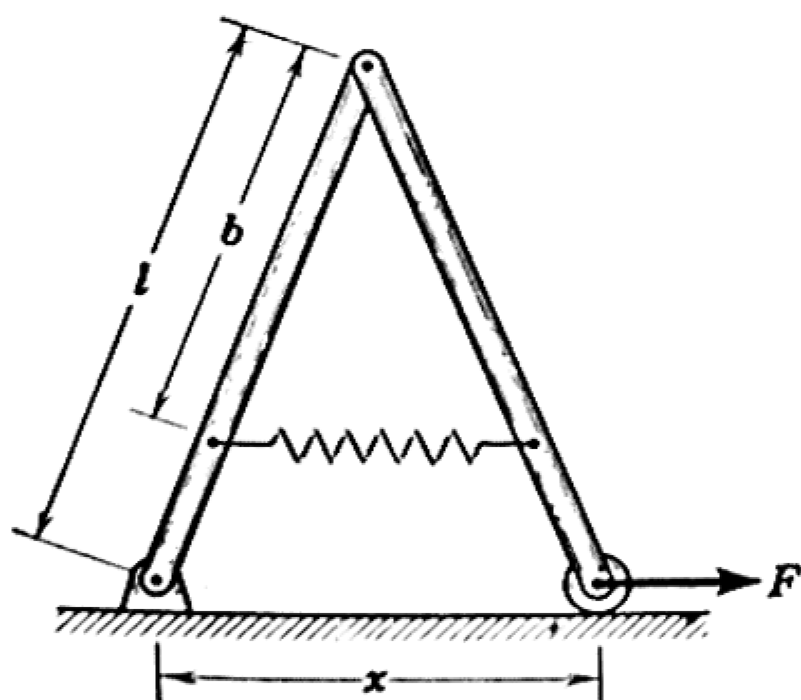
$$[\delta U = \delta V_e] \qquad -2Wa\sin\theta\,\delta\theta = -4ka^2\left(\cos\theta - \frac{1}{\sqrt{2}}\right)\sin\theta\,\delta\theta,$$

$$\cos\theta = \frac{1}{\sqrt{2}} + \frac{W}{2ka}, \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}} + \frac{W}{2ka}\right). \qquad \textit{Ans.}$$

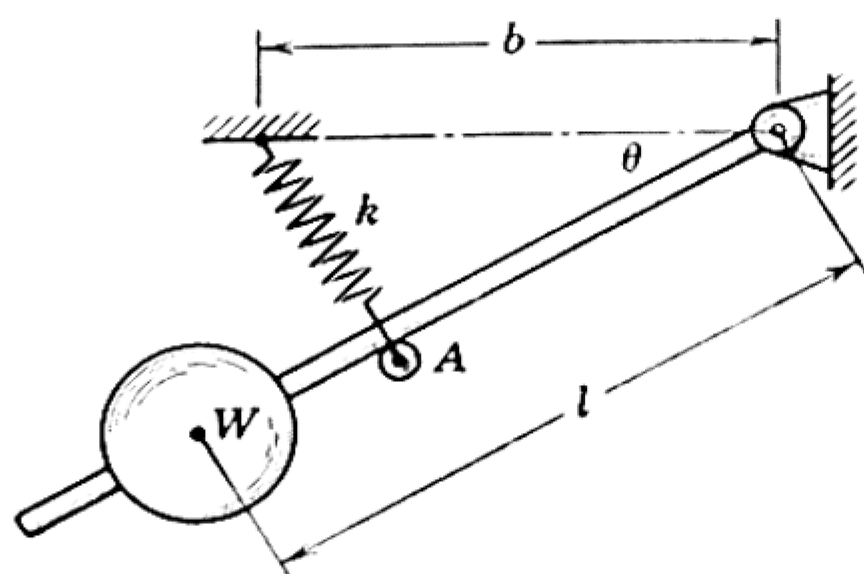
A second solution is given by $\sin\theta = 0$ which would require absence from mechanical interference.

PROBLEMS

590. The spring-connected links of Prob. 305 are shown again here. Determine the value of x for equilibrium under the action of the force F. The spring has a stiffness k and has zero tension when $x = a$.



Prob. 590

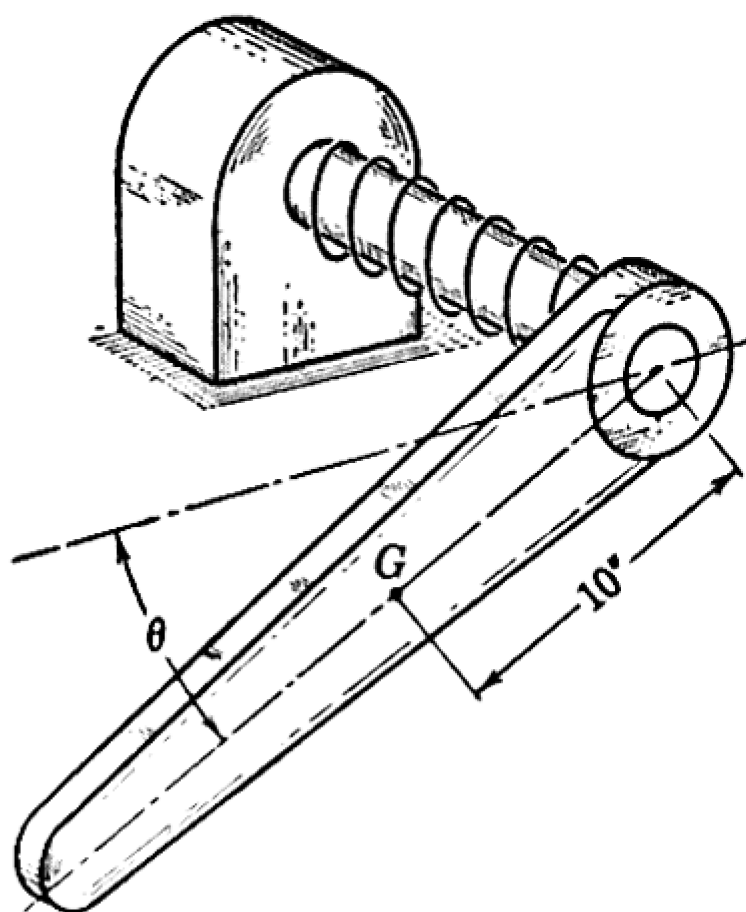


Prob. 591

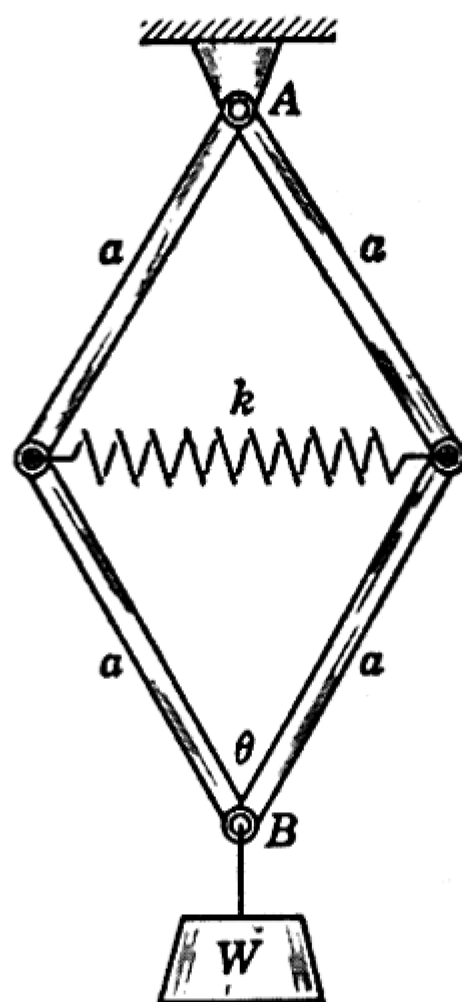
591. The small roller A moves so that the spring assumes a position of minimum stretch for any value of θ. When $\theta = 0$, the spring force is zero. Find θ for the natural equilibrium position.

Ans. $\theta = \sin^{-1}\frac{Wl}{kb^2}$

592. The figure for Prob. 164 is repeated here. Determine the angle θ for equilibrium. The spring constant is 4000 lb.in./rev., the center of gravity of the 50 lb. lever is at G, and the energy of the spring is zero for the horizontal position $\theta = 0$.



PROB. 592

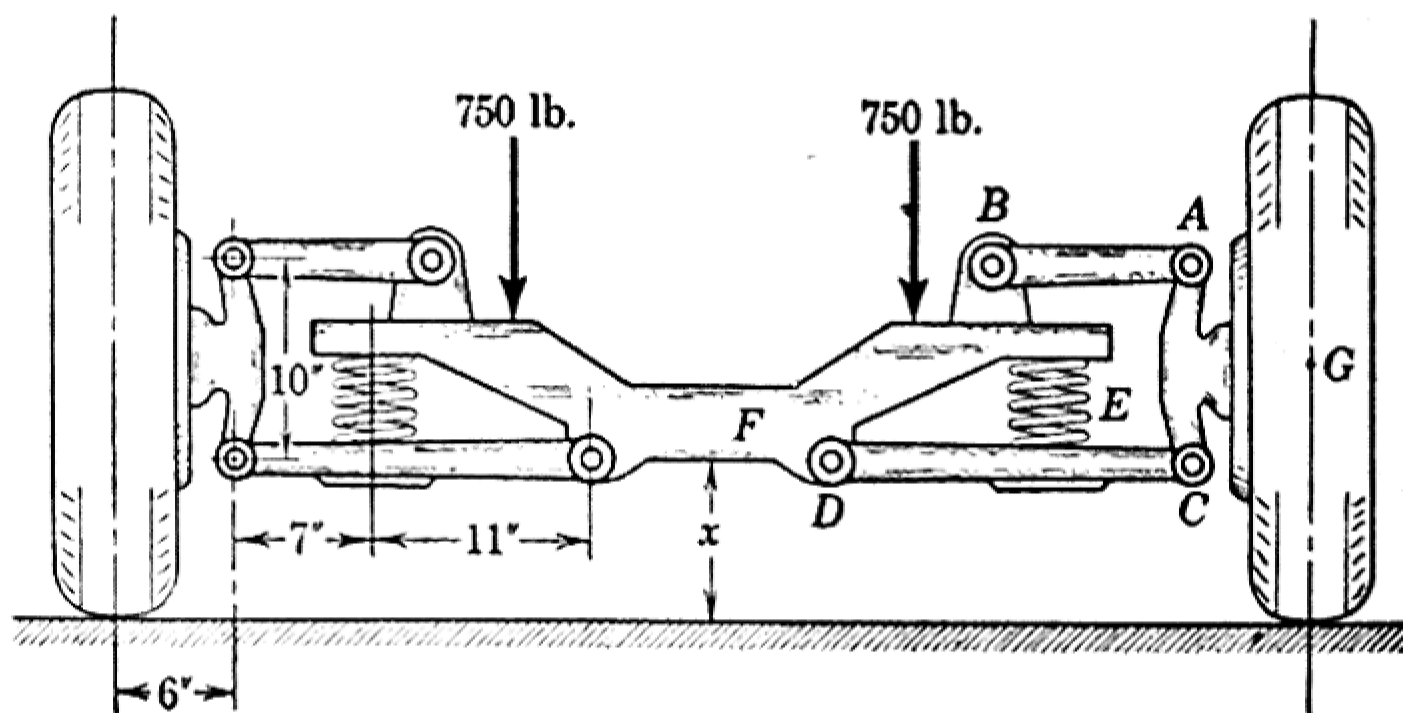


PROB. 593

593. When the weight W is removed from the mechanism shown, points A and B coincide and the spring compression is reduced to zero. Determine the spring constant k if equilibrium occurs for $\theta = 90$ deg.

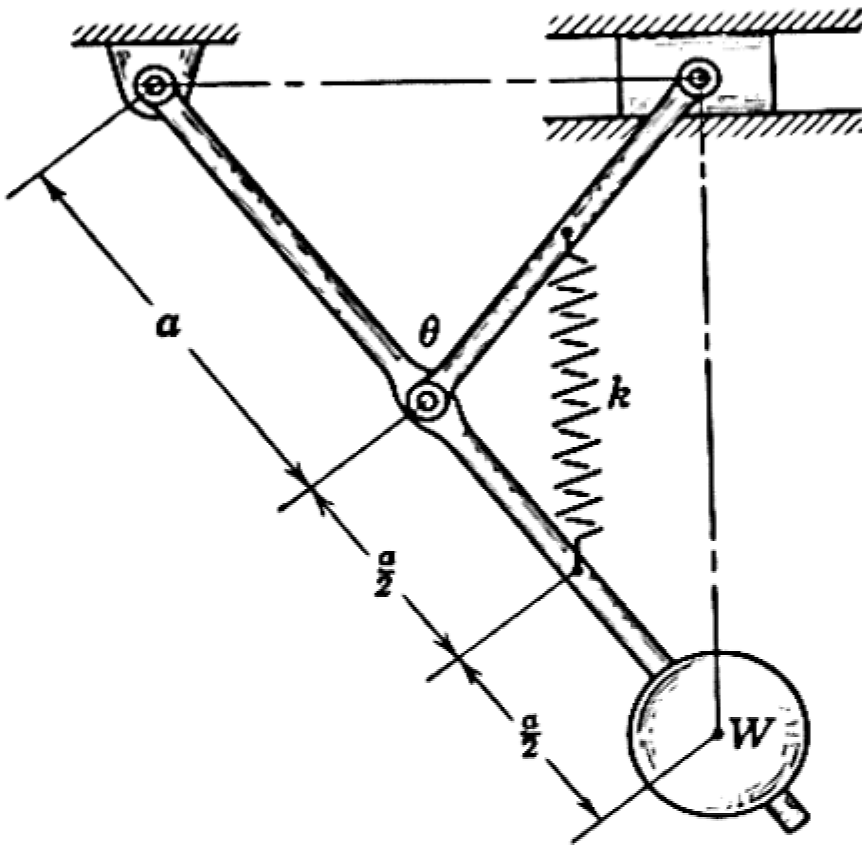
$$Ans.\quad k = \frac{W\sqrt{2}}{2a(\sqrt{2} - 1)}$$

594. The sketch of the automobile front-end suspension of Prob. 319 is repeated here. If the frame F must be jacked up so that $x = 14.5$ in. in order to relieve the compression in the coil springs, determine the value of x for equilibrium when the jack is removed. Assume that the line through AC remains vertical and that the weight of the frame is included in the vertical loading of 1500 lb. The constant of each spring is 409 lb./in.

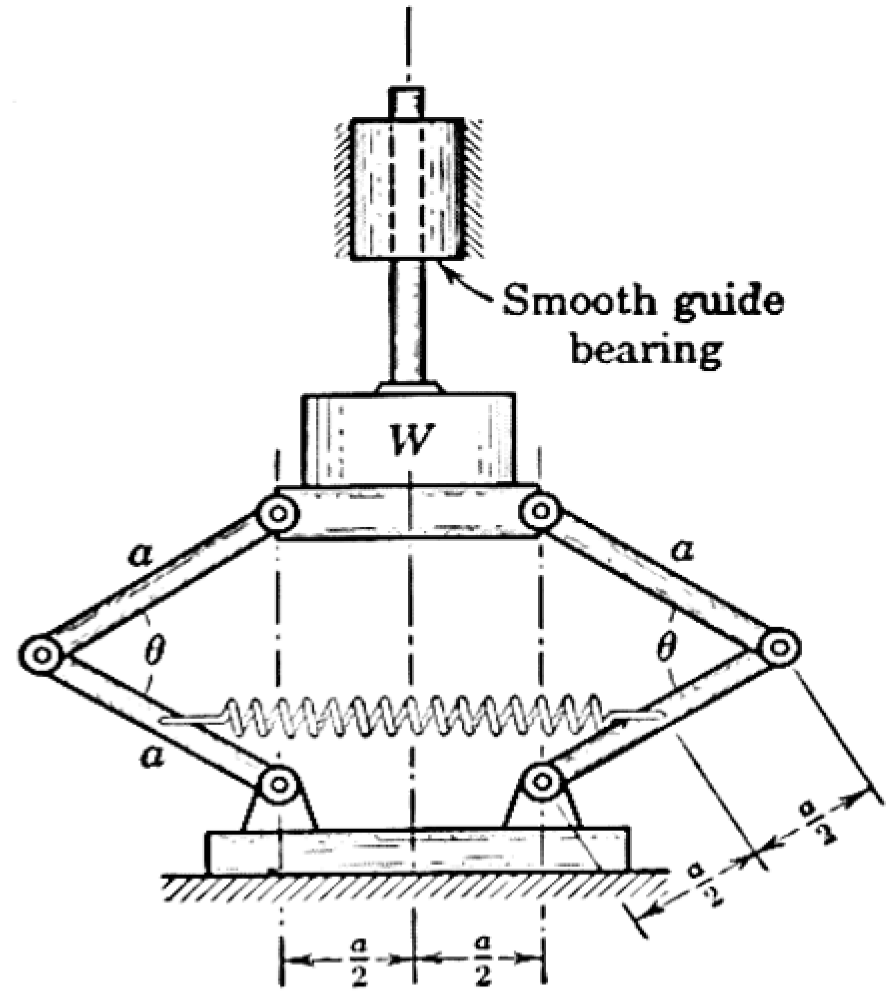


PROB. 594

595. When $\theta = 180$ deg. the spring, whose constant is k, is unextended. Determine the angle θ for equilibrium under the action of the weight W. Neglect any friction in the guide.



PROB. 595

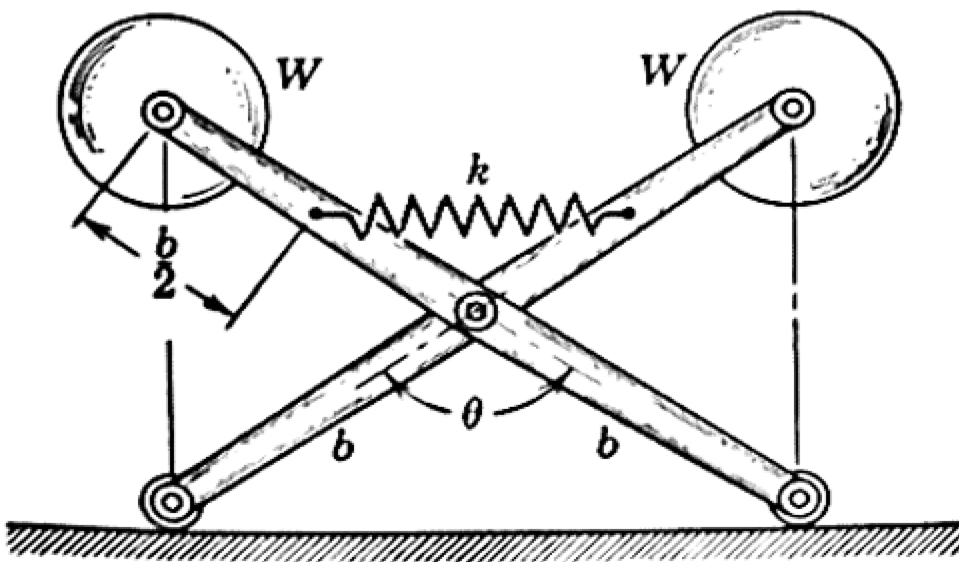


PROB. 596

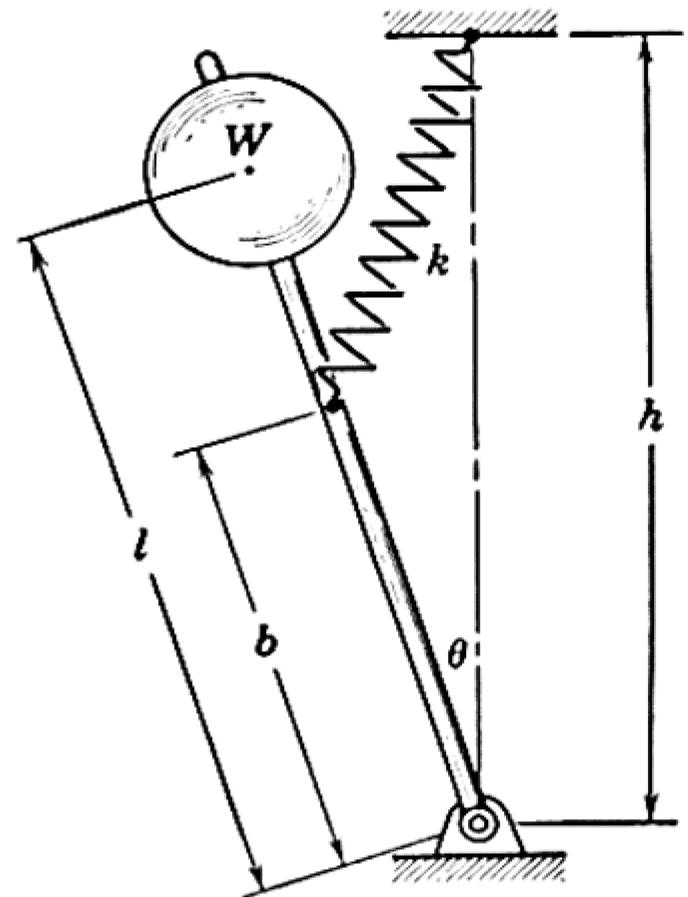
596. The mechanism of Prob. 323 is reproduced here. Determine the angle θ for the equilibrium positions. The force in the spring is zero when $\theta = 180$ deg., and the weights of the links are negligible.

597. The mechanism shown is constructed so that when $\theta = 0$ there is no mechanical interference and the spring force is zero. If the spring constant is k, determine θ for the position of equilibrium other than at $\theta = 0$. Neglect the weights of the arms.

Ans. $\theta = 2\cos^{-1}\dfrac{4W}{kb}$



PROB. 597



PROB. 598

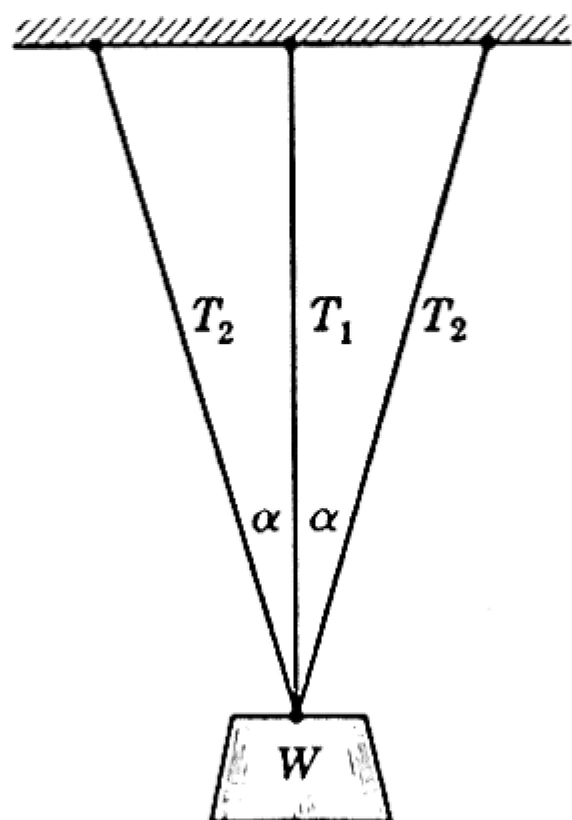
• **598.** Find the length c of the spring for the natural position of equilibrium of the mechanism shown. The unstretched length of the spring is $h - b$ and its stiffness is k.

$$Ans.\quad c = \frac{h - b}{1 - \dfrac{Wl}{kbh}}$$

* **599.** Replace the bar connecting the two wheels of Prob. 160 by a light spring of stiffness k and unstretched length l. Determine the value of x and the corresponding elongation Δ of the spring for equilibrium.

$$\textit{Ans.}\quad x = W_1\left(\frac{l}{\sqrt{W_1^2 + W_2^2}} + \frac{1}{k\sqrt{2}}\right),\ \Delta = \frac{1}{k}\sqrt{\frac{W_1^2 + W_2^2}{2}}$$

* **600.** A load W is suspended by the three elastic wires which differ only in length. When the load is removed, the wires are straight and meet at a point. It is known that the central wire alone requires a force K to stretch it a unit amount. Determine the vertical deflection x of W when suspended by the three wires and calculate the tensions in the wires. (*Hint:* Assume the deflection to be so small that the angle α remains essentially unchanged. Treat each wire as a stiff spring with the appropriate constant.)



PROB. 600

$$\textit{Ans.}\quad x = \frac{W}{K(1 + 2\cos^3\alpha)},$$

$$T_1 = \frac{W}{1 + 2\cos^3\alpha},$$

$$T_2 = \frac{W\cos^2\alpha}{1 + 2\cos^3\alpha}$$

64. Mechanical Efficiency. Because of energy loss by friction the ratio of useful work derived from a machine to the work done on the machine during the same interval is always less than unity. This ratio is a measure of the efficiency e of a machine. Thus

$$e = \frac{\text{useful work done by machine}}{\text{work done on machine}}.$$

The mechanical efficiency of simple machines which have one degree of freedom and which operate in a uniform manner may be determined by the principle of virtual work by evaluating the numerator and denominator of the expression for e during a virtual displacement. If a machine operates in a uniform manner with time, any elastic changes due to elongations or compressions which occurred during the initial application of the loads will remain constant during the continuous operation and will not be involved in the expressions.

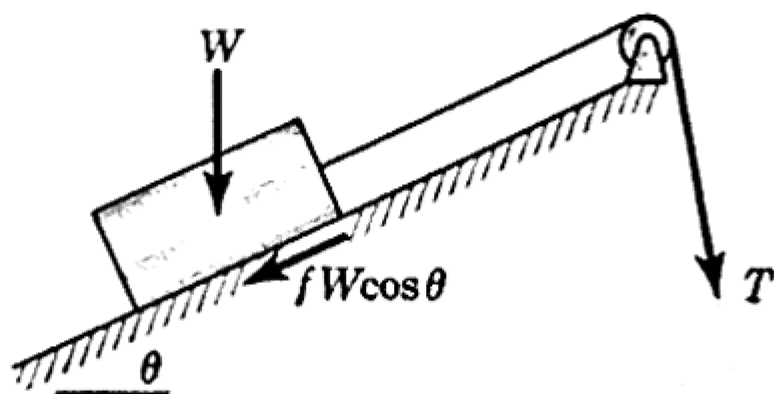


FIG. 81

As an example consider the work done by the tension T in the cable, Fig. 81, in causing the body to slide up the inclined plane at constant speed. The active-force diagram discloses only three forces which do work on the system composed of

the body and cable. During a virtual movement δs up the plane the work done on the body is

$$T\,\delta s = (W \sin\theta + fW \cos\theta)\,\delta s.$$

The useful work done by the machine consists of the work in raising the weight W the vertical distance $\delta s \sin\theta$ and is

$$W\,\delta s \sin\theta.$$

The efficiency of the inclined plane is then

$$e = \frac{W\,\delta s \sin\theta}{(W \sin\theta + fW \cos\theta)\,\delta s} = \frac{1}{1 + f/\tan\theta}.$$

As a second example of mechanical efficiency consider the screw jack described in Art. 53 and shown in Fig. 70. Equation (38) gives the force P applied at a radius a necessary to raise the weight W, where the screw has a mean radius r and a helix angle α, and where the friction angle is ϕ. This relation is

$$Pa = Wr \tan(\alpha + \phi).$$

During a small rotation $\delta\theta$ of the screw in a direction to raise the load the moment Pa does work in the amount of

$$Pa\,\delta\theta = Wr\,\delta\theta \tan(\alpha + \theta).$$

The work done by the machine is the work done in raising the load through the vertical distance $(\delta\theta/2\pi)L = r\,\delta\theta \tan\alpha$, where L is the lead or advancement per revolution of the screw. Thus the efficiency of the screw is

$$e = \frac{Wr\,\delta\theta \tan\alpha}{Wr\,\delta\theta \tan(\alpha + \phi)} = \frac{\tan\alpha}{\tan(\alpha + \phi)}.$$

As the friction in the threads is decreased, the friction angle ϕ becomes smaller, and the efficiency approaches unity.

65. Potential Energy. Energy represents the capacity to do work. A body or system of bodies is said to have *potential energy* when, by virtue of its elastic state or position, it is capable of doing work on some other body. There are these two types of mechanical potential energy. Potential energy V_e associated with the elastic state has been discussed in Art. 63 and is $V_e = \frac{1}{2}kx^2$, where k is the constant ratio of applied force to resulting deformation x. This expression represents recoverable energy provided the limit of the elasticity of the spring material has not been exceeded. The second type of potential energy V_g is that associated with the position of a body in a field of force. More exactly it is the work

done on a body in changing its position in the force field to which it is subjected. The most common force field is, of course, the earth's gravitational field, and for experiments near the surface of the earth the intensity may be assumed to be constant. Thus for the body of weight W, shown in Fig. 82, the work done by a force (not shown) in raising the body a height h above some arbitrary datum plane is Wh, which is its potential energy of position V_g relative to this datum plane. When the body is a distance h below the plane, its potential energy is $-Wh$. Inasmuch as it is always the change in potential energy which is involved, the reference plane may be chosen at any convenient altitude. This potential energy of position for a given body is determined only by the resulting change in height h and not by the path followed in arriving at the altitude h. The energy of position is potentially available since, in allowing the body to be lowered the height h, it will do Wh units of work on the body which lowers it.

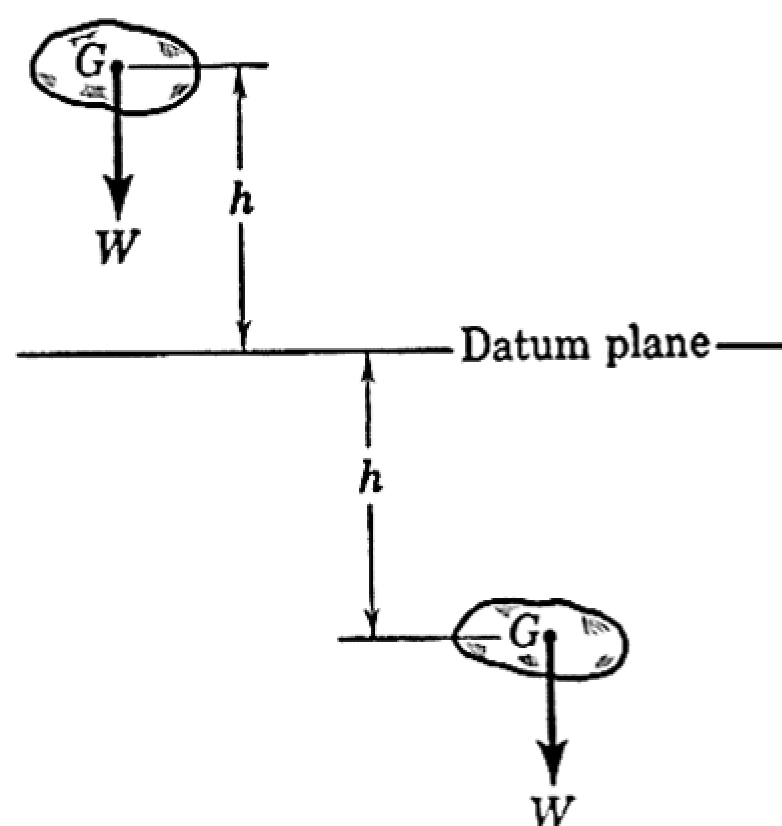


Fig. 82

The total mechanical potential energy V of a body or system of bodies is the sum of the recoverable elastic energy and the energy of position and is

$$V = V_e + V_g.$$

66. Energy Criterion for Equilibrium. A well-known principle of mechanics states that the natural position of equilibrium for any system of bodies without friction is that position which makes the total potential energy a minimum. Water seeks its lowest position; a marble in a bowl will come to rest when its potential energy is a minimum; a hanging coil spring will come to rest in the position where the sum of its potential energies of position and extension is the lowest possible value. The validity of this principle will now be shown and demonstrated by example.

As shown in Art. 65, the work done by gravity forces may be accounted for by the term V_g representing potential energy of position. The work done by externally applied active forces may also be accounted for in the potential energy term V_g. In Fig. 83 the force F may be replaced by the equivalent system composed of the weight F which applies the same force through the tension in the cord. During a virtual movement δs of A in the direction of F, the force F does positive work by the amount $F\,\delta s$, whereas in the equivalent system the potential energy V_g decreases by

the amount $F\ \delta s$. Hence the work done by an external active force may be accounted for in the potential energy term if the algebraic sign is reversed.

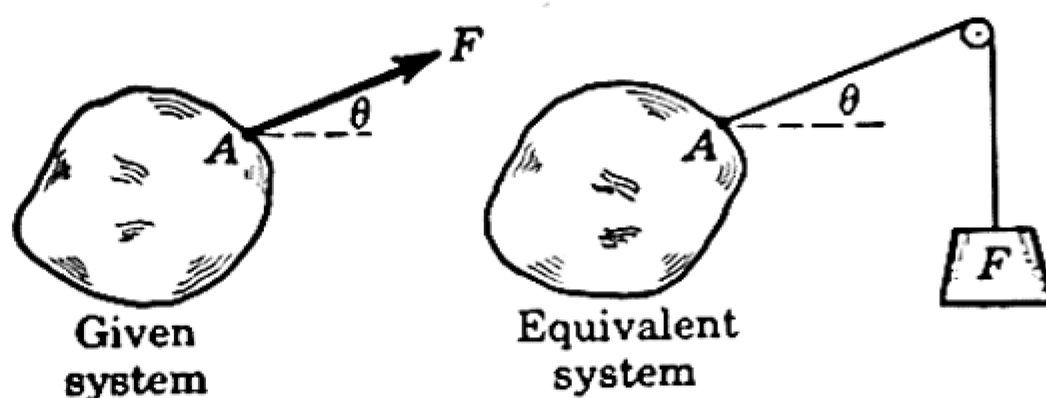


FIG. 83

For a real system without friction where the effect of all external active forces is expressed by the potential energy term, the equilibrium requirement, Eq. (46), may now be written:

$$\delta V = 0. \tag{47}$$

Thus the variation of the total potential energy V evaluated for a virtual displacement must be zero for equilibrium. Equation (47) is another way of expressing the equilibrium requirement for frictionless systems.

As an example of the foregoing concept consider the simple system in equilibrium shown in Fig. 84, consisting of the spring which is deflected

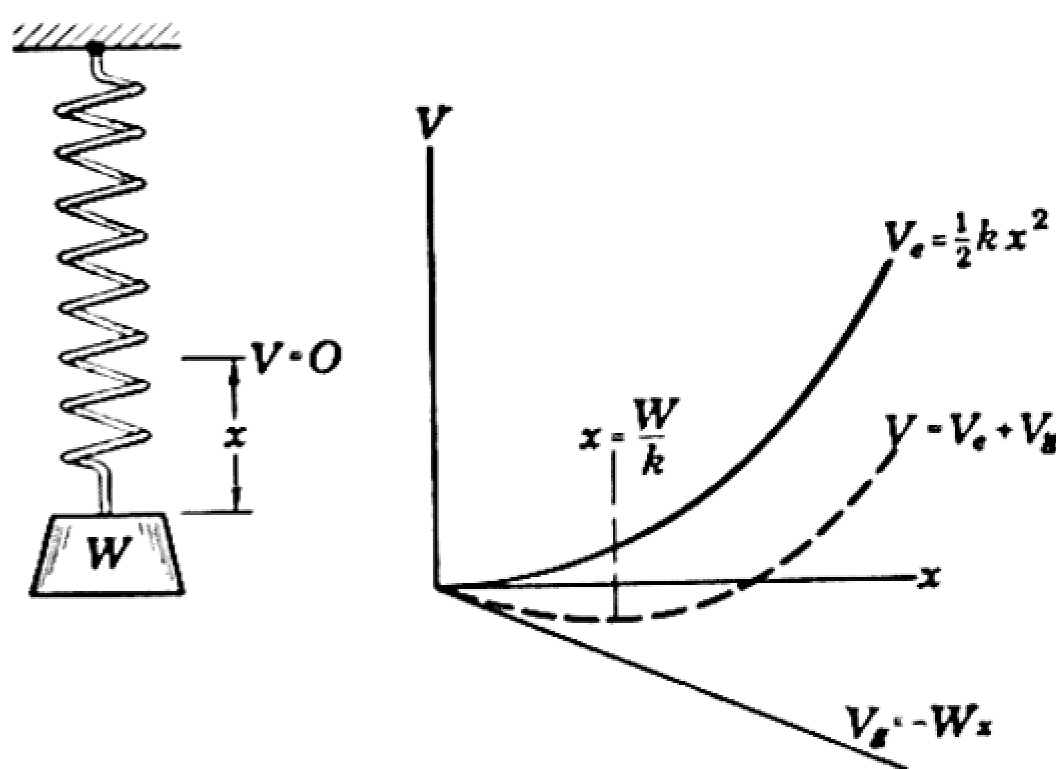


FIG. 84

an amount x due to the attached weight W. By taking $x = 0$ arbitrarily as the position for which $V = 0$, the potential energy for the displacement x is

$$V = V_e + V_g = \tfrac{1}{2}kx^2 - Wx.$$

The condition for equilibrium given by Eq. (47) requires

$$\delta V = kx\ \delta x - W\ \delta x = 0, \qquad x = \frac{W}{k}.$$

This value agrees with the result which can be obtained by inspection. The contribution of V_e and V_g to the total potential energy V for various possible values of x is also shown in Fig. 84, and the value $x = W/k$ is seen to be a minimum.

For systems with n degrees of freedom the variation of V with respect to each of the n coordinates separately, the remaining ones held constant, must be evaluated and set equal to zero. This procedure will give n equations in the n unknown coordinate values. If x_n represents one of the n coordinates, the partial derivative

$$\frac{\partial V}{\partial x_n} = 0$$

must be evaluated for each of the n coordinates.

The principle of zero variation in potential energy for equilibrium may be applied to *conservative systems* only, i.e., systems free from kinetic friction forces which do negative work. A system with kinetic friction will approach the position of minimum potential energy but will not reach it.

67. Stability of Equilibrium. The requirement of zero variation of the potential energy for equilibrium, as stated by Eq. (47), is equivalent to the requirement

$$\frac{dV}{dx} = 0 \tag{47a}$$

for a system with a single degree of freedom, where x is the coordinate defining the position of the system. Mathematically this equation defines the condition for a stationary value of V, a maximum, a minimum, or a point of inflection in the curve of V versus x. Although the natural equilibrium configuration usually represents a minimum value of V, there are the other two possibilities for stationary values which also represent equilibrium positions. Figure 85 shows an example of each

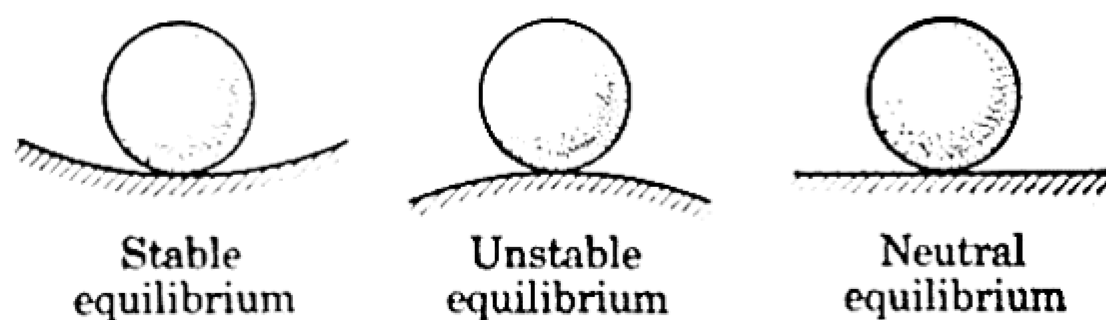


Fig. 85

type of equilibrium for the simple case of a sphere supported by surfaces of three different curvatures. The equilibrium is said to be *stable, unstable*, or *neutral* according to whether the potential energy is a *minimum* or a *maximum* or exhibits a *point of inflection*. Equilibrium is stable if a

slight displacement from the equilibrium position results in an *increase* of potential energy and a corresponding tendency to reduce this energy by returning to the former position. Equilibrium is unstable if a slight displacement results in a *decrease* of potential energy and a corresponding tendency to reduce this energy still further by additional movement away from the equilibrium position. In the case of neutral equilibrium a slight displacement causes neither a tendency to return to nor a tendency to move away from the equilibrium position.

The potential energy criteria for the three types of equilibrium are represented in Fig. 86 for the case of a single degree of freedom where the

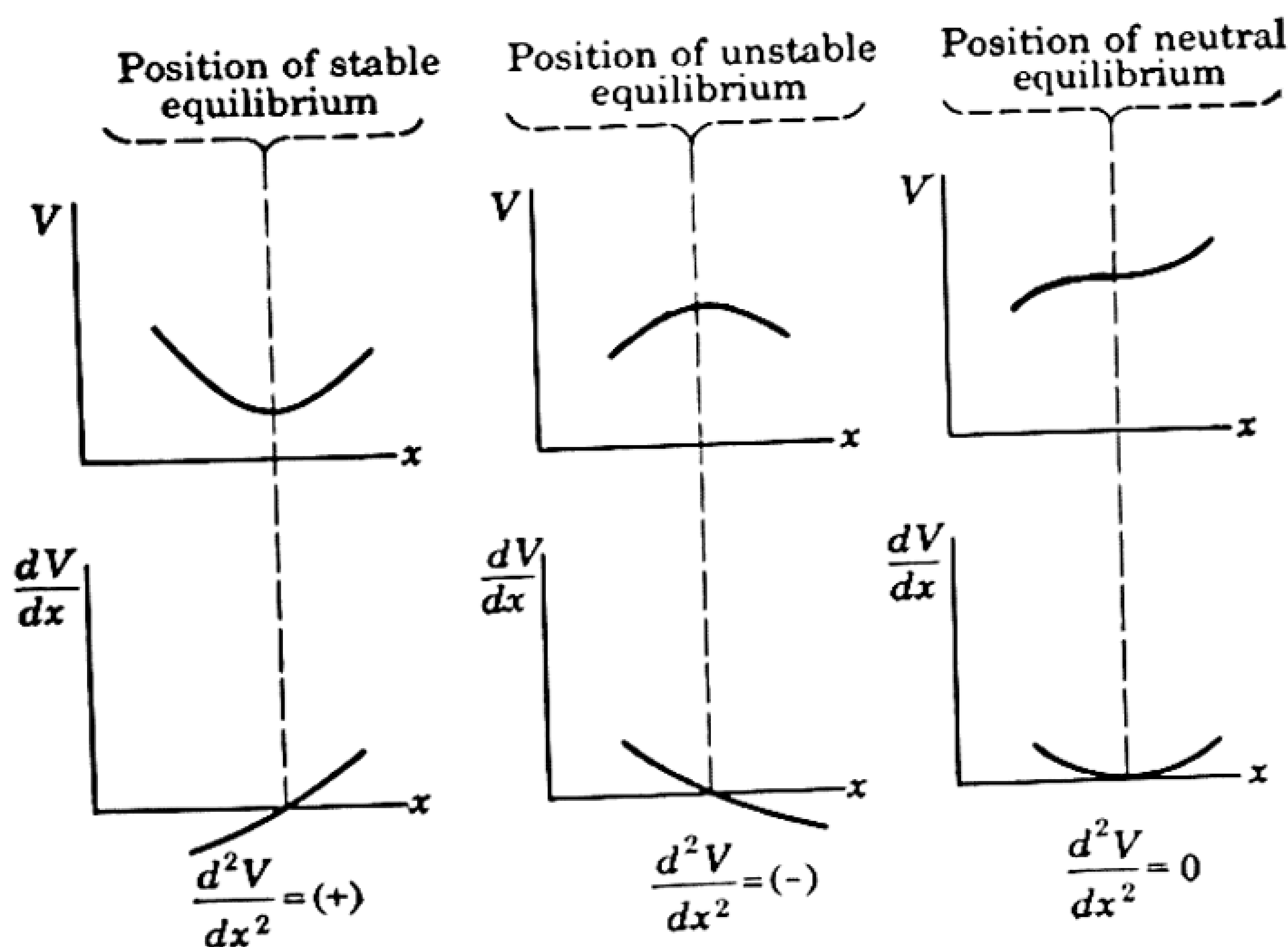


FIG. 86

coordinate defining the configuration is x. Once the total potential energy V is expressed as a function of the defining coordinate, the equilibrium position is determined by Eq. (47a). The type of equilibrium is determined by the second derivative of V and is stable, unstable, or neutral according to whether d^2V/dx^2 is positive, negative, or zero at the equilibrium position.

SAMPLE PROBLEMS

601. Solve Prob. 589 by considering the total potential energy of the system.

Solution: An arbitrary datum plane for zero potential energy of position V_g must first be selected. Let this plane pass through A so that the potential energy of the two weights is

$$V_g = -2Wa\cos\theta.$$

The elastic potential energy due to the extension of the spring is

$$V_e = \tfrac{1}{2}kx^2 = \tfrac{1}{2}k\left(2a\cos\theta - \frac{2a}{\sqrt{2}}\right)^2.$$

The total potential energy of the system is then

$$V = V_g + V_e = -2Wa\cos\theta + \tfrac{1}{2}k\left(2a\cos\theta - \frac{2a}{\sqrt{2}}\right)^2.$$

The equilibrium state is defined by

$$[\delta V = 0] \qquad -2Wa(-\sin\theta\,\delta\theta) + k\left(2a\cos\theta - \frac{2a}{\sqrt{2}}\right)(-2a\sin\theta\,\delta\theta) = 0,$$

$$\left[W - 2ka\left(\cos\theta - \frac{1}{\sqrt{2}}\right)\right]\sin\theta = 0.$$

There are two solutions to this equation,

$$\sin\theta = 0 \qquad \text{and} \qquad \cos\theta - \frac{1}{\sqrt{2}} = \frac{W}{2ka}.$$

The first solution gives $\theta = 0$ and defines a perfectly valid equilibrium position, assuming the device was constructed in such a way that $\theta = 0$ could be realized without mechanical interference. Clearly, though, this position is not the natural equilibrium position as any small movement would cause the device to move away from this position. The second solution gives the result

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}} + \frac{W}{2ka}\right)$$

and represents the natural equilibrium position. For this position the total energy V is a minimum.

There is very little difference between the method of solution used here and that employed in Prob. 589, and the equivalence of the two should be studied.

602. The total potential energy V of a conservative mechanical system with a single degree of freedom in the coordinate x is given by

$$V = 4x^3 - 6x^2 + 3.$$

Determine the equilibrium positions and indicate the stability.

Solution: Equilibrium requires

$$\left[\delta V = 0 \quad \text{or} \quad \frac{dV}{dx} = 0\right] \qquad \frac{dV}{dx} = 12x^2 - 12x = 0,$$

$$x = 0, \qquad x = 1.$$

The stability is found from the second derivative. Thus,

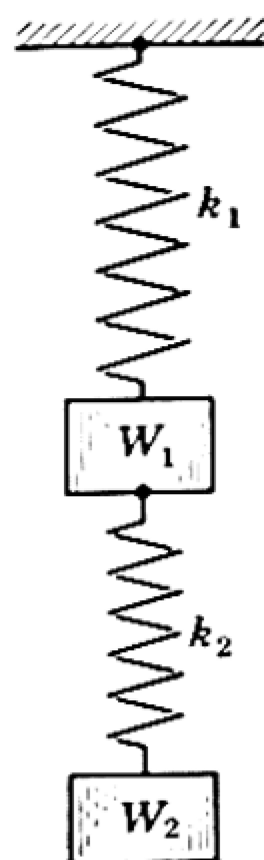
$$\frac{d^2V}{dx^2} = 24x - 12.$$

For $x = 0$, $d^2V/dx^2 = -12$, and, since the sign is negative, this position is unstable. For $x = 1$, $d^2V/dx^2 = +12$, and this position is stable since the second derivative is positive.

PROBLEMS

Solve the following problems by considering the total potential energy of each system.

603. The two weights W_1 and W_2 are suspended by two springs of constants k_1 and k_2 as shown. Determine from energy considerations the total deflection x of W_2 from the position of zero force in both springs to the equilibrium position.



PROB. 603

604. The potential energy V of a conservative system is given by $V = ax^2/2 + bx^3/3$, where x is the coordinate specifying the position of the system. Determine the equilibrium positions and the stability if a and b are positive.

Ans. $x = -a/b$ unstable, $x = 0$ stable

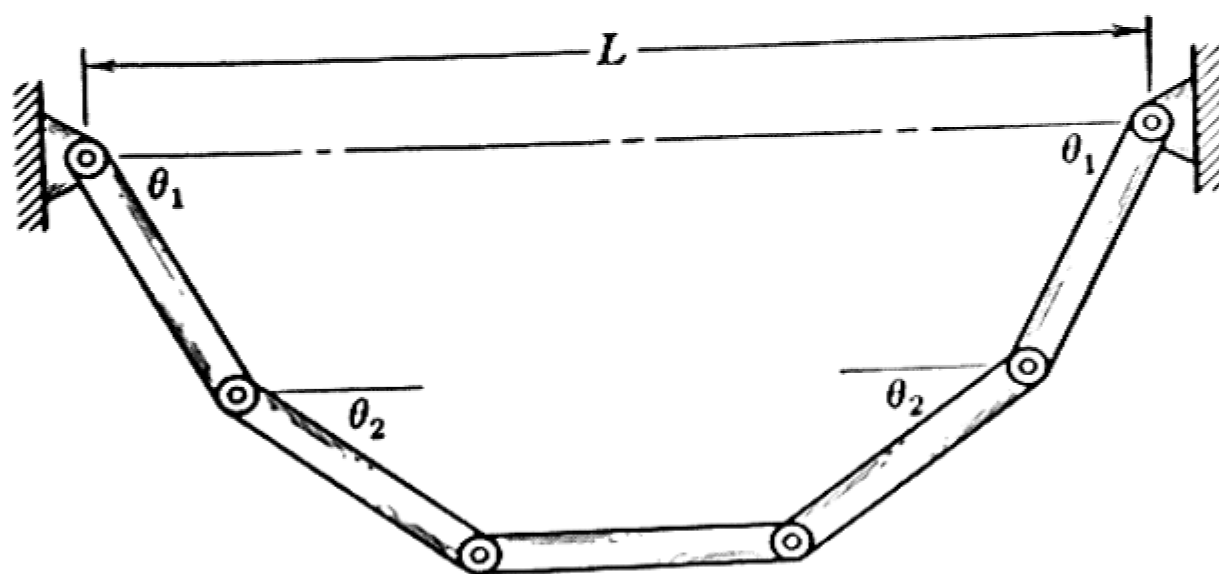
605. A conservative mechanical system has a total potential energy of $V = 15x^2 - 5x^3 + 30$, where x is the coordinate which specifies the position of the system. Find the equilibrium positions and the type of stability.

606. Solve Prob. 595 by considering the total potential energy of the system.

* **607.** Solve Prob. 597 by considering the total energy of the system and check the stability criteria for both equilibrium positions.

* **608.** Five uniform links each of weight W and length l are suspended as shown. Specify the relation between θ_1 and θ_2 which must hold for equilibrium. (*Hint:* By symmetry it may be seen that there is only one degree of freedom.)

Ans. $\tan \theta_1 = 2 \tan \theta_2$

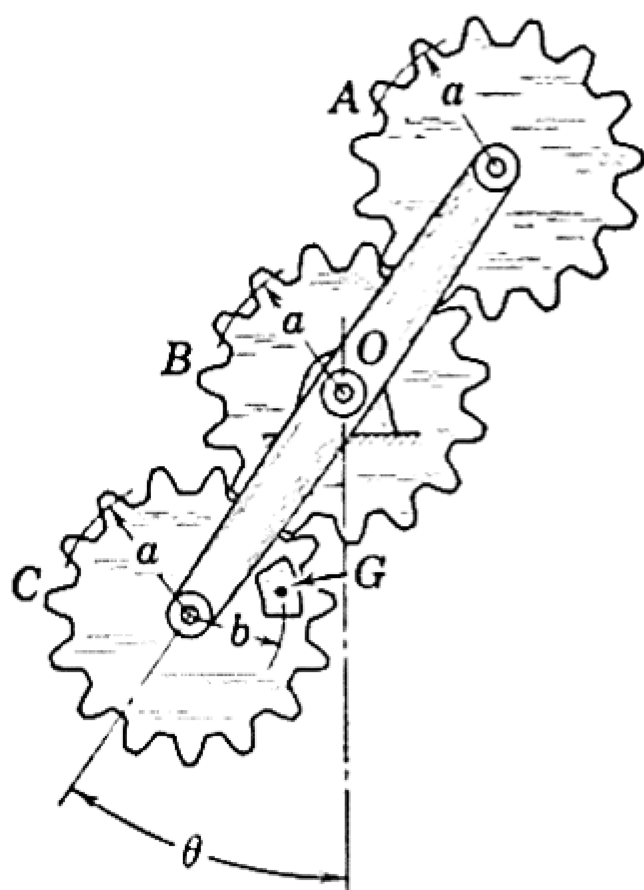


PROB. 608

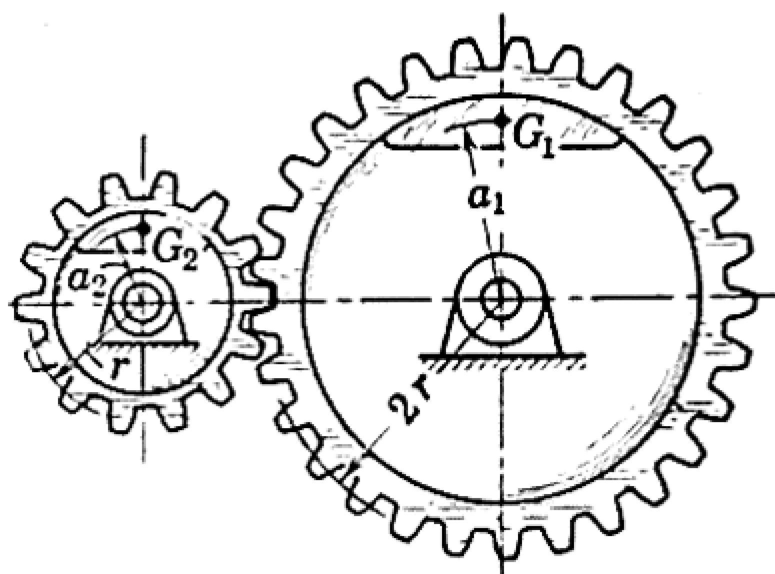
* **609.** In the mechanism shown gear B is fixed and cannot rotate. Gears A and C, together with the attached arm, are free to rotate about B. Gear C carries an eccentric weight whose center of gravity is at G. The center of gravity of the remainder of the system is at O. When $\theta = 0$, the point G is on

the vertical center line through O. Determine the angle θ for equilibrium under the action of the unbalanced weight for $b > a/2$. What will happen if $b < a/2$?

Ans. $\theta = \cos^{-1} \frac{a}{2b}$



PROB. 609



PROB. 610

* **610.** The two gears carry equal eccentric weights W with centers of gravity at G_1 and G_2 and turn freely on their bearings until they find their stable equilibrium position. Find the angle θ measured from the vertical through which the larger gear rotates from the position shown to the equilibrium position. Discuss the stability.

Ans. $\theta = \cos^{-1}\left(\frac{-a_1}{4a_2}\right)$ for $a_1 < 4a_2$, $\theta = \pi$ for $a_1 > 4a_2$

Appendix A

Moments of Inertia

I. MOMENTS OF INERTIA OF AREAS

A1. Definitions. In the analysis of the distribution of stress over the cross-sectional areas of structural and machine members an expression of the form $\int y^2\,dA$ is encountered, where y is the distance from an element dA of the area to an axis which is either in or normal to the plane of the area. Expressions of this type also appear in other engineering problems. By reason of the frequent occurrence of these integrals it is convenient to develop them for some of the more common areas and to tabulate the results.

The integral to which reference is made is generally called the *moment of inertia* of the area about the axis in question. A more fitting term is the *second moment of area* since the first moment $y\,dA$ is multiplied again by the moment arm y to obtain the result for the element dA. The word *inertia* appears in the terminology by reason of the similarity between the mathematical form of the integrals for second moments of areas and those for the resultant moments of the inertia forces in the case of rotating bodies. The moment of inertia of an area is a purely mathematical property of the area and in itself has no physical significance.

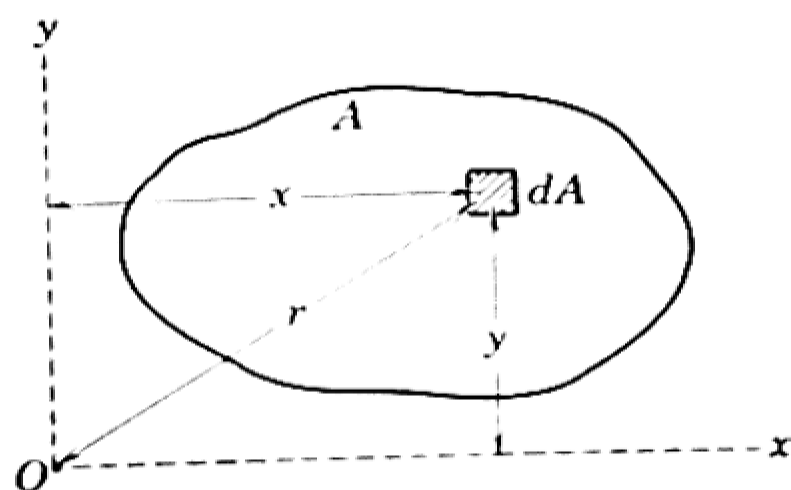


FIG. A1

Consider the area A in the x–y plane, Fig. A1. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively. Therefore the moments of inertia of A about the same axes are

$$I_x = \int y^2\,dA, \qquad \text{(A1)}$$

$$I_y = \int x^2\,dA,$$

where the integration covers the entire area. The moment of inertia of

dA about the pole O (z-axis) is, by similar definition, $dJ_z = r^2\,dA$, and the moment of inertia of the entire area about O is

$$J_z = \int r^2\,dA. \tag{A2}$$

The expressions defined by Eqs. (A1) are known as *rectangular* moments of inertia, whereas the expression of Eq. (A2) is known as the *polar* moment of inertia. Since $x^2 + y^2 = r^2$, it is clear that

$$J_z = I_x + I_y. \tag{A3}$$

A polar moment of inertia for an area whose boundaries are more simply described in rectangular coordinates than in polar coordinates is easily calculated with the aid of Eq. (A3).

It should be noted that the moment of inertia of an element involves the square of the distance from the inertia axis to the element. An element whose coordinate is negative contributes as much to the moment of inertia as does an element with a positive coordinate of the same magnitude. Consequently the moment of inertia of an area about any axis is always a positive quantity. In contrast, the first moment of the area, which was involved in the computations of centroids, could be either positive or negative.

The dimensions of moments of inertia of areas are clearly L^4, where L stands for the dimension of length. Thus the units for area moments of inertia are expressed as quartic inches (in.^{4}) or quartic feet (ft.^{4}).

The choice of the coordinates to use for the calculation of moments of inertia is important. Rectangular coordinates should be used for shapes whose boundaries are most easily expressed in these coordinates. Polar coordinates will usually simplify problems involving boundaries which are easily described in r and θ. The choice of an element of area which simplifies the integration as much as possible is also important. These considerations are quite analogous to those discussed and illustrated in Chapter V in the calculation of centroids.

A2. Radius of Gyration. The moment of inertia of an area is a measure of the distribution of the area from the axis in question. Assume all the area A, Fig. A2, to be concentrated into a strip of negligible thickness at a distance k_x from the x-axis such that the product k_x^2A equals the moment of inertia about the axis. The distance k_x, called the *radius of gyration*, is then a measure of the distribution of area from the inertia axis. By definition, then, for any axis

$$I = k^2A \qquad \text{or} \qquad k = \sqrt{\frac{I}{A}}. \tag{A4}$$

When this definition is substituted in each of the three terms in Eq. (A3), there results

$$k_z^2 = k_x^2 + k_y^2. \tag{A5}$$

Thus the square of the radius of gyration about a polar axis equals the sum of the squares of the radii of gyration about the two corresponding rectangular axes.

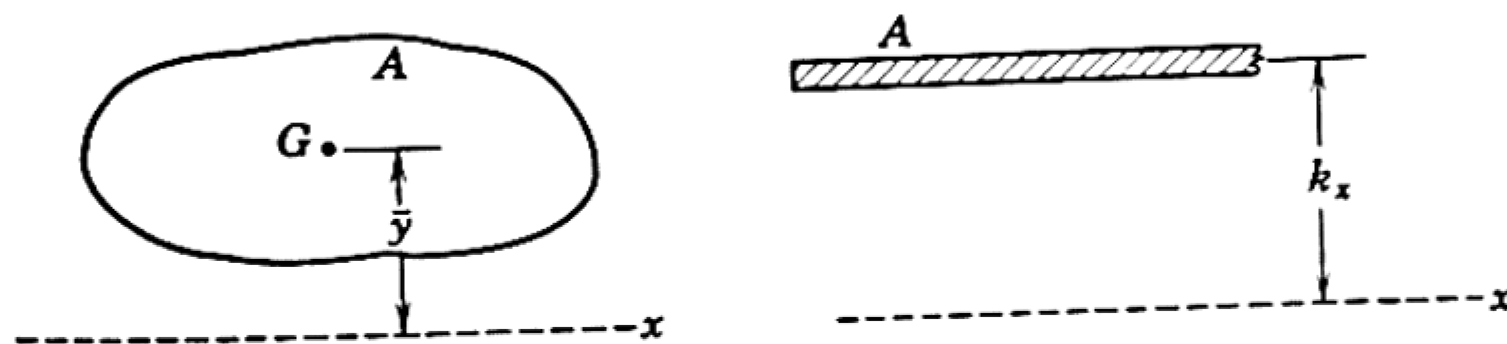


Fig. A2

It is imperative that there be no confusion between the coordinate to the centroid of the area and the radius of gyration k. The square of the centroidal distance, Fig. A2, is $\bar{y}^2$ and is the square of the mean value of the distances y from the elements dA to the axis. The quantity k_x^2, on the other hand, is the mean of the squares of these distances. The moment of inertia is *not* equal to $A\bar{y}^2$ since the square of the mean is not equal to the mean of the squares.

A3. Transfer of Axes. The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig. A3 the x_o–y_o axes pass through the centroid G of the area. Let it be desired to determine the moments of inertia of the area about the parallel x–y axes. By definition the moment of inertia of the element dA about the x-axis is

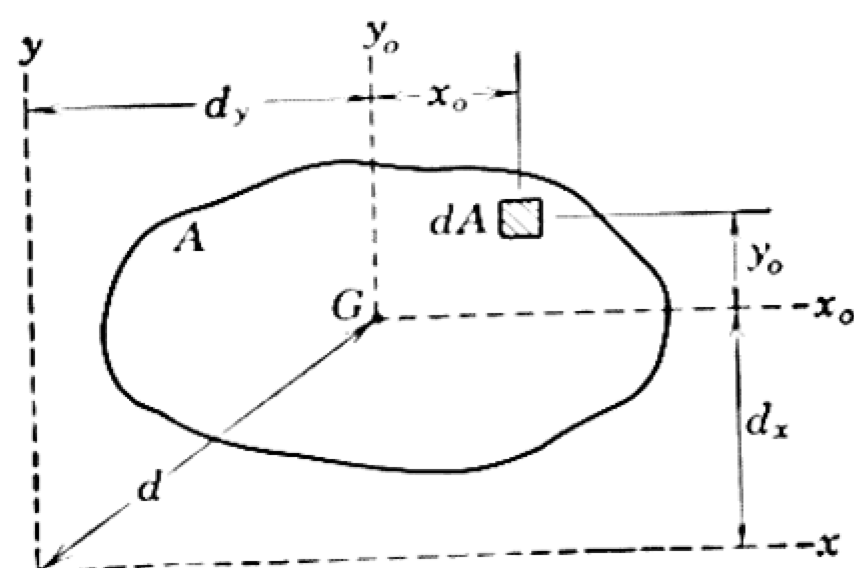


Fig. A3

$$dI_x = (y_o + d_x)^2\,dA.$$

Expanding and integrating give

$$I_x = \int y_o^2\,dA + 2d_x\int y_o\,dA + d_x^2\int dA.$$

The first integral is the moment of inertia $\bar{I}_x$ about the centroidal x_o-axis. The second integral is zero since $A\bar{y}_o = \int y_o\,dA$ and $\bar{y}_o$ is auto-

matically zero. The third integral is simply Ad_x^2. Thus the expression for I_x and the similar expression for I_y become

$$I_x = \bar{I}_x + Ad_x^2,$$
$$I_y = \bar{I}_y + Ad_y^2. \qquad \text{(A6)}$$

By Eq. (A3) the sum of these two equations gives

$$J_z = \bar{J}_z + Ad^2. \qquad \text{(A6a)}$$

Equations (A6) and (A6*a*) are the so-called *parallel-axis theorems.* Two points in particular should be noted. First, the axes between which the transfer is made must be parallel, and, second, one of the axes must pass through the centroid of the area.

If a transfer between two parallel axes neither one of which passes through the centroid is desired, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis.

The parallel-axis theorems also hold for radii of gyration. With substitution of the definition of k into Eqs. (A6), the transfer relation becomes

$$k^2 = \bar{k}^2 + d^2, \qquad \text{(A6b)}$$

where $\bar{k}$ is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes. The axes may be either in the plane or normal to the plane of the area.

A summary of the moment of inertia relations for some of the common plane figures is given in Table B5, Appendix B.

SAMPLE PROBLEMS

A1. Determine the moments of inertia of the rectangular area about the centroidal x_o–y_o axes, the centroidal polar axis G, the x-axis, and the polar axis O.

Solution: For the calculation of the moment of inertia $\bar{I}_x$ about the x_o-axis a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus

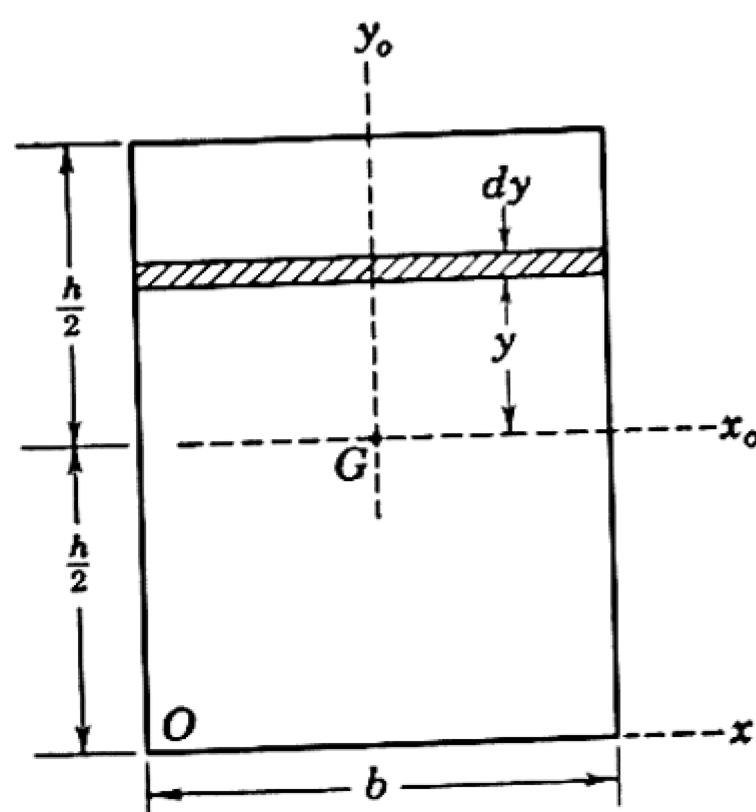
$$\left[I_x = \int y^2\,dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}\,bh^3. \qquad \textit{Ans.}$$

By interchanging symbols the moment of inertia about the centroidal y_o-axis is

$$\bar{I}_y = \tfrac{1}{12}\,hb^3. \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[J_z = I_x + I_y] \qquad \bar{J}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2). \qquad \textit{Ans.}$$



PROB. A1

By the parallel-axis theorem the moment of inertia about the x-axis is

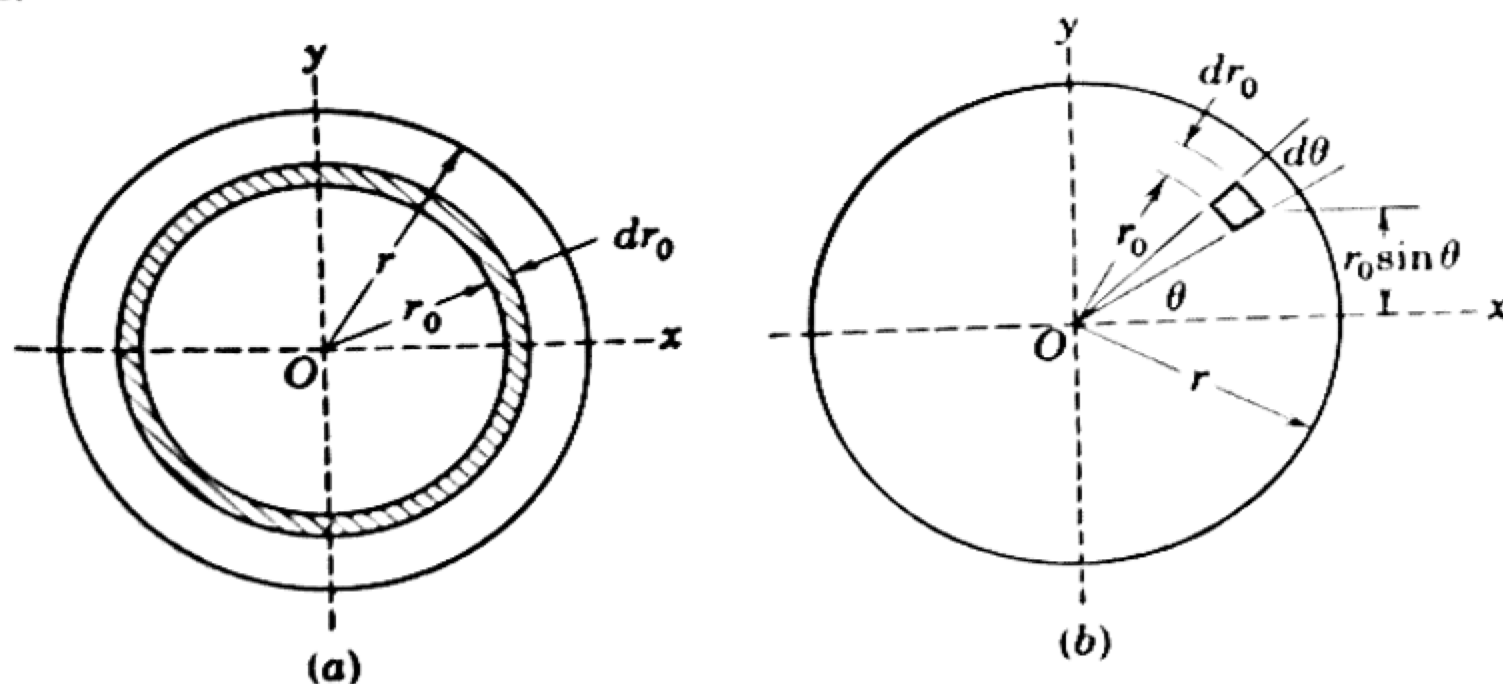
$[I_x = \bar{I}_x + Ad_x^2]$ $$I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2. \qquad \textit{Ans.}$$

The polar moment of inertia about O may also be obtained by the parallel-axis theorem. Thus

$[J_z = \bar{J}_z + Ad^2]$ $$J_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right],$$

$$J_z = \tfrac{1}{3}A(b^2 + h^2). \qquad \textit{Ans.}$$

A2. Calculate the moments of inertia of the area of the circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



PROB. A2

Solution: An element of area in the form of a circular ring, shown in the a-part of the figure, may be used for the calculation of the moment of inertia about the

polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_o\, dr_o$, and thus

$$\left[J_z = \int r^2\, dA\right] \qquad J_z = \int_0^r r_o^2(2\pi r_o\, dr_o) = \frac{\pi r^4}{2} = \tfrac{1}{2}Ar^2. \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{J}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}}. \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. (A3)

$$[J_z = I_x + I_y] \qquad I_x = \tfrac{1}{2}J_z = \frac{\pi r^4}{4} = \tfrac{1}{4}Ar^2. \qquad \textit{Ans.}$$

The radius of gyration about the diametral axis is

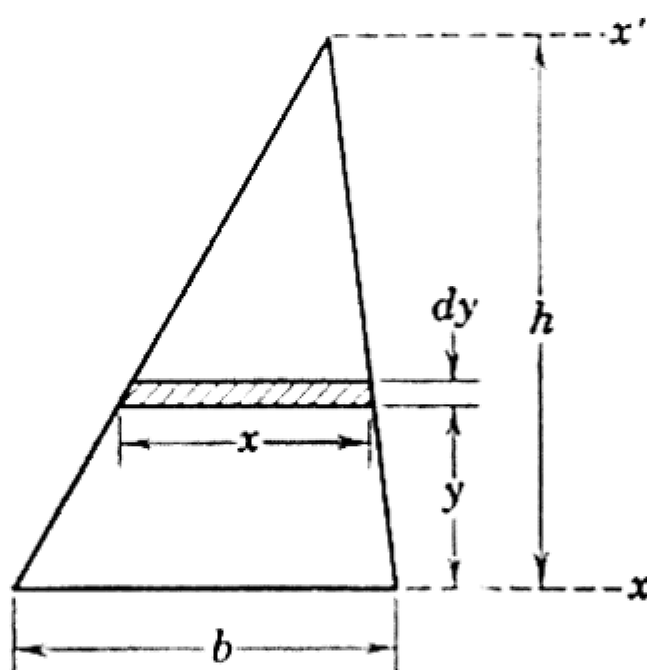
$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2}. \qquad \textit{Ans.}$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_o\, dr_o\, d\theta$ shown in the b-part of the figure. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^r (r_o \sin\theta)^2 r_o\, dr_o\, d\theta,$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta = \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4}. \qquad \textit{Ans.}$$

A3. Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



PROB. A3

Solution: A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\, dy = [(h - y)b/h]\, dy$. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^h y^2 \frac{h - y}{h} b\, dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12}. \qquad \textit{Ans.}$$

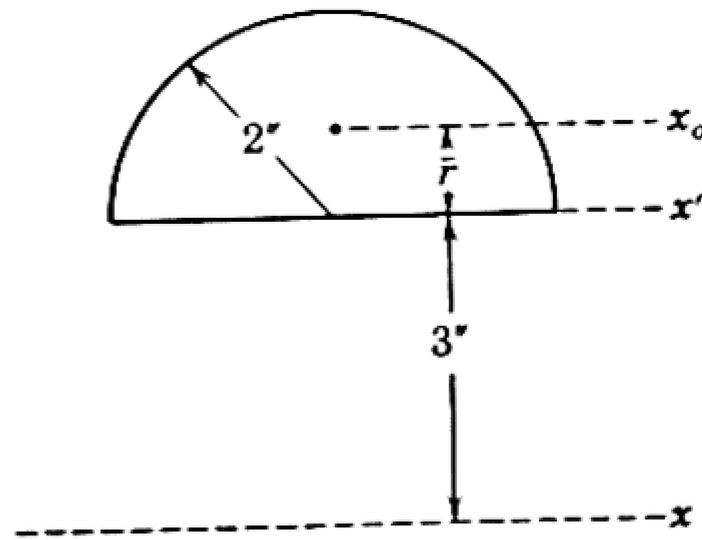
By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}. \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4}. \qquad \textit{Ans.}$$

A4. Determine the moment of inertia about the x-axis of the semicircular area shown.



PROB. A4

Solution: The moment of inertia of the semicircular area about the x'-axis is one half of that for a complete circle about the same axis. Thus from the results of Prob. A2

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{2^4\pi}{8} = 2\pi \text{ in.}^4$$

The moment of inertia $\bar{I}$ about the parallel centroidal axis x_o is obtained next. Transfer is made through the distance $\bar{r} = 4r/3\pi = (4 \times 2)/3\pi = 8/(3\pi)$ in. by the parallel-axis theorem. Hence

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2\pi - \left(\frac{2^2\pi}{2}\right)\left(\frac{8}{3\pi}\right)^2 = 1.755 \text{ in.}^4$$

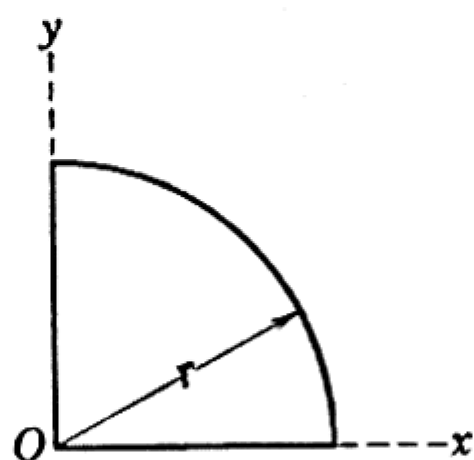
Finally, transfer is made from the centroidal x_o-axis to the x-axis, which gives

$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755 + \left(\frac{2^2\pi}{2}\right)\left(3 + \frac{8}{3\pi}\right)^2,$$

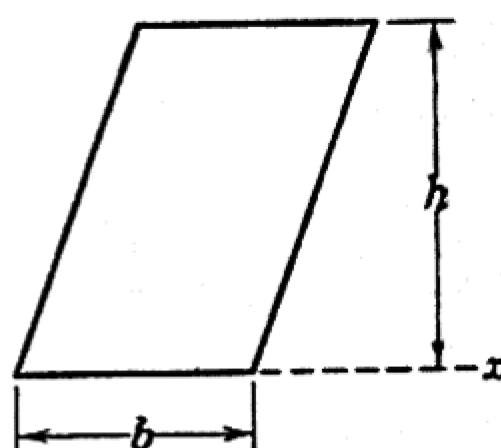
$$= 1.755 + 93.1 = 94.9 \text{ in.}^4 \qquad \textit{Ans.}$$

PROBLEMS

A5. Find the moments of inertia of the area of the quarter circle shown about the diametral x-axis and the polar axis through O. *Ans.* $I_x = \frac{\pi r^4}{16}, J_z = \frac{\pi r^4}{8}$



PROB. A5



PROB. A6

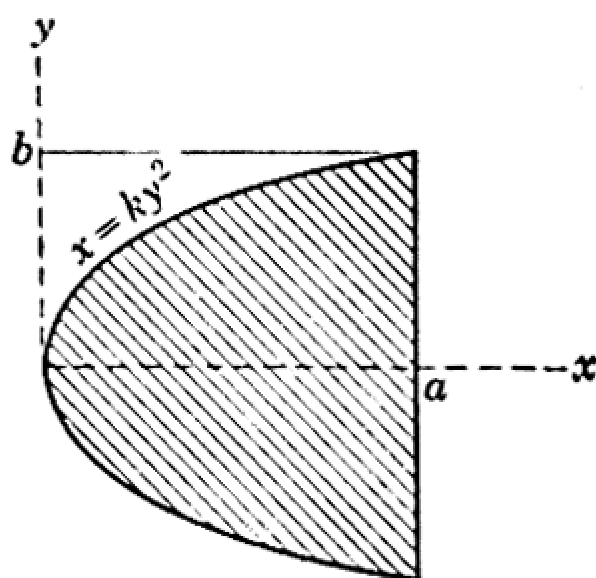
A6. Find the moments of inertia of the area of the parallelogram about the base (x-axis) and about a parallel centroidal axis. (Compare with Sample Prob. A1.)

A7. Approximate the result for the moment of inertia of a circular area about a diameter by dividing the circle into strips parallel to the inertia axis and of width $r/5$. Treat the moment of inertia of each strip as its area times the square of the distance from its center to the axis.

A8. Find the radius of gyration k of a square of side b about one diagonal.

Ans. $k = \frac{b}{2\sqrt{3}}$

A9. Find the moment of inertia of the shaded area about the x-axis by taking a horizontal strip of area dA. *Ans.* $I_x = \frac{4}{15}ab^3$



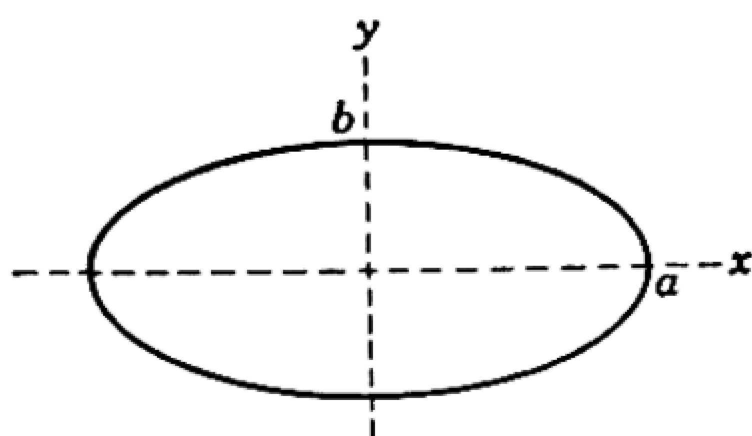
PROB. A9

A10. Solve Prob. A9 by choosing a vertical strip of area dA and applying the results of Sample Prob. A1 to this element.

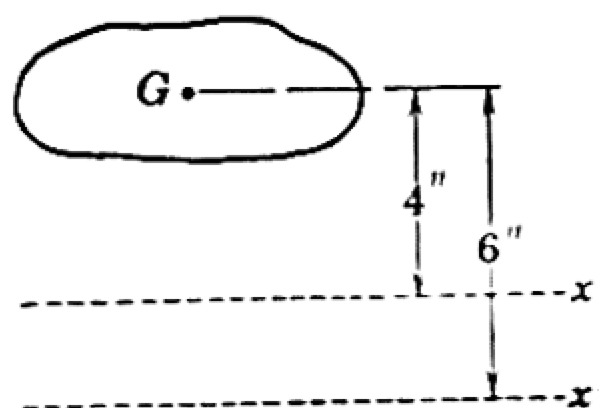
A11. Find the moment of inertia of the figure in Prob. A9 about the y-axis.

Ans. $I_y = \frac{4}{7}a^3b$

A12. Calculate the moments of inertia of the elliptical area about the major axes and the central polar axis.



PROB. A12

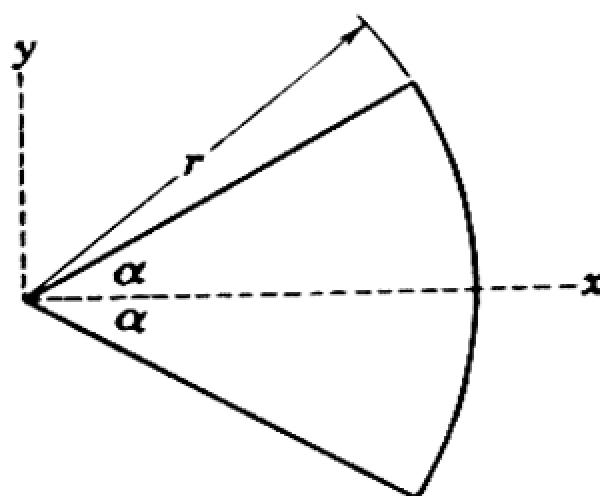


PROB. A13

A13. The x- and x'-axes are located from the centroid G of the irregular area as shown. If the moments of inertia about these axes are $I_x = 1800$ in.^{4} and $I_{x'} = 2200$ in.^{4}, determine the area A of the figure. *Ans.* $A = 20$ in.^{2}

A14. Use the results of Probs. A9 and A11 to find the polar moment of inertia J of the figure in Prob. A9 about the point $(a, 0)$. (The centroid is a distance $3a/5$ from the origin, and the area of the figure is $4ab/3$.)

A15. Determine the moments of inertia of the area of the circular sector about the x- and y-axes. *Ans.* $I_x = \frac{r^4}{4}\left(\alpha - \frac{\sin 2\alpha}{2}\right)$, $I_y = \frac{r^4}{4}\left(\alpha + \frac{\sin 2\alpha}{2}\right)$



PROB. A15

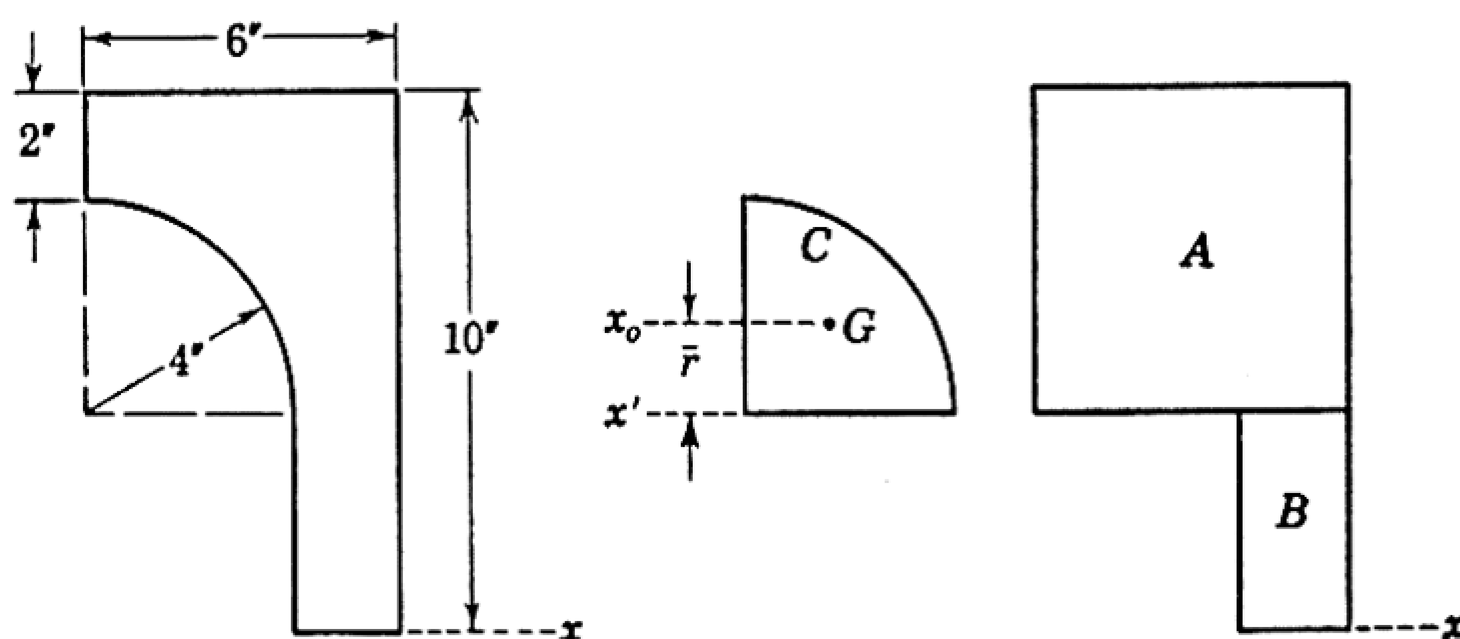
A16. The area of a circular ring of inside radius r and outside radius $r + \Delta r$ is approximately equal to the circumference at the mean radius times the thickness Δr. The polar moment of inertia of the ring may be approximated by multiplying this area by the square of the mean radius. What per cent error is involved if $\Delta r = r/10$? *Ans.* Error = 0.226%

A4. Composite Areas. The moment of inertia of a composite area about a particular axis is the algebraic sum of the moments of inertia of the various parts about the same axis. The results of the problems in Art. A3 and the tabulation of results in Table B5, Appendix B, may be used to determine the moments of inertia for component parts of the shapes given. It is often convenient to regard a composite area as composed of positive and negative parts. The moment of inertia of a negative area is a minus quantity.

When the section is composed of a large number of parts, it is convenient to tabulate the results for the parts in terms of the area A, centroidal moment of inertia $\bar{I}$, distance d from the centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2. For any one of the parts the desired moment of inertia is $\bar{I} + Ad^2$, and thus for the entire section the desired moment of inertia may be expressed as $I = \Sigma\bar{I} + \Sigma Ad^2$.

SAMPLE PROBLEM

A17. Compute the moment of inertia and radius of gyration about the x-axis for the cross section shown.



PROB. A17

Solution: The composite area may be considered as composed of the two rectangles A and B and the negative quarter circular area C. For the rectangle A the moment of inertia about the x-axis is

$[I = \bar{I} + Ad^2]$ $$I_x = \tfrac{1}{12} \times 6 \times 6^3 + 6^2 \times 7^2 = 1872 \text{ in.}^4$$

The moment of inertia of B about the x-axis is

$$I_x = \tfrac{1}{3} \times 2 \times 4^3 = 42.67 \text{ in.}^4$$

The moment of inertia of the negative quarter circle C about its horizontal diameter is

$$I_{x'} = -\tfrac{1}{4} \times \tfrac{1}{4}\pi \times 4^4 = -50.27 \text{ in.}^4$$

Transfer of this result through the distance $\bar{r} = 4r/3\pi = (4 \times 4)/3\pi = 1.697$ in. gives for the centroidal moment of inertia of C

$[\bar{I} = I - Ad^2]$ $$\bar{I} = -50.27 - \left(-\frac{\pi}{4} \times 4^2\right)(1.697)^2 = -14.07 \text{ in.}^4$$

The moment of inertia of C may now be found with respect to the x-axis, and the transfer from the centroidal axis gives

$[I = \bar{I} + Ad^2]$ $$I_x = -14.07 + \left(-\frac{\pi}{4} \times 4^2\right)(4 + 1.697)^2 = -422 \text{ in.}^4$$

The moment of inertia of the net section about the x-axis is the sum of moments of inertia of its component parts. Thus

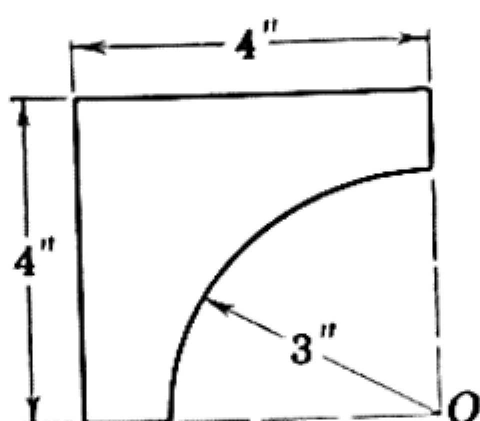
$$I_x = 1872 + 42.7 - 422 = 1493 \text{ in.}^4, \qquad \textit{Ans.}$$

and

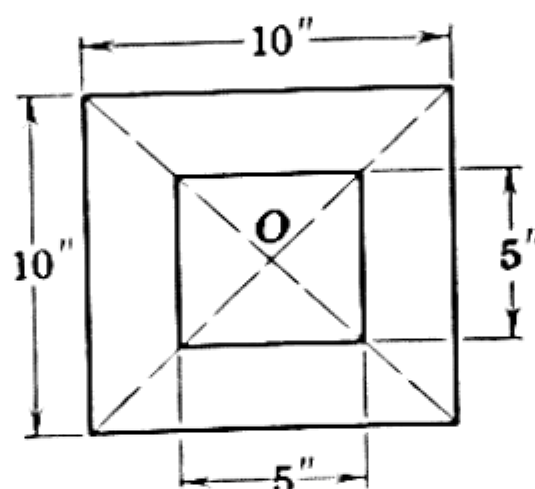
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1493}{31.43}} = 6.89 \text{ in.} \qquad \textit{Ans.}$$

PROBLEMS

A18. Determine the polar moment of inertia J for the section about point O.
Ans. $J = 138.9$ in.^{4}



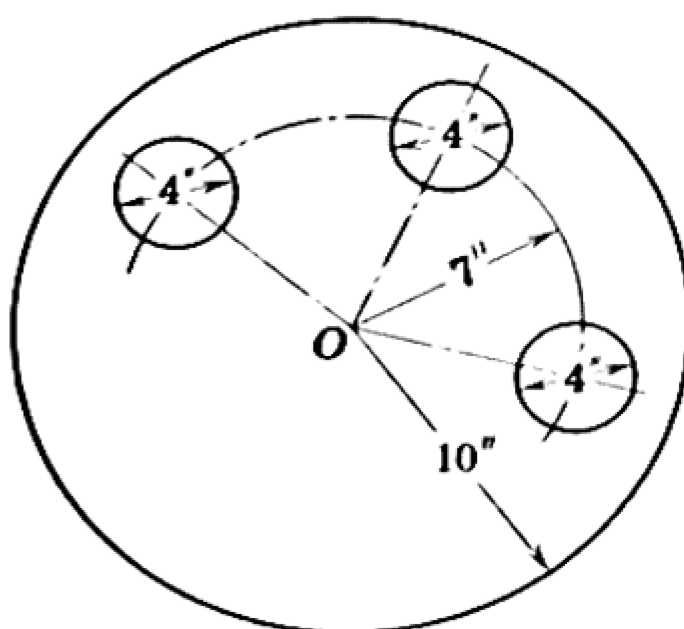
Prob. A18



Prob. A19

A19. Find the polar moment of inertia J about point O for the cross section bounded by the two squares.

A20. Find the polar moment of inertia of the net area about O.

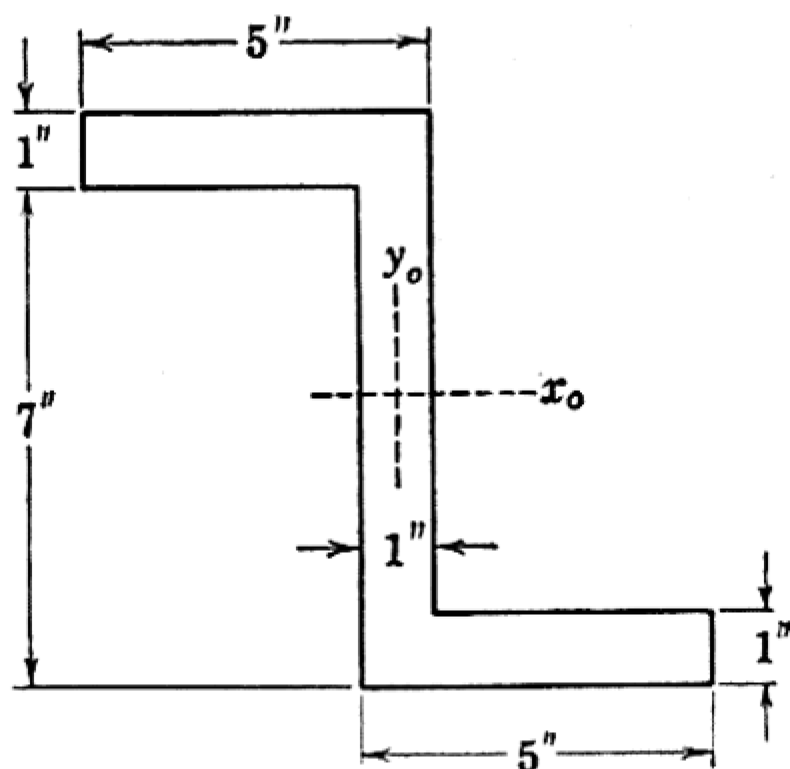


Prob. A20

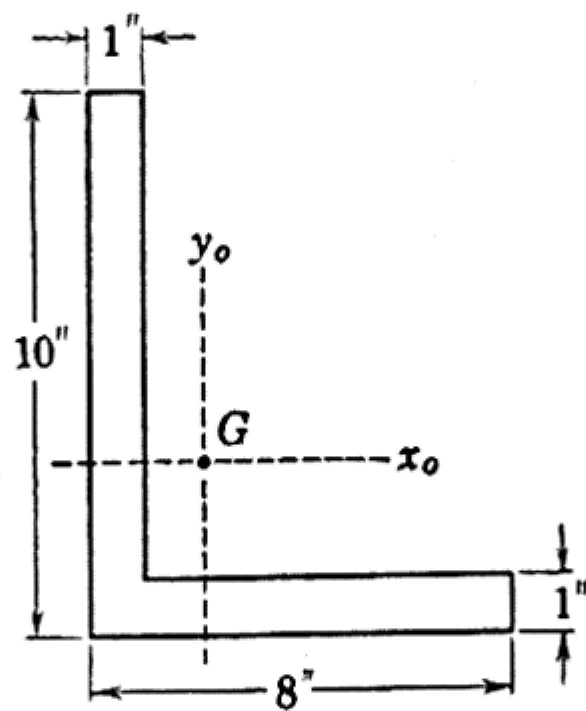
A21. Determine the moment of inertia of the area of a rectangle of sides a and b about a diagonal.
Ans. $I = \dfrac{a^3b^3}{6(a^2 + b^2)}$

A22. Find the moment of inertia about the x-axis of the area between the curves $x = y^2$ and $x = y$ from $x = 0$ to $x = 1$, where x and y are in inches.

A23. Determine the moments of inertia of the *Z*-section about the centroidal x_o- and y_o-axes.



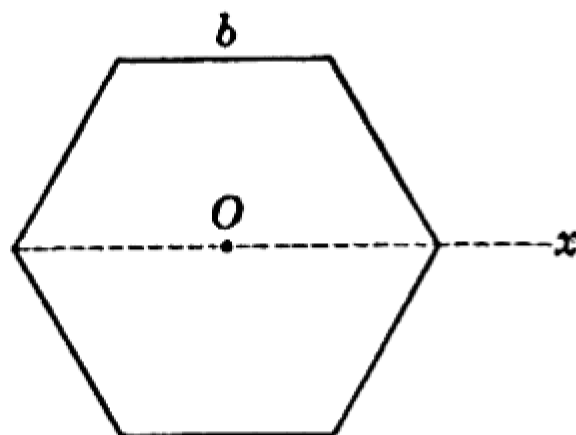
PROB. A23



PROB. A24

A24. Determine the moment of inertia of the angle section about its horizontal centroidal axis x_o. *Ans.* $\bar{I}_x = 167.3$ in.^{4}

A25. Determine the moment of inertia of the area of the hexagon of side b about the x-axis.

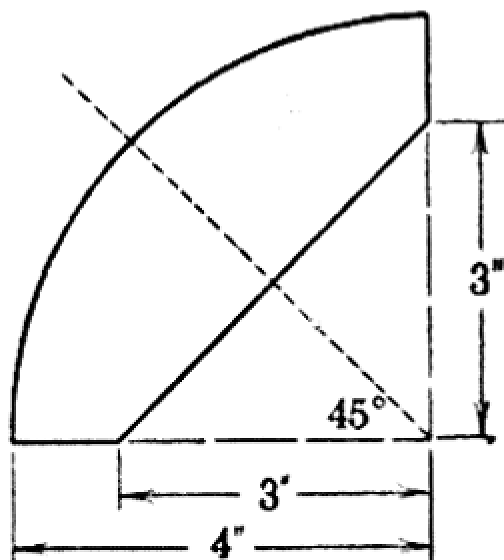


PROB. A25

A26. Determine the polar moment of inertia J of the hexagonal area of Prob. A25 about O.

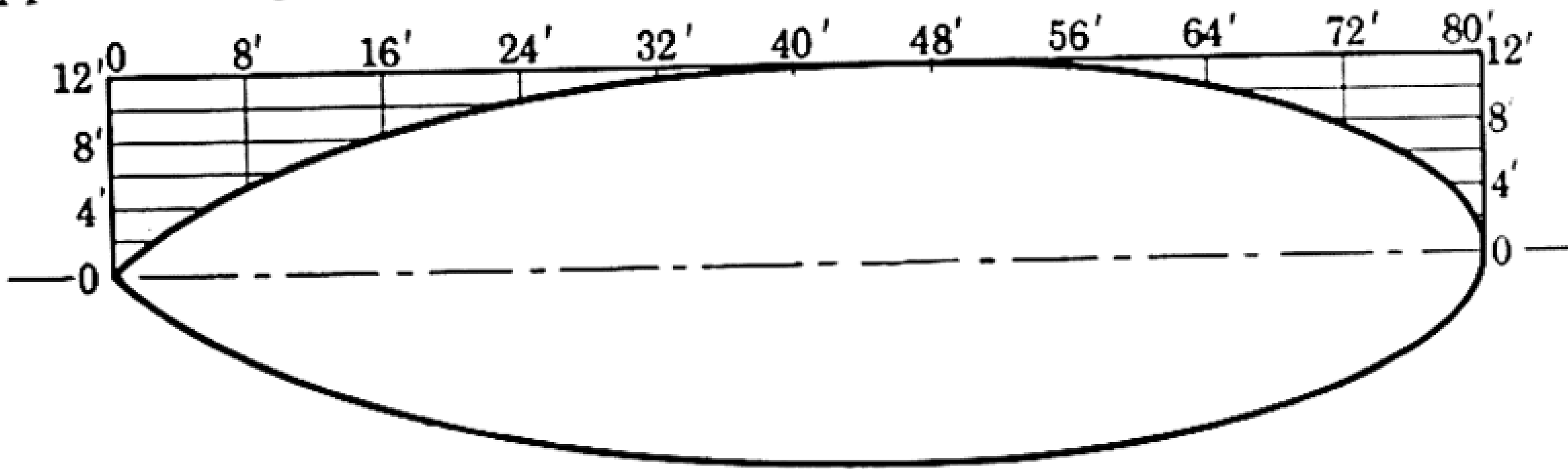
Ans. $J = \dfrac{5\sqrt{3}}{8} b^4$

A27. Determine the radius of gyration of the section about the 45 deg. axis of symmetry. (*Hint:* Use the results of Prob. A15.)



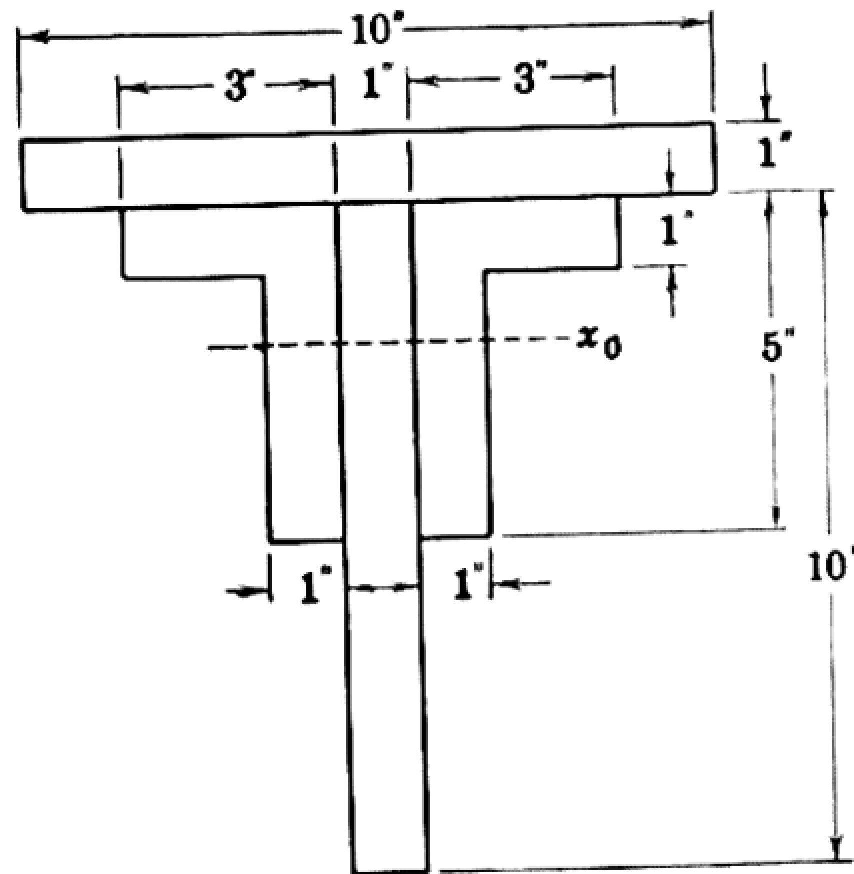
PROB. A27

A28. In the calculation of the stability of a ship's hull it is necessary to know the moment of inertia about the longitudinal center line of the area of the horizontal cross section of the hull at the waterline. Estimate this moment of inertia for the waterline shape reproduced here by dividing the area into a number of approximating strips. *Ans.* $I \cong 53{,}000$ ft.^{4}



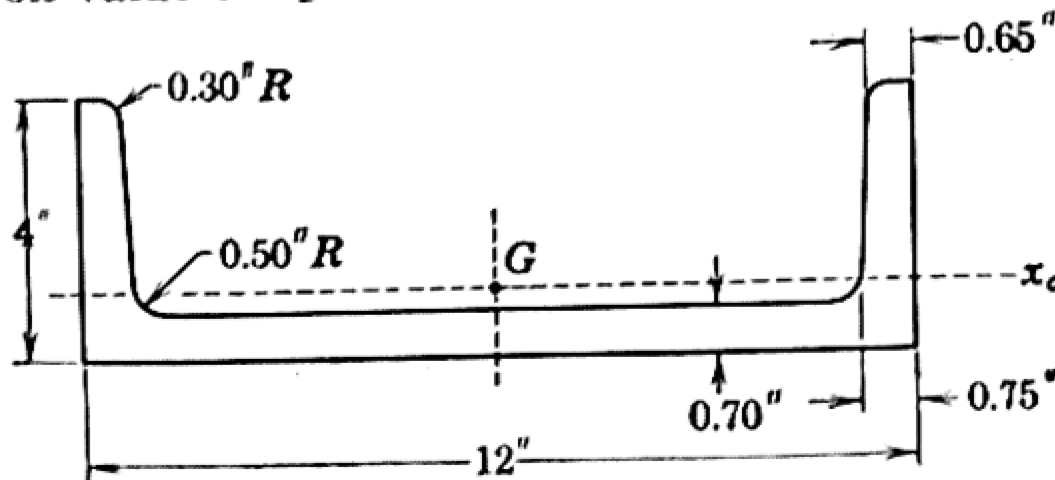
Prob. A28

A29. Determine the moment of inertia of the built-up structural section about its centroidal x_o-axis.



Prob. A29

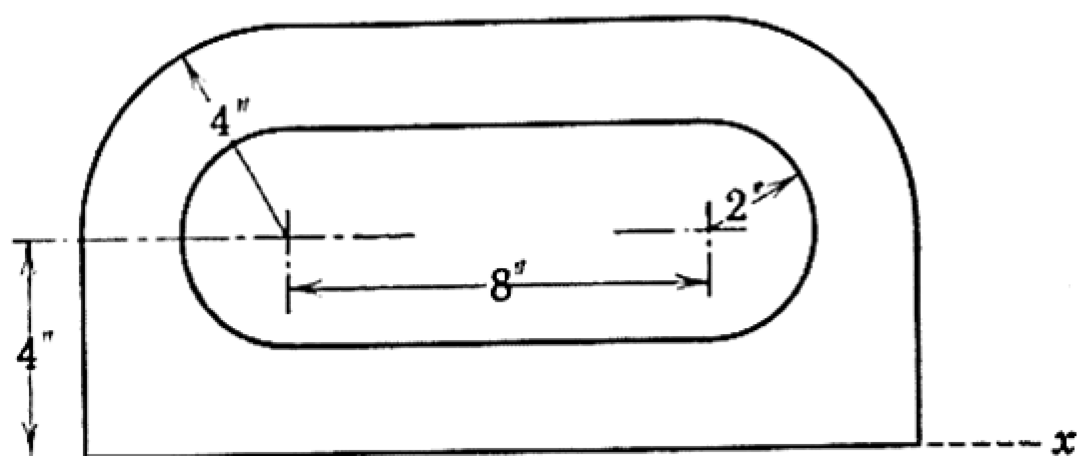
A30. Calculate the moment of inertia of the standard 12 × 4 in. channel section about the centroidal x_o-axis. Neglect the fillets and radii and compare with the handbook value of $\bar{I}_x = 16.0$ in.^{4}



Prob. A30

A31. Determine the moment of inertia of the cross section shown about the x-axis.

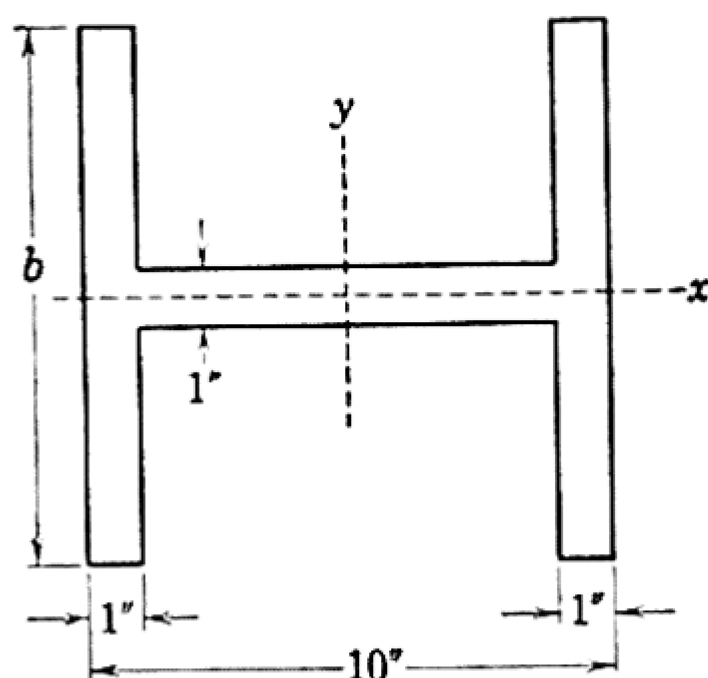
Ans. $I_x = 1611 \text{ in.}^4$



PROB. A31

A32. Determine the flange width b for the H-beam section such that the moments of inertia about the central x- and y-axes will be equal.

Ans. $b = 16.1$ in.



PROB. A32

A5. Product of Inertia. In certain problems involving unsymmetrical cross sections an expression occurs which has the form

$$dP_{xy} = xy\,dA,$$

$$P_{xy} = \int xy\,dA, \tag{A7}$$

where x and y are the coordinates of the element of area dA. The quantity P_{xy} is called the *product of inertia* of the area A about the x–y axes. Unlike moments of inertia, the product of inertia can be positive or negative.

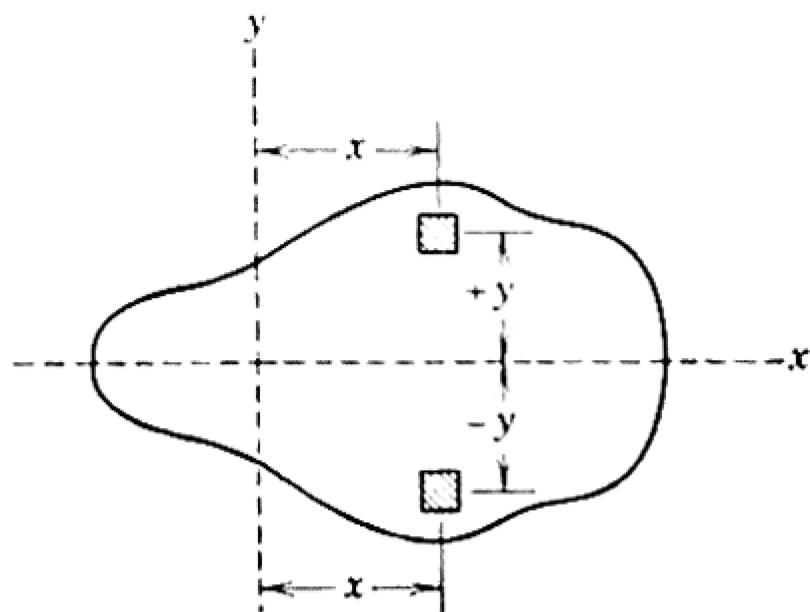


FIG. A4

By reference to Fig. A4 it can be seen for an axis of symmetry, such as the x-axis, that the sum of the terms $x(-y)\,dA$ and $x(+y)\,dA$ due to symmetrically placed elements vanishes. Since the entire area may

be considered as composed of pairs of such elements, it follows that the product of inertia vanishes.

A transfer-of-axis theorem exists for products of inertia which is similar to that for moments of inertia. By definition the product of inertia of the area A in Fig. A3 about the x- and y-axes in terms of the coordinates x_o, y_o to the centroidal axes is

$$P_{xy} = \int (x_o + d_y)(y_o + d_x)\, dA,$$

$$= \int x_o y_o\, dA + d_x \int x_o\, dA + d_y \int y_o\, dA + d_x d_y \int dA,$$

$$\boldsymbol{P_{xy} = \bar{P}_{xy} + d_x d_y A,} \tag{A8}$$

where $\bar{P}_{xy}$ is the product of inertia with respect to the centroidal x_o–y_o axes which are parallel to the x–y axes.

A6. Inclined Axes. It is often necessary to calculate the moment of inertia of an area about inclined axes. This consideration leads directly to the important problem of determining the axes about which the moment of inertia is a maximum and a minimum.

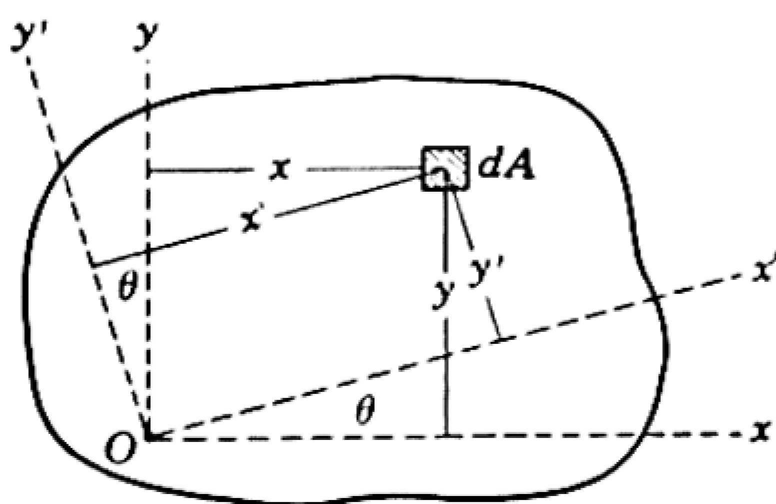


Fig. A5

In Fig. A5 the moments of inertia of the area about the x'- and y'-axes are

$$I_{x'} = \int y'^2\, dA = \int (y \cos \theta - x \sin \theta)^2\, dA,$$

$$I_{y'} = \int x'^2\, dA = \int (y \sin \theta + x \cos \theta)^2\, dA.$$

Expanding and substituting the trigonometric identities,

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}, \qquad \cos^2 \theta = \frac{1 + \cos 2\theta}{2},$$

and the defining relations for I_x, I_y, P_{xy} give

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - P_{xy}\sin 2\theta,$$
$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + P_{xy}\sin 2\theta. \qquad \text{(A9)}$$

In a similar manner

$$P_{x'y'} = \int x'y'\,dA = \frac{I_x - I_y}{2}\sin 2\theta + P_{xy}\cos 2\theta. \qquad \text{(A9}a\text{)}$$

Adding Eqs. (A9) gives $I_{x'} + I_{y'} = I_x + I_y = J_z$, the polar moment of inertia about O, which checks the result of Eq. (A3).

The angle which makes $I_{x'}$ and $I_{y'}$ a maximum or a minimum may be determined by setting the derivative of either $I_{x'}$ or $I_{y'}$ with respect to θ equal to zero. Thus

$$\frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2P_{xy}\cos 2\theta = 0.$$

Denoting this critical angle by α gives

$$\tan 2\alpha = \frac{2P_{xy}}{I_y - I_x}. \qquad \text{(A10)}$$

Equation (A10) gives two values for 2α which differ by π since $\tan 2\alpha = \tan (2\alpha + \pi)$. Consequently the two solutions for α will differ by $\pi/2$. One value defines the axis of maximum moment of inertia, and the other

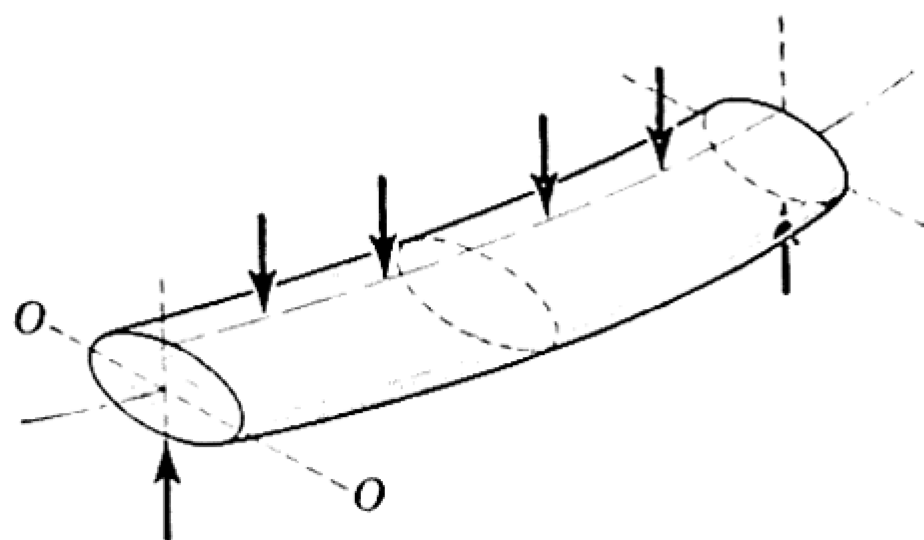


FIG. A6

value defines the axis of minimum moment of inertia. These two rectangular axes are known as the *principal axes of inertia*. Substitution of Eq. (A10) in Eq. (A9a) shows that the product of inertia is zero for principal axes of inertia. A beam of oval cross section loaded transversely, Fig. A6, if free to rotate about its longitudinal axis, will turn until the horizontal axis of its cross section is the minimum axis of inertia O–O.

The relations in Eqs. (A9), (A9a), and (A10) may be represented graphically by a diagram known as Mohr's circle. For given values of I_x, I_y, and P_{xy} the corresponding values of $I_{x'}$, $I_{y'}$, and $P_{x'y'}$ may be determined from the diagram for any desired angle θ. A horizontal axis for the measurement of moments of inertia and a vertical axis for the measurement of products of inertia are first selected, Fig. A7. Next, point A, which has the coordinates (I_x, P_{xy}), and point B, which has the coordinates $(I_y, -P_{xy})$, are located. A circle is drawn with these two points

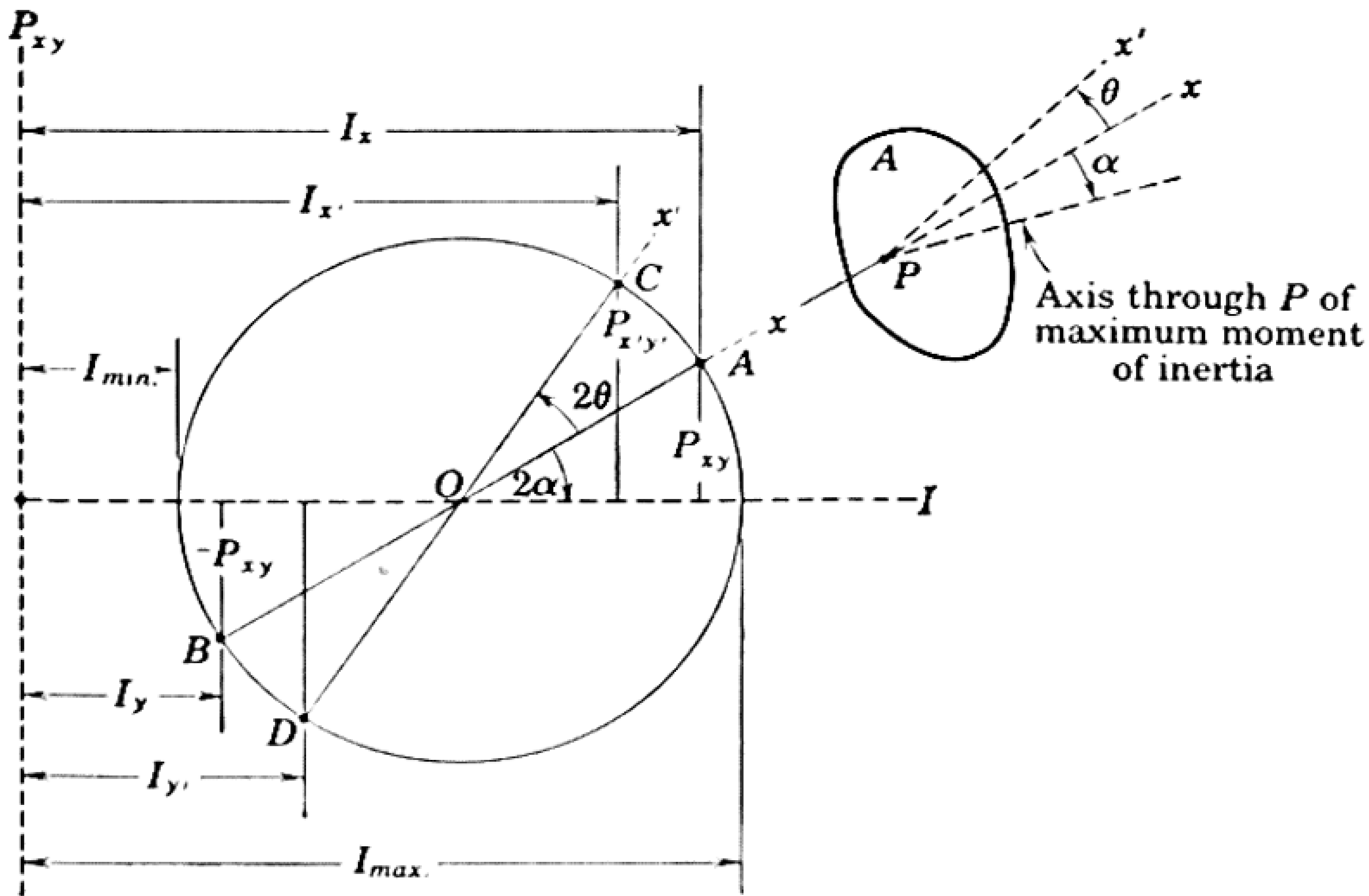
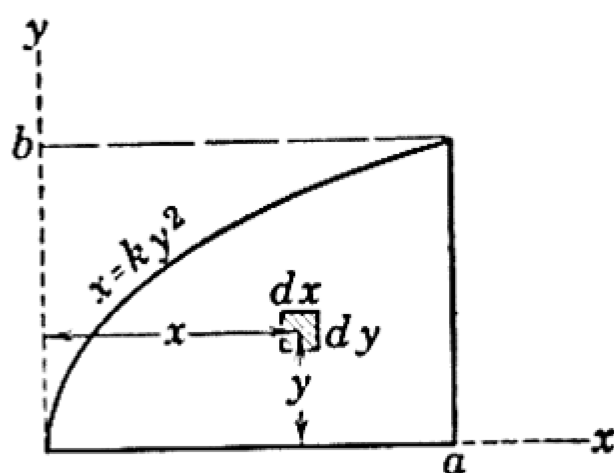


FIG. A7

as the extremities of a diameter. The angle from the radius OA to the horizontal axis is 2α or twice the angle from the x-axis of the area in question to the axis of maximum moment of inertia. Both the angle on the diagram and the angle on the area are measured in the same sense as shown. The coordinates of any point C are $(I_{x'}, P_{x'y'})$, and those of the corresponding point D are $(I_{y'}, -P_{x'y'})$. Also the angle between OA and OC is 2θ or twice the angle from the x-axis to the x'-axis. Again both angles are measured in the same sense as shown. It may be verified from the trigonometry of the circle that Eqs. (A9), (A9a), and (A10) agree with the statements made.

SAMPLE PROBLEMS

A33. Determine the product of inertia for the area under the parabola shown.

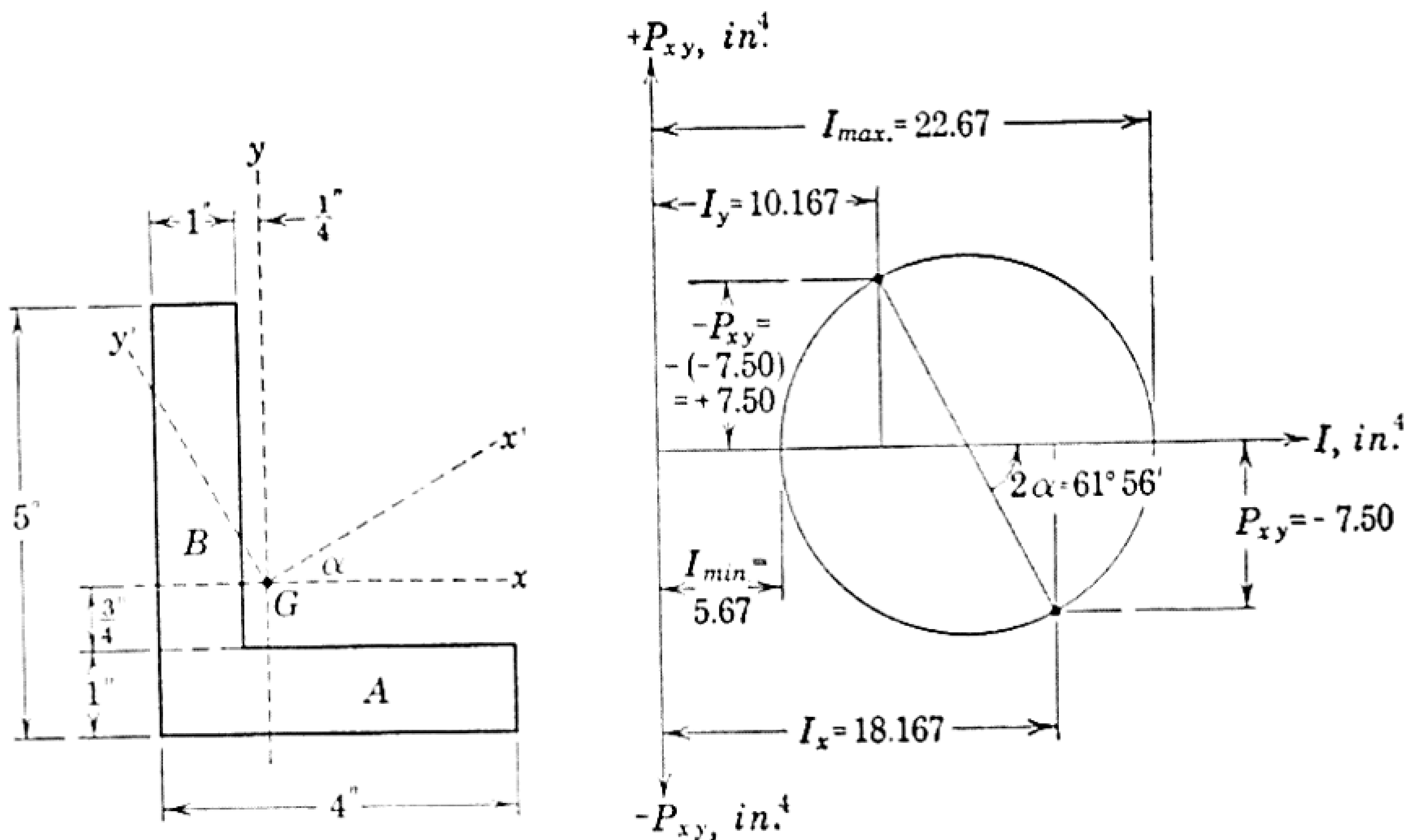


PROB. A33

Solution: The equation of the curve becomes $x = ay^2/b^2$. The product of inertia for the element $dA = dx\,dy$ is $dP_{xy} = xy\,dx\,dy$ and for the entire area is

$$P_{xy} = \int_0^b \int_{ay^2/b^2}^a xy\,dx\,dy = \int_0^b \frac{1}{2}\left(a^2 - \frac{a^2y^4}{b^4}\right) y\,dy = \tfrac{1}{6}\,a^2b^2. \qquad \textit{Ans.}$$

A34. Locate the principal centroidal axes of inertia with their corresponding maximum and minimum moments of inertia for the angle section.



PROB. A34

Solution: The centroid G is easily located as shown. The product of inertia for each rectangle about its own centroidal axes parallel to the x- and y-axes is zero by symmetry. Thus the product of inertia for part A is

$[P_{xy} = \bar{P}_{xy} + d_x d_y A]$ $\qquad$ $P_{xy} = 0 + (-\frac{5}{4})(+\frac{3}{4})(4) = -3.75 \text{ in.}^4$

Likewise for B,

$[P_{xy} = \bar{P}_{xy} + d_x d_y A]$ $\qquad$ $P_{xy} = 0 + (\frac{5}{4})(-\frac{3}{4})(4) = -3.75 \text{ in.}^4$

For the complete angle

$$P_{xy} = -3.75 - 3.75 = -7.50 \text{ in.}^4$$

The moments of inertia for part A are

$$[I = \bar{I} + Ad^2] \qquad I_x = \tfrac{1}{12} \times 4 \times 1^3 + (\tfrac{5}{4})^2 \times 4 = 6.583 \text{ in.}^4,$$

$$I_y = \tfrac{1}{12} \times 1 \times 4^3 + (\tfrac{3}{4})^2 \times 4 = 7.583 \text{ in.}^4$$

In similar manner the moments of inertia for part B are $I_x = 11.583$ in.^{4}, $I_y = 2.583$ in.^{4} Thus for the entire section

$$I_x = 6.583 + 11.583 = 18.167 \text{ in.}^4,$$

$$I_y = 7.583 + 2.583 = 10.167 \text{ in.}^4$$

The inclination of the principal axes of inertia is given by Eq. (A10). Therefore

$$\left[\tan 2\alpha = \frac{2P_{xy}}{I_y - I_x}\right] \qquad \tan 2\alpha = \frac{-2 \times 7.50}{10.167 - 18.167} = 1.875,$$

$$2\alpha = 61° \, 56', \qquad \alpha = 30° \, 58'. \qquad \textit{Ans.}$$

From Eqs. (A9) the principal moments of inertia are

$$I_{\text{max.}} = I_{x'} = \frac{18.167 + 10.167}{2} + \frac{18.167 - 10.167}{2} \times 0.4705 + 7.50 \times 0.8824$$

$$= 22.67 \text{ in.}^4; \qquad \textit{Ans.}$$

$$I_{\text{min.}} = I_{y'} = \frac{18.167 + 10.167}{2} - \frac{18.167 - 10.167}{2} \times 0.4705 - 7.50 \times 0.8824$$

$$= 5.67 \text{ in.}^4 \qquad \textit{Ans.}$$

These results may also be obtained graphically by construction of the Mohr circle as shown to the right of the angle in the figure.

PROBLEMS

A35. Determine the product of inertia P_{xy} of the area of a rectangle about x- and y-axes coinciding with two adjacent sides of lengths a and b. The rectangle lies in the first quadrant.

A36. Obtain the product of inertia for the area of the quarter circle shown with Prob. A5 about the x- and y-axes by direct integration. *Ans.* $P_{xy} = \frac{r^4}{8}$

A37. Solve Prob. A21 for the moment of inertia about a diagonal of the rectangle of sides a and b by the method of this article.

A38. The moments of inertia of an area with respect to the principal axes of inertia x, y through a point P are $I_x = 32.0$ in.^{4} and $I_y = 12.0$ in.^{4} With the aid of Mohr's circle determine the moment of inertia $I_{x'}$ and the product of inertia $P_{x'y'}$ for the area about axes x', y' through P and rotated 15 deg. clockwise from the axes x, y. *Ans.* $I_{x'} = 30.66$ in.^{4}, $P_{x'y'} = -5$ in.^{4}

A39. The moments of inertia of an area about axes x, y through a point P are $I_x = 62.8$ in.^{4} and $I_y = 148.2$ in.^{4} The product of inertia P_{xy} is negative, and the minimum moment of inertia about the axis through P is 42.8 in.^{4} Determine with the aid of Mohr's circle the maximum moment of inertia and the angle α measured positive counterclockwise from the x-axis to the axis of maximum moment of inertia.

A40. Determine the maximum and minimum moments of inertia about centroidal axes for the Z-section of Prob. A23 and indicate the counterclockwise angle α made by the axis of maximum moment of inertia with the x_o-axis.

Ans. $\bar{I}_{\text{max.}} = 181.9$ in.^{4}, $\bar{I}_{\text{min.}} = 20.7$ in.^{4}, $\alpha = 30° 8'$

A41. Determine the maximum and minimum moments of inertia about centroidal axes for the angle section of Prob. A24 and indicate the counterclockwise angle α made by the axis of maximum moment of inertia with the x_o-axis.

II. MOMENTS OF INERTIA OF MASS

A7. Definitions. The mass moment of inertia of a body is a measure of the inertial resistance to rotational acceleration. In Fig. A8 the body of mass m is caused to rotate about the axis O–O with an angular acceleration α. An element of mass dm has a component of acceleration tangent to its circular path equal to $r\alpha$, and the resultant tangential force on this element equals the force $r\alpha\, dm$. The moment of this force about the axis O–O is $r^2\alpha\, dm$. The sum of the moments of these forces for all elements is $\int r^2\alpha\, dm$. For a rigid body α is the same for all radial lines in the body and may be taken outside the integral sign. The remaining integral is known as the moment of inertia I of the mass m and is

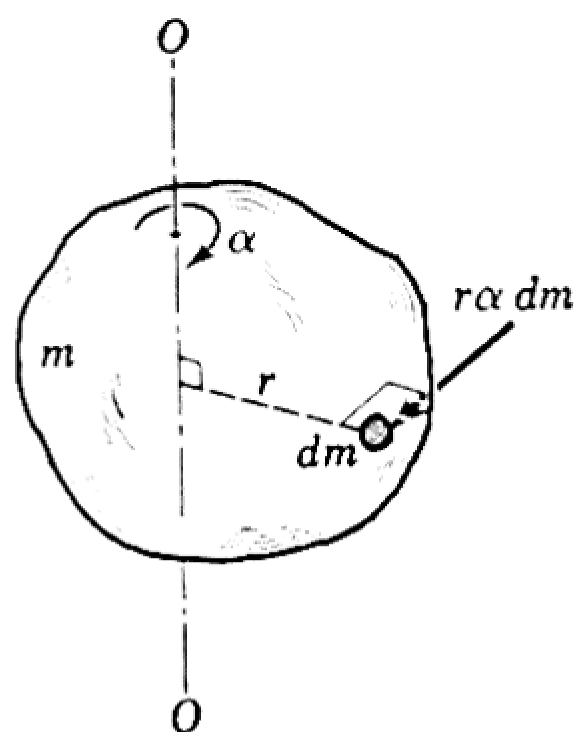


Fig. A8

$$I = \int r^2\, dm. \qquad \textbf{(A11)}$$

This integral represents an important property of a body and is involved in the force analysis of any body which has rotational acceleration. Just as the mass m of a body is a measure of the resistance to translational acceleration, the moment of inertia is a measure of resistance to rotational acceleration due to the mass or inertia of the body.

If the mass density ρ is constant throughout the body, the moment of inertia becomes

$$I = \rho \int r^2\, dV,$$

where dV is the element of volume. In this case the integral by itself defines a purely geometrical property of the body. When the mass density is not constant but is expressed as a function of the coordinates of the body, it must be left within the integral sign and its effect accounted for in the integration process.

If the body is a wire or slender rod of length L and mass ρ per unit length, the moment of inertia about an axis becomes $I = \int r^2\rho\, dL$, where r is the perpendicular distance from the element dL to the axis in question. If the body is a thin flat plate of area A and mass ρ per unit area, the moment of inertia is $I = \int r^2\rho\, dA$. When ρ is constant over the plate, the expression becomes $I = \rho\int r^2\, dA$. Thus the moment of inertia of the plate equals the mass per unit area times the *area* moment of inertia, described in Part I of this appendix for axes in or normal to the plane of the area.

In general the coordinates which best fit the boundaries of the body should be used in the integration. It is particularly important to make a good choice of the element of volume dV. An element of lowest possible order should be chosen, and the correct expression for the moment of inertia of the element about the axis involved should be used. For example, in finding the moment of inertia of a right circular cone about its central axis, a cylindrical element in the form of a circular slice of infinitesimal thickness should be used. The differential moment of inertia for this element is the correct expression for the moment of inertia of a circular cylinder of infinitesimal thickness about its central axis.

The dimensions of mass moments of inertia are (mass) $\times$ (distance)^2 and are usually expressed in the units *lb. ft. sec.*^{2} Frequently the units *ft.*^{2} *slugs* are used, where the slug is taken as the unit of mass.

A8. Radius of Gyration. The radius of gyration k of a mass m about an axis for which the moment of inertia is I is

$$k = \sqrt{\frac{I}{m}} \qquad \text{or} \qquad I = k^2m. \tag{A12}$$

Thus k is a measure of the distribution of mass of a given body about the axis in question, and its definition is analogous to the definition for the radius of gyration for second moments of area. If all the mass m could be concentrated at a distance k from the axis, the correct moment of inertia would be k^2m. The moment of inertia of a body about a particular axis is frequently indicated by specifying the radius of gyration of the

body about the axis and the weight of the body. The moment of inertia is then calculated from Eq. (A12).

A9. Transfer of Axes. If the moment of inertia of a body is known about a centroidal axis, it may be determined easily about any parallel axis. To prove this statement consider the two parallel axes in Fig. A9, one of which is a centroidal axis through the center of gravity G. The radial distances from the two axes to any element of mass dm are r_o and r, and the separation of the axes is d. Substituting the law of cosines $r^2 = r_o^2 + d^2 + 2r_o d \cos\theta$ into the definition for the moment of inertia about the noncentroidal axis gives

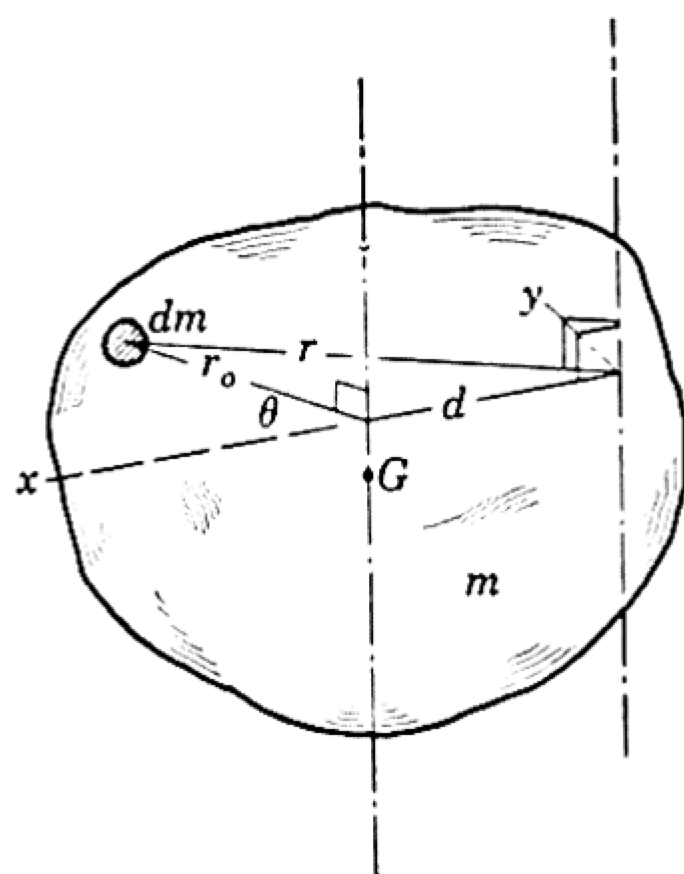


Fig. A9

$$I = \int r^2\,dm = \int (r_o^2 + d^2 + 2r_o d \cos\theta)\,dm,$$

$$= \int r_o^2\,dm + d^2 \int dm + 2d \int y\,dm.$$

The first integral is the moment of inertia $\bar{I}$ about the centroidal axis, the second integral is md^2, and the third integral equals zero since the y-coordinate of the center of gravity with respect to an origin at G is zero. Thus the parallel-axis theorem is

$$I = \bar{I} + md^2. \qquad \textbf{(A13)}$$

It must be remembered that the transfer cannot be made unless one axis passes through the center of gravity and unless the axes are parallel. When the expressions for the radii of gyration are substituted in Eq. (A13), there results

$$k^2 = \bar{k}^2 + d^2, \qquad \textbf{(A13a)}$$

which is the parallel-axis theorem for obtaining the radius of gyration k about an axis a distance d from a parallel centroidal axis for which the radius of gyration is $\bar{k}$.

A10. Product of Inertia. In a few problems of advanced mechanics the integrals

$$I_{xy} = \int xy\,dm, \qquad I_{yz} = \int yz\,dm, \qquad I_{xz} = \int xz\,dm$$

are useful. These integrals are called the products of inertia of the mass m. They may be either positive or negative. In general, a three-dimensional body has three moments of inertia about the three mutually perpendicular coordinate axes and three products of inertia about the

three coordinate planes. For an unsymmetrical body of any shape it is found that for a given origin of coordinates there is one orientation of axes for which the products of inertia vanish. These axes are called the *principal axes of inertia.* The corresponding moments of inertia about these axes are known as the *principal moments of inertia* and include the maximum possible value, the minimum possible value, and an intermediate value for any orientation of axes about the given origin.

A11. Moment of Inertia with Respect to a Plane. The moment of inertia of a body with respect to a plane is useful in some problems primarily as an aid to the calculation of the moment of inertia with respect to a line. The moment of inertia with respect to the y–z plane is defined as $\int x^2\,dm$ and that with respect to the x–z plane is $\int y^2\,dm$. Since $x^2 + y^2 = r^2$, where r is the distance from dm to the z-axis, the moment of inertia I_z about the z-axis is

$$I_z = \int r^2\,dm = \int x^2\,dm + \int y^2\,dm.$$

Similar expressions may be written for the two other axes.

A summary of some of the more useful formulas for mass moments of inertia is given in Table B6, Appendix B.

SAMPLE PROBLEMS

A42. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder of mass m and radius r about its central axis O–O.

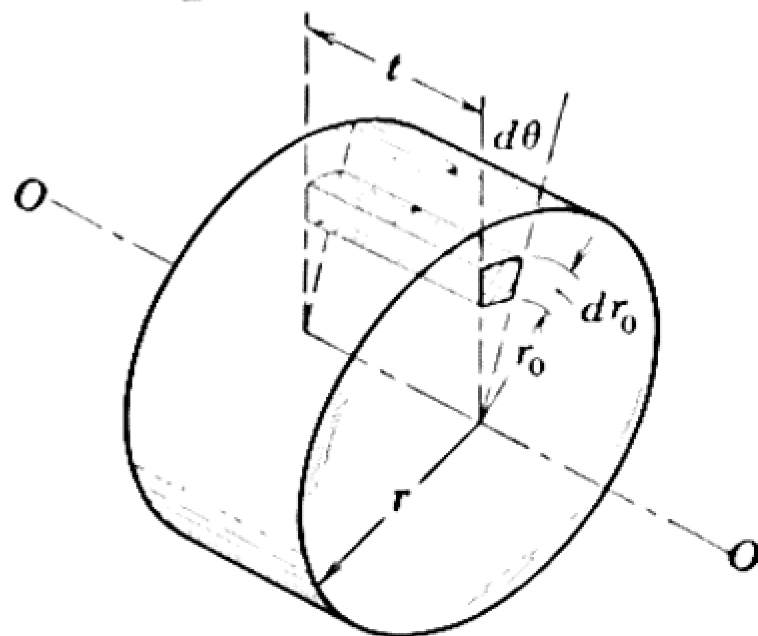
Solution: An element of mass in cylindrical coordinates is $dm = \rho\,dV = \rho t r_o\,dr_o\,d\theta$. The moment of inertia about the axis of the cylinder is

$$I = \int r_o^2\,dm = \rho t \int_0^{2\pi}\int_0^{r} r_o^3\,dr_o\,d\theta = \rho t\,\frac{\pi r^4}{2} = \tfrac{1}{2}\,mr^2. \qquad Ans.$$

The radius of gyration is

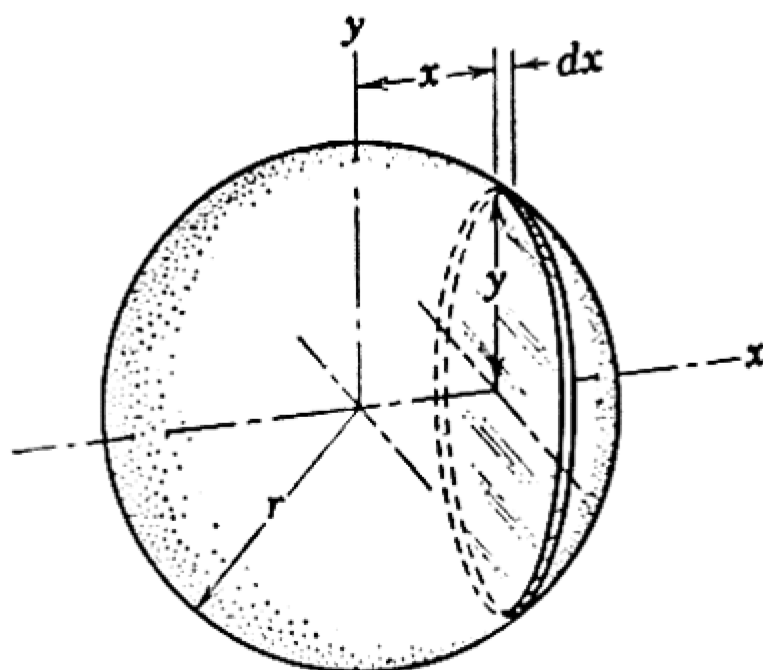
$$k = \sqrt{\frac{I}{m}} = \frac{r}{\sqrt{2}}. \qquad Ans.$$

The result $I = \frac{1}{2}mr^2$ applies *only* to a solid homogeneous circular cylinder and cannot be used for any other wheel of circular periphery.



PROB. A42

A43. Determine the moment of inertia and radius of gyration of a homogeneous solid sphere of mass m and radius r about a diameter.



PROB. A43

Solution: A circular slice of radius y and thickness dx is chosen as the volume element. From the results of Prob. A42 the moment of inertia about the x-axis of the elemental cylinder is

$$dI_x = \frac{1}{2}(dm)y^2 = \frac{1}{2}(\pi\rho y^2\,dx)y^2 = \frac{\pi\rho}{2}(r^2 - x^2)^2\,dx,$$

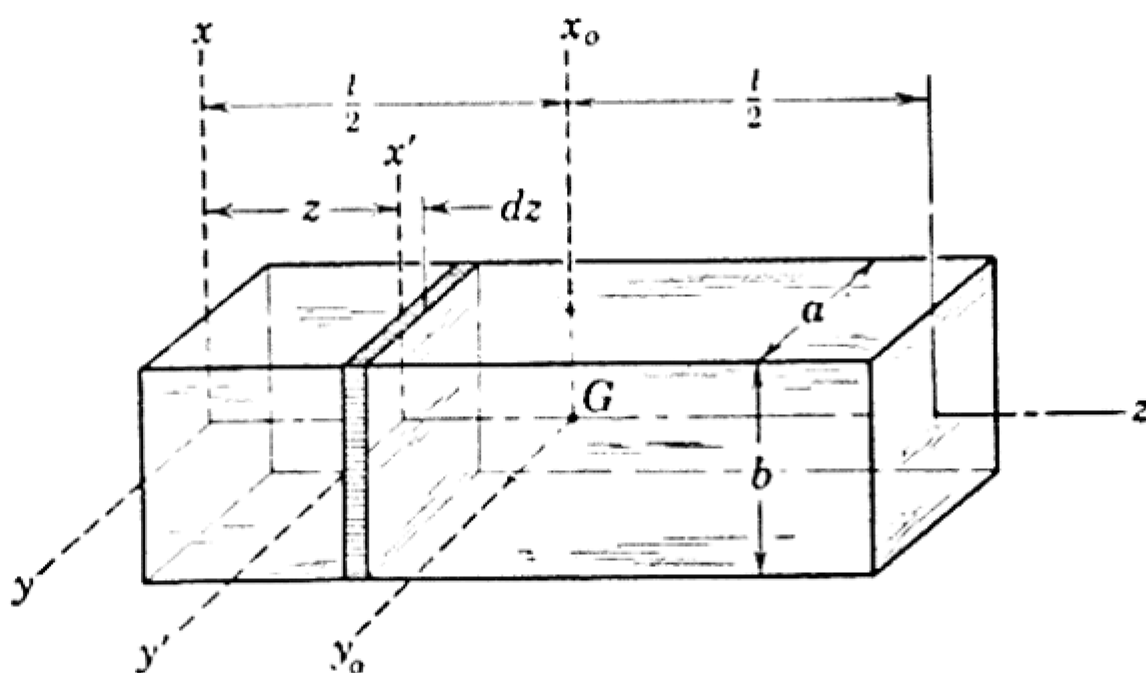
where ρ is the constant mass density of the sphere. The total moment of inertia about the x-axis is

$$I_x = \frac{\pi\rho}{2}\int_{-r}^{r}(r^2 - x^2)^2\,dx = \tfrac{8}{15}\pi\rho r^5 = \tfrac{2}{5}mr^2. \qquad \textit{Ans.}$$

The radius of gyration is

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{2}{5}}\,r. \qquad \textit{Ans.}$$

A44. Determine the moments of inertia of the homogeneous rectangular parallelepiped of mass m about the centroidal x_o- and z-axes and about the x-axis through one end.



PROB. A44

Solution: A transverse slice of thickness dz is selected as the element of volume. The moment of inertia of this slice of infinitesimal thickness equals the moment of inertia of the area of the section times the mass per unit area $\rho\, dz$. Thus the moment of inertia of the transverse slice about the y'-axis is

$$dI_{y'} = (\rho\, dz)(\tfrac{1}{12}ab^3),$$

and that about the x'-axis is

$$dI_{x'} = (\rho\, dz)(\tfrac{1}{12}a^3b).$$

As long as the element is a plate of differential thickness, the principle of Eq. (A3) may be applied to give

$$dI_z = dI_{x'} + dI_{y'} = (\rho\, dz)\frac{ab}{12}(a^2 + b^2).$$

These expressions may now be integrated to obtain the desired results.

The moment of inertia about the z-axis is

$$I_z = \int dI_z = \frac{\rho ab}{12}(a^2 + b^2)\int_0^l dz = \tfrac{1}{12}\, m(a^2 + b^2), \qquad \textit{Ans.}$$

where m is the mass of the block. By interchanging symbols the moment of inertia about the x_o-axis is

$$I_{x_o} = \tfrac{1}{12}m(a^2 + l^2). \qquad \textit{Ans.}$$

The moment of inertia about the x-axis may be found by the parallel-axis theorem, Eq. (A13). Thus

$$I_x = I_{x_o} + m\left(\frac{l}{2}\right)^2 = \tfrac{1}{12}\, m\,(a^2 + 4l^2). \qquad \textit{Ans.}$$

This last result may be obtained by expressing the moment of inertia of the elemental slice about the x-axis and integrating the expression over the length of the bar. Again by the parallel-axis theorem

$$dI_x = dI_{x'} + z^2\, dm = (\rho\, dz)(\tfrac{1}{12}a^3b) + z^2\rho ab\, dz,$$

$$= \rho ab\left(\frac{a^2}{12} + z^2\right) dz.$$

Integrating gives the result obtained previously,

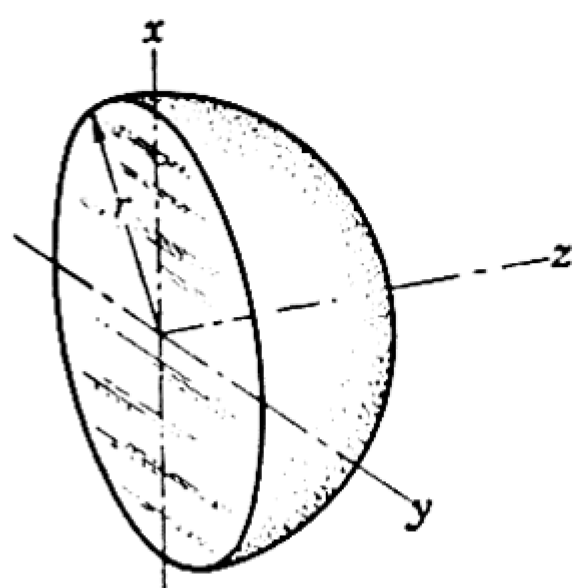
$$I_x = \rho ab\int_0^l \left(\frac{a^2}{12} + z^2\right) dz = \frac{\rho abl}{3}\left(l^2 + \frac{a^2}{4}\right) = \tfrac{1}{12}\, m\,(a^2 + 4l^2).$$

The expression for I_x may be simplified for a long prismatical bar or slender rod whose transverse dimensions are small compared with the length. In this case a^2 may be neglected compared with $4l^2$, and the moment of inertia of such a slender bar about an axis through one end normal to the bar becomes $I = \frac{1}{3}ml^2$. By the same approximation the moment of inertia about a centroidal axis normal to the bar is $I = \frac{1}{12}ml^2$.

PROBLEMS

A45. A bar 10 in. long has a square cross section 1 in. on a side. Determine the percentage error e in using the approximate formula $I = \frac{1}{3}ml^2$ for the moment of inertia about an axis normal to the bar and through the center of one end parallel to an edge. (See Prob. A44.)

Ans. $e = 0.249\%$



PROB. A47

A46. The moment of inertia of a solid homogeneous cylinder of radius r about an axis parallel to the central axis of the cylinder may be obtained approximately by multiplying the mass of the cylinder by the square of the distance d between the two axes. What per cent error e results if (a) $d = 10r$, (b) $d = 2r$?

A47. From the results of Prob. A43 state without computation the moments of inertia of the solid homogeneous hemisphere of mass m about the x- and z-axes.

A48. Determine the moment of inertia of a circular ring of mass m and inside and outside radii r_1 and r_2, respectively, about its central polar axis.

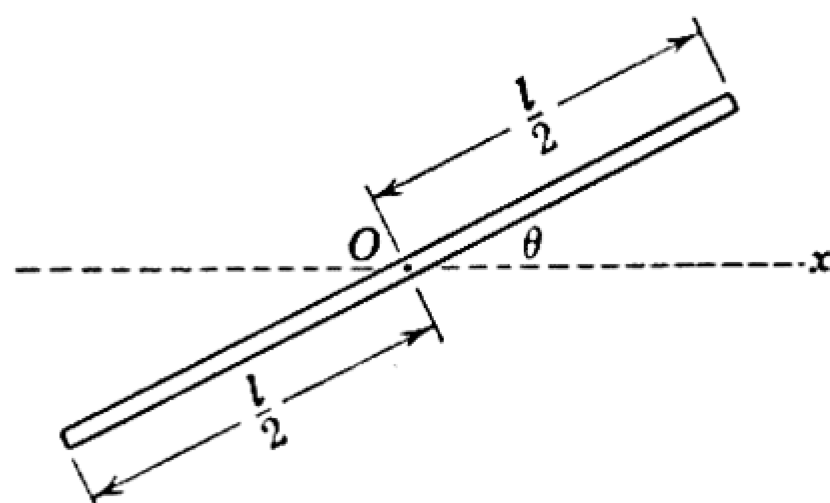
Ans. $I = \frac{1}{2}m(r_2^2 + r_1^2)$

A49. Calculate the moment of inertia of a homogeneous right circular cone of mass m and base radius r about the cone axis. *Ans.* $I = \frac{3}{10}mr^2$

A50. Without integrating determine from the results of Probs. A43 and A49 the moments of inertia about the z-axis for (a) the spherical wedge of Prob. 354 and (b) the conical wedge of Prob. 351. Each wedge has a mass m.

Ans. (a) $I_z = \frac{2}{5}ma^2$, (b) $I_z = \frac{3}{10}mr^2$

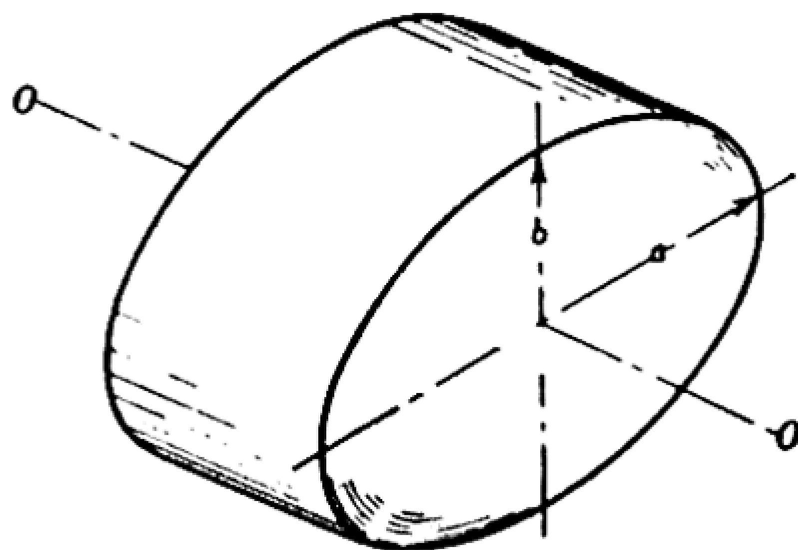
A51. Find the moment of inertia of the slender rod of mass m about the x-axis.



PROB. A51

A52. The moment of inertia of a body with respect to the x–y plane is 0.202 lb. ft. sec.^{2}, and that with respect to the y–z plane is 0.440 lb. ft. sec.^{2} The radius of gyration about the y-axis is 1.20 ft. Find the weight W of the body.

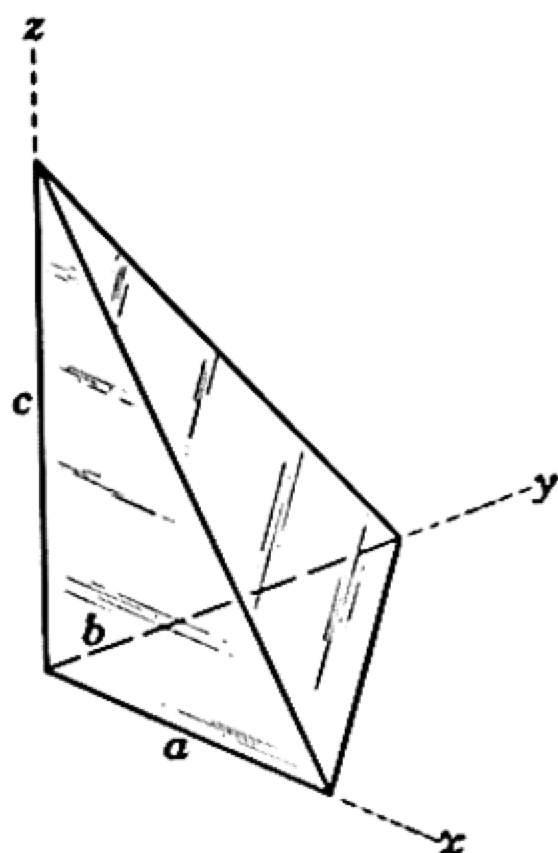
A53. Determine the moment of inertia of the elliptical cylinder of mass m about the cylinder axis O–O. *Ans.* $I = \frac{1}{4}m(a^2 + b^2)$



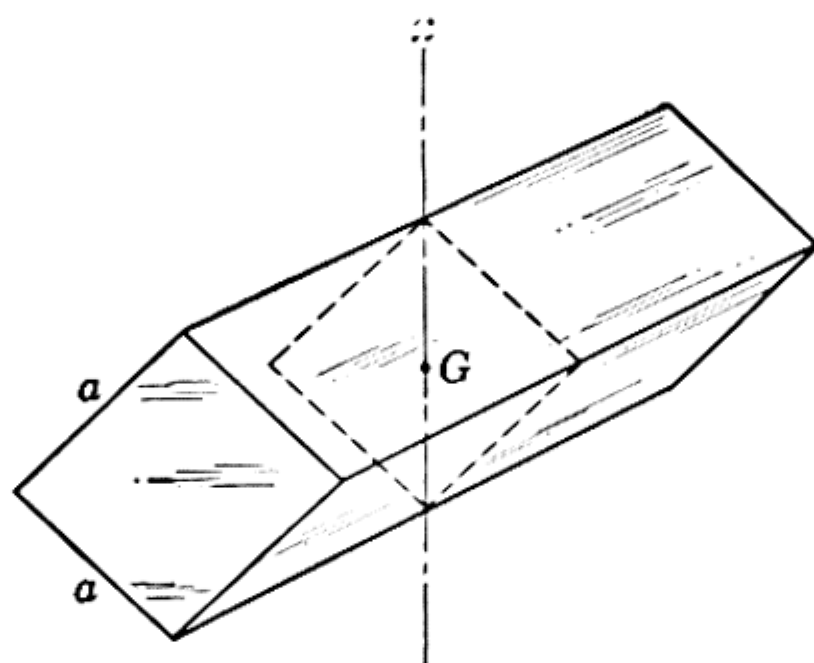
PROB. A53

A54. Determine the moment of inertia about the z-axis of the homogeneous solid paraboloid of revolution of mass m shown with Prob. 338.

A55. Find the moment of inertia of the tetrahedron of mass m about the z-axis.



Prob. A55

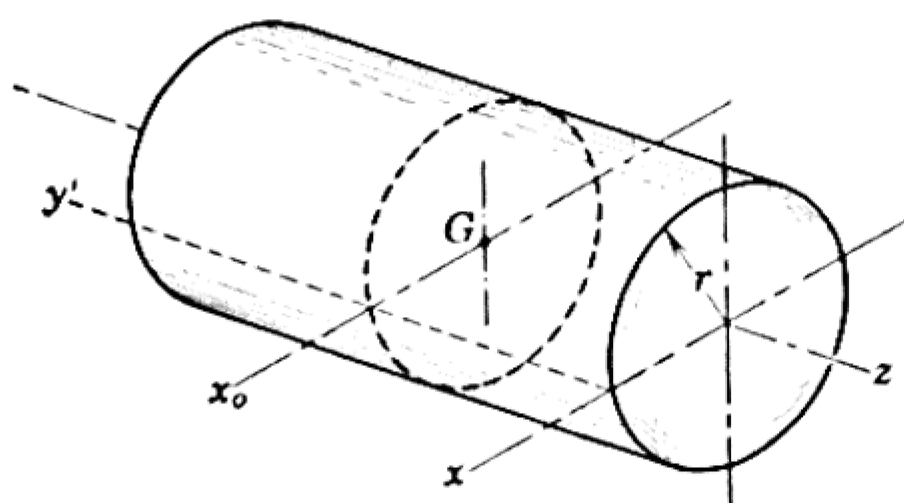


Prob. A56

A56. The homogeneous bar of square cross section has a mass m. Determine the moment of inertia of the bar about the centroidal x-axis shown which is a diagonal of the square section. *Ans.* $I_x = \frac{1}{12}m(a^2 + l^2)$

A57. Determine the moments of inertia of the homogeneous right circular cylinder of mass m about the x_o-, x-, and y'-axes shown.

Ans. $I_{x_o} = \frac{1}{12}m(3r^2 + l^2)$, $I_x = \frac{1}{12}m(3r^2 + 4l^2)$, $I_{y'} = \frac{3}{2}mr^2$

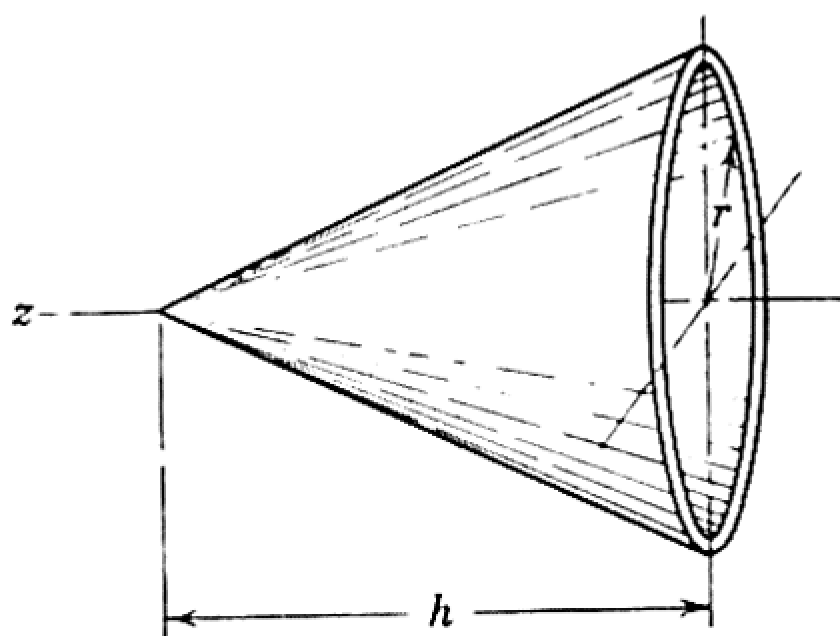


Prob. A57

A58. The density of a sphere of radius r varies uniformly with the radius from ρ_o at the center to twice that value at the surface. Determine the moment of inertia of the sphere about a diameter in terms of the mass m of the sphere.

A59. Determine the moments of inertia of the half spherical shell shown with Prob. 344 with respect to the x- and z-axes. The mass of the shell is m, and its thickness is negligible compared with the radius r. *Ans.* $I_x = I_z = \frac{2}{3}mr^2$

A60. Determine the moment of inertia of the conical shell of mass m about the axis of rotation. Wall thickness is negligible. *Ans.* $I_z = \frac{1}{2}mr^2$

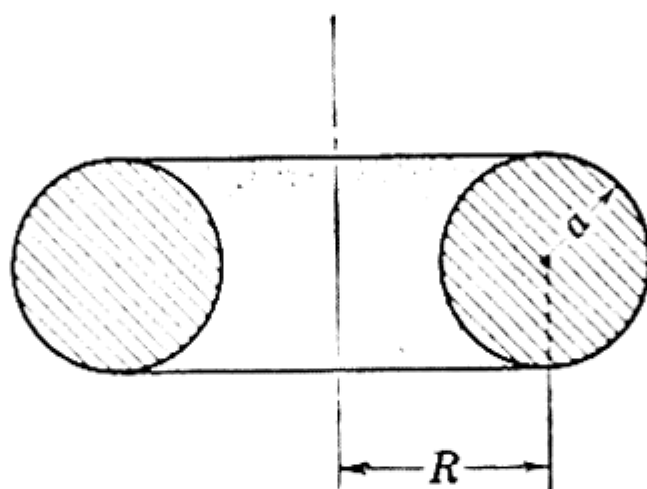


PROB. A60

* **A61.** Determine the moment of inertia about the z-axis of the bell-shaped shell of uniform small thickness described in Prob. 356 if the mass is m.

Ans. $I_z = \dfrac{15\pi - 44}{6(\pi - 2)} ma^2$

* **A62.** Determine the moment of inertia about the generating axis of a complete ring of circular section (torus) with the dimensions shown in the sectional view. *Ans.* $I = m(R^2 + \frac{3}{4}a^2)$



PROB. A62

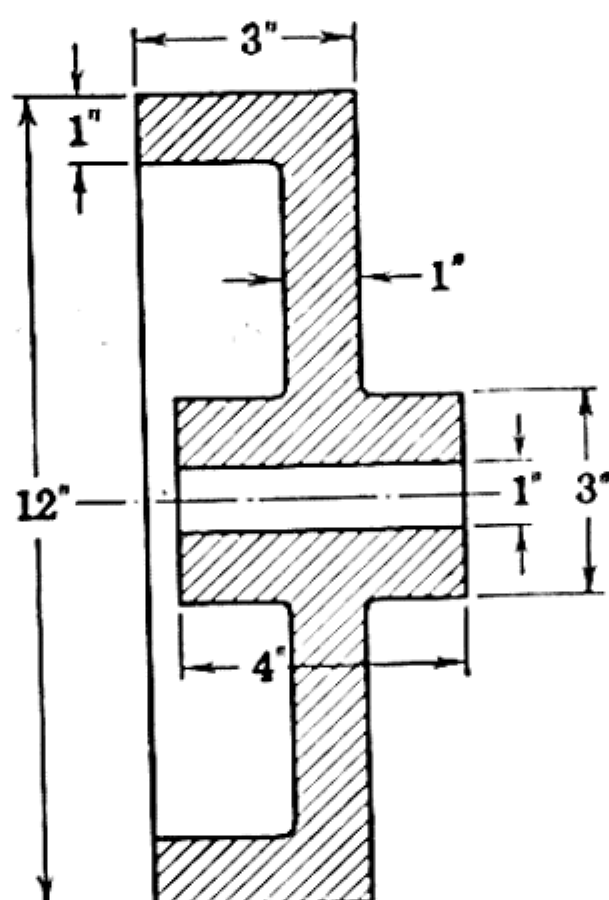
A12. Composite Bodies. The defining integral, Eq. (A11), involves the square of the distance from the axis to the element and so is always positive. Thus, as in the case of area moments of inertia, the mass moment of inertia of a composite body is the sum of the moments of inertia of the individual parts about the same axis. It is often convenient to consider a composite body as defined by positive volumes and negative volumes. The moment of inertia of a negative element, such as a hole, must be considered a minus quantity.

PROBLEMS

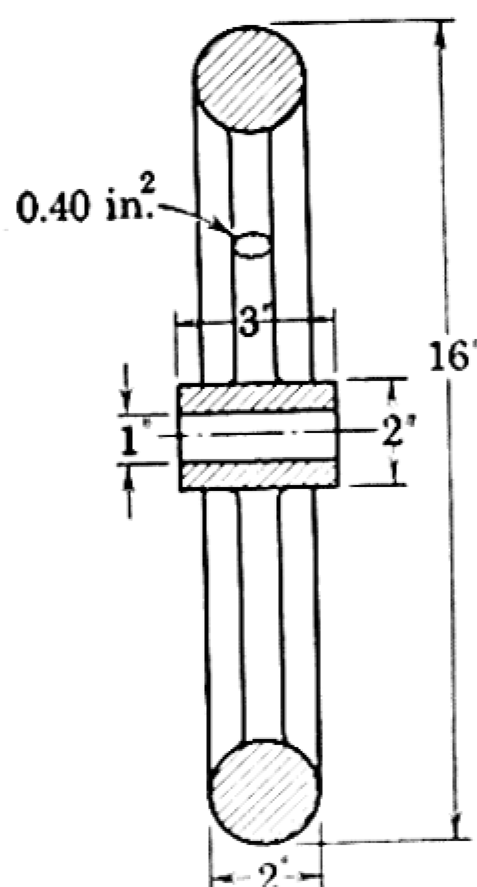
A63. Calculate the moment of inertia about the z-axis of the cylinder with the hemispherical cavity shown with Prob. 346 if the net mass is m.

Ans. $I_z = \frac{7}{10}ma^2$

A64. Calculate the moment of inertia about the central axis of the aluminum rotor shown in section.



PROB. A64

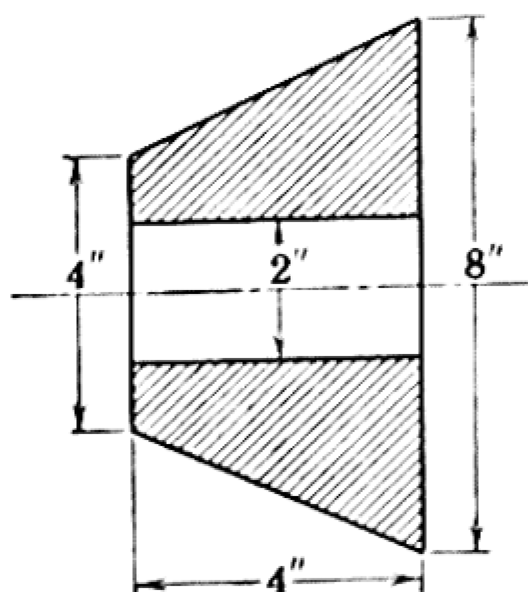


PROB. A65

A65. Calculate the moment of inertia of the steel handwheel about its axis. There are six spokes, each of which has a uniform cross-sectional area of 0.40 in.^{2}

Ans. $I = 0.431$ lb. ft. sec.^{2}

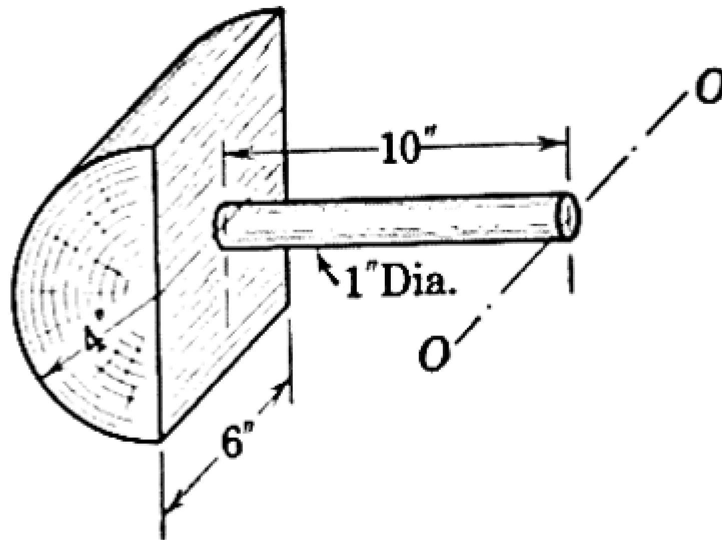
A66. Determine the radius of gyration of the homogeneous rotor, shown in section, about its central axis. *Ans.* $k = 2.43$ in.



PROB. A66

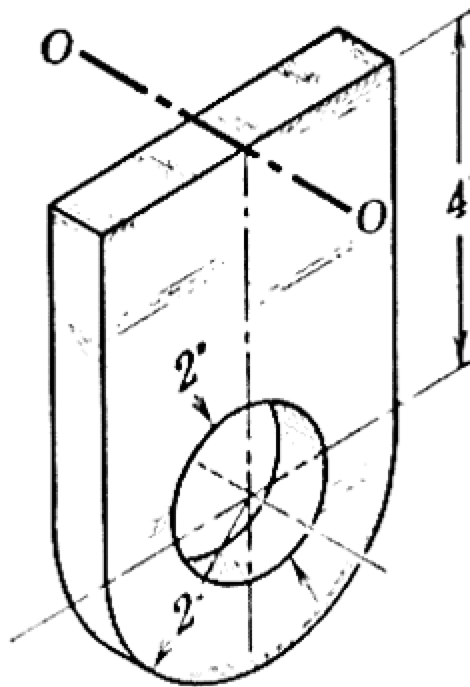
A67. The slender rod bent into the shape shown in Prob. 366 weighs 0.52 lb./ft. Determine the moment of inertia of the rod about the x-axis.

A68. Determine the moment of inertia of the mallet with respect to the axis O-O. The head is made from hard wood weighing 65 lb./ft.^{3}, and the handle is made from steel weighing 0.283 lb./in.^{3} *Ans.* $I_O = 0.1896$ lb. ft. sec.^{2}



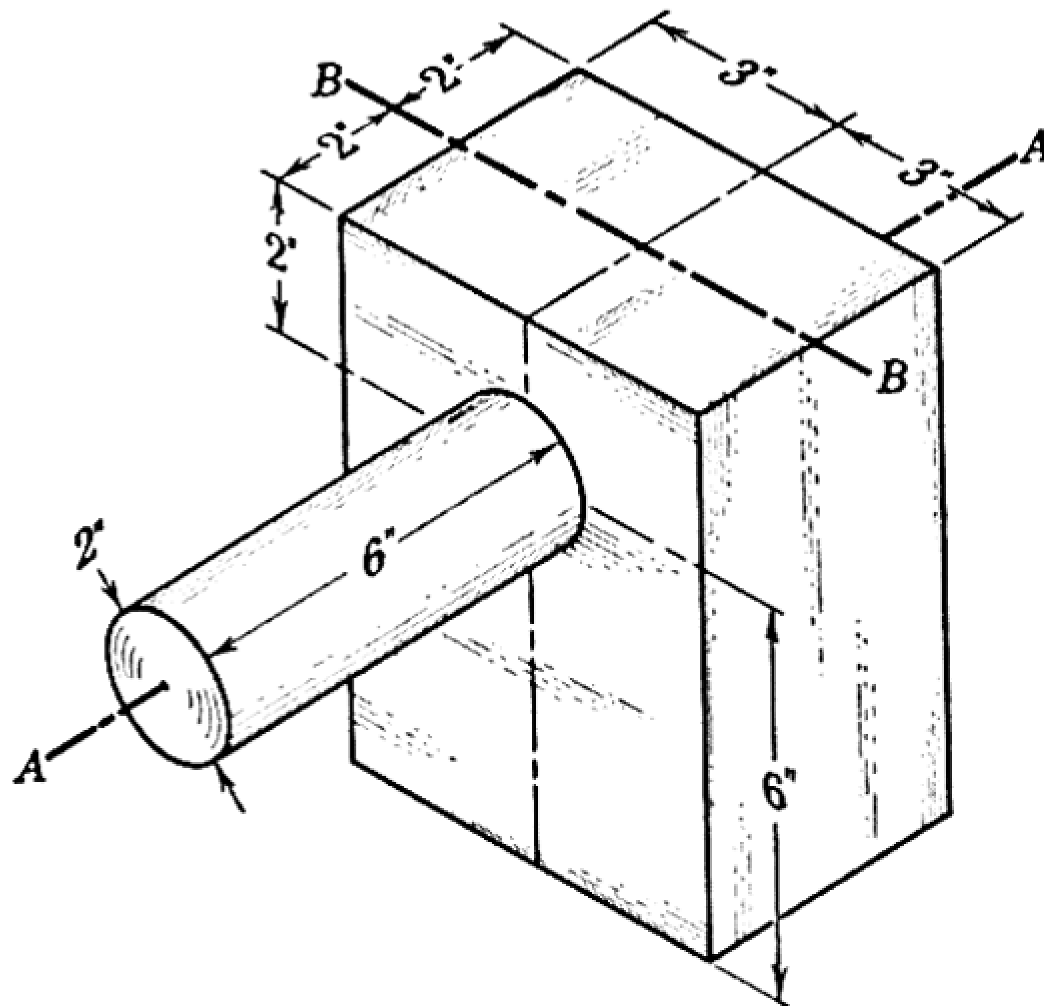
PROB. A68

A69. The part shown weighs 3.22 lb. Determine its moment of inertia about the axis O-O.



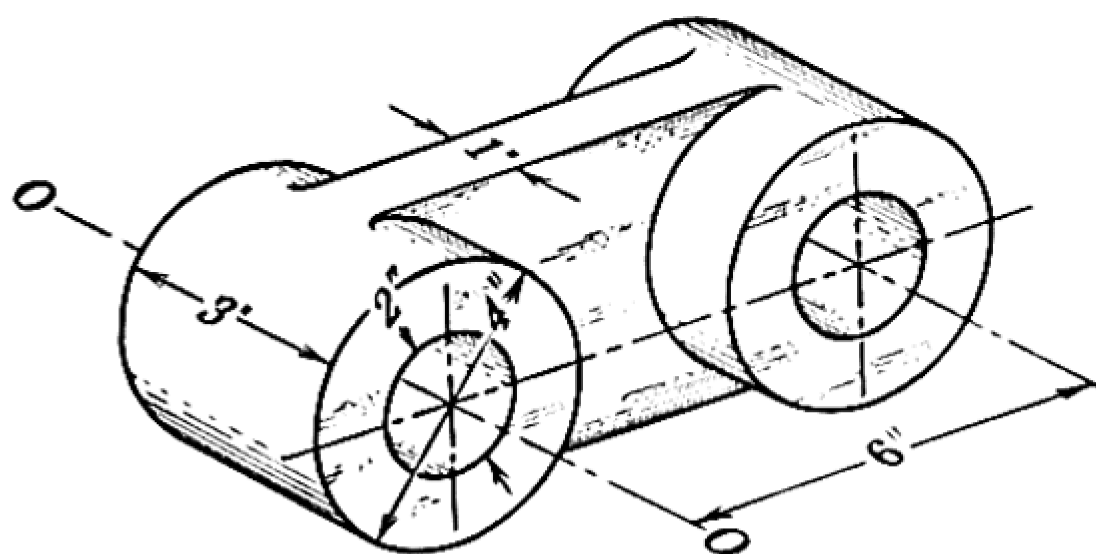
PROB. A69

* **A70.** Determine the moments of inertia of the steel body shown about axes A and B. *Ans.* $I_A = 0.1446$ lb. ft. sec.^{2}, $I_B = 0.302$ lb. ft. sec.^{2}



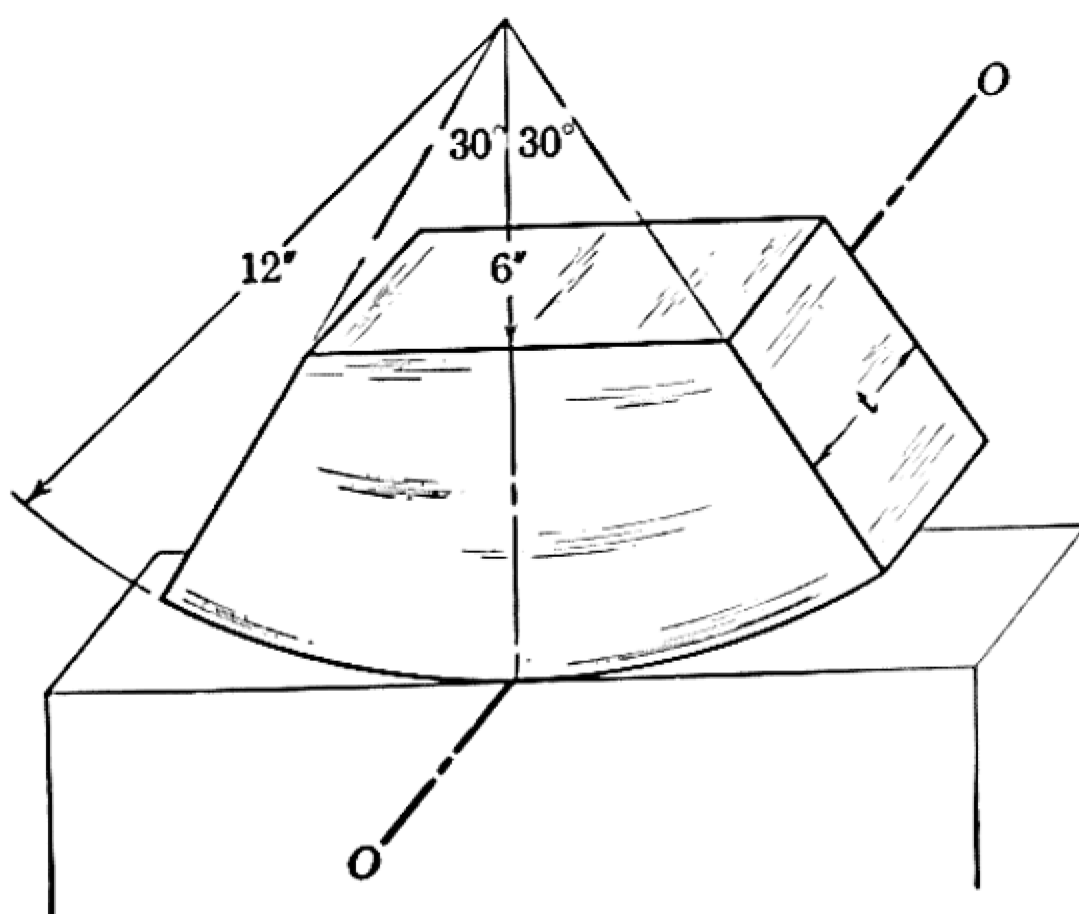
PROB. A70

* **A71.** Determine the radius of gyration of the symmetrical steel link about the axis *O–O*. *Ans.* $k = 4.36$ in.



Prob. A71

* **A72.** The desired moment of inertia of the steel rocker about the *O–O* axis is 0.204 lb. ft. sec.² Determine the necessary thickness *t*. *Ans.* $t = 3.17$ in.



Prob. A72

Appendix B

Useful Tables

TABLE B1. DENSITIES, lb./ft.^{3}

Aluminum	168	Mercury	847
Concrete (av.)	150	Oil (av.)	56
Copper	556	Steel	489
Earth (wet, av.)	110	Water (fresh)	62.4
(dry, av.)	80	(salt)	64
Ice	56	Wood (soft, pine)	30
Iron (cast)	450	(hard, oak)	50
Lead	710		

TABLE B2. COEFFICIENTS OF FRICTION

The coefficients in the following table represent typical values only. Actual coefficients for a given situation will depend on the exact nature of the contacting surfaces. A variation of the order of 25 to 100 per cent from these values could be expected in an actual problem, depending on prevailing conditions of cleanliness, roughness, pressure, lubrication, and velocity.

Contacting Surfaces	Coefficient of Static Friction	Coefficient of Kinetic Friction
Metal on metal (dry)	0.2	0.1
Metal on metal (greasy)	0.1	0.05
Rubber or leather on wood or metal (dry)	0.4	0.3
Hardwood on metal (dry)	0.6	0.4
Hardwood on metal (greasy)	0.2	0.1
Hemp on metal (dry)	0.3	0.2
Wire rope on iron pulley (dry)	0.2	0.15
Rubber tires on smooth pavement (dry)	0.9	0.8
Asbestos brake lining on cast iron	0.4	0.3
Metal on ice	...	0.02
Steel on wet grindstone	...	0.7
Cast-iron brake shoes on steel railway tires		
(10 m.p.h.)	...	0.3
(30 m.p.h.)	...	0.2
(60 m.p.h.)	...	0.05

	Coefficient of Rolling Friction, f_r
Pneumatic tires on smooth pavement	0.02
Steel tires on steel rails	0.006

Table B3. Useful Mathematical Relations

A. Series (expression in bracket following series indicates range of convergence)

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \quad [x^2 < 1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad [x^2 < \infty]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad [x^2 < \infty]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad [x^2 < \infty]$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad [x^2 < \infty]$$

B. Differentials

$$\frac{dx^n}{dx} = nx^{n-1}, \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\lim_{\Delta x \to 0} \sin \Delta x = \sin dx = \tan dx = dx$$

$$\lim_{\Delta x \to 0} \cos \Delta x = \cos dx = 1$$

$$\frac{d \sin x}{dx} = \cos x, \qquad \frac{d \cos x}{dx} = -\sin x, \qquad \frac{d \tan x}{dx} = \sec^2 x$$

$$\frac{d \sinh x}{dx} = \cosh x, \qquad \frac{d \cosh x}{dx} = \sinh x, \qquad \frac{d \tanh x}{dx} = \operatorname{sech}^2 x$$

C. Integrals

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{x} = \log x$$

$$\int \sqrt{a + bx}\, dx = \frac{2}{3b}\sqrt{(a + bx)^3}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}$$

$$\int \frac{x\, dx}{a + bx} = \frac{1}{b^2}[a + bx - a \log (a + bx)]$$

$$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a} \quad \text{or} \quad \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}$$

$$\int \sqrt{x^2 \pm a^2}\, dx = \tfrac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2 \log (x + \sqrt{x^2 \pm a^2})]$$

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

$$\int x\sqrt{a^2 - x^2}\, dx = -\tfrac{1}{3}\sqrt{(a^2 - x^2)^3}$$

$$\int x^2\sqrt{a^2 - x^2}\, dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

$$\int x^3\sqrt{a^2 - x^2}\, dx = -\tfrac{1}{5}(x^2 + \tfrac{2}{3}a^2)\sqrt{(a^2 - x^2)^3}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int x\sqrt{x^2 \pm a^2}\, dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}$$

$$\int x^2\sqrt{x^2 \pm a^2}\, dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log (x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

$$\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$$

$$\int \sin x\, dx = -\cos x$$

$$\int \cos x\, dx = \sin x$$

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \sin^3 x\, dx = -\frac{\cos x}{3}(2 + \sin^2 x)$$

$$\int \cos^3 x\, dx = \frac{\sin x}{3}(2 + \cos^2 x)$$

$$\int x \sin x\, dx = \sin x - x \cos x$$

$$\int x \cos x\, dx = \cos x + x \sin x$$

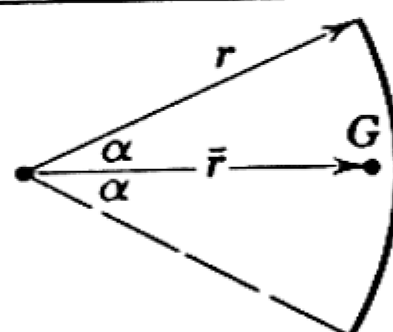
$$\int \sinh x\, dx = \cosh x$$

$$\int \cosh x\, dx = \sinh x$$

TABLE B4. CENTROIDS

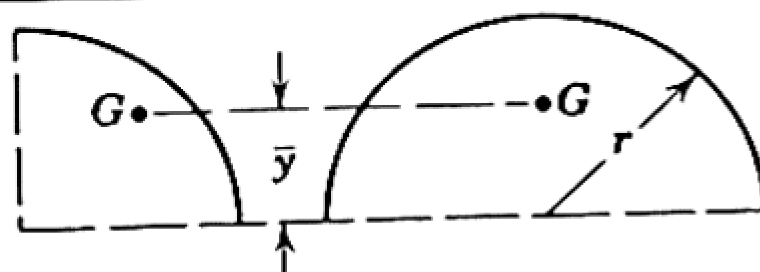
Arc Segment

$$\bar{r} = \frac{r \sin \alpha}{\alpha}$$



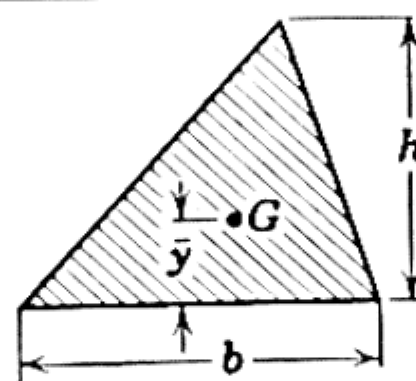
Quarter and Semicircular Arcs

$$\bar{y} = \frac{2r}{\pi}$$



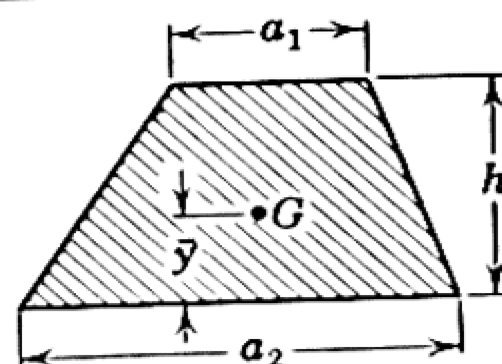
Triangular Area

$$\bar{y} = \frac{h}{3}$$



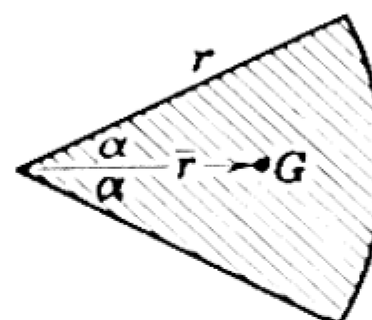
Trapezoidal Area

$$\bar{y} = \frac{1}{3}\frac{2a_1 + a_2}{a_1 + a_2}h$$



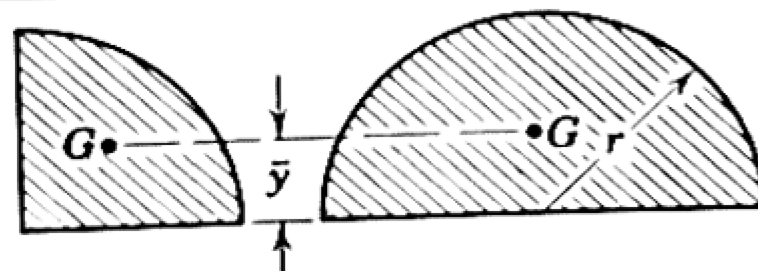
Area of Circular Sector

$$\bar{r} = \frac{2}{3}\frac{r \sin \alpha}{\alpha}$$



Quarter and Semicircular Areas

$$\bar{y} = \frac{4r}{3\pi}$$



Area of Elliptical Quadrant

$$\bar{x} = \frac{4a}{3\pi}$$

$$\bar{y} = \frac{4b}{3\pi}$$

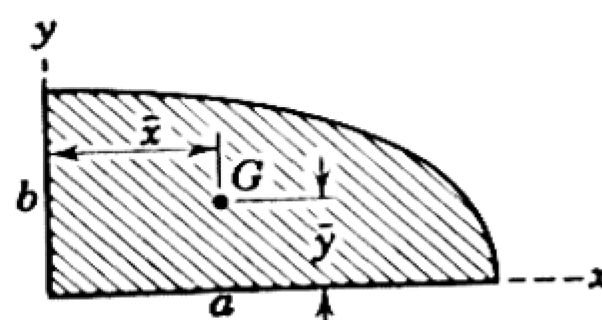
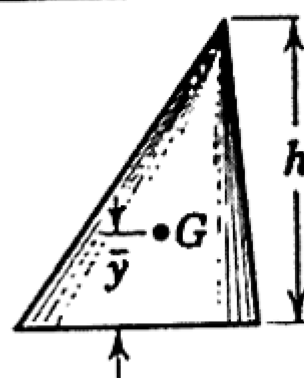


TABLE B4. CENTROIDS—*Continued*

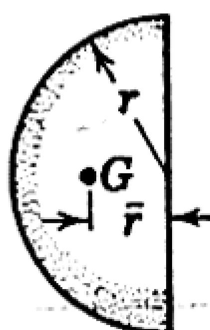
Lateral Area of Cone or Pyramid

$$\bar{y} = \frac{h}{3}$$



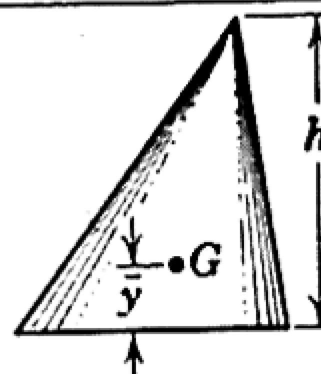
Area of Hemisphere, or Hemispherical Shell

$$\bar{r} = \frac{r}{2}$$



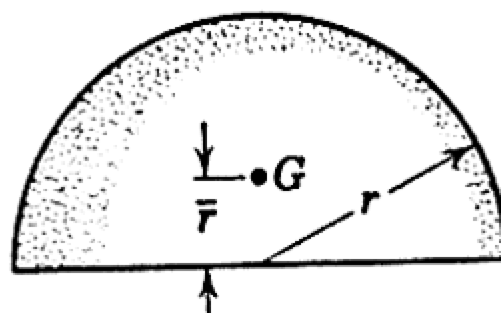
Volume of Cone or Pyramid

$$\bar{y} = \frac{h}{4}$$



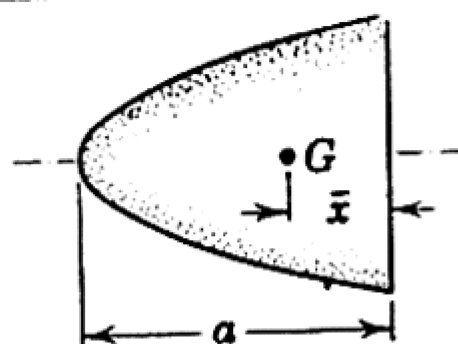
Hemispherical Volume

$$\bar{r} = \frac{3r}{8}$$



Volume of Paraboloid of Revolution

$$\bar{x} = \frac{a}{3}$$



Volume of Half Ellipsoid of Revolution

$$\bar{x} = \frac{3a}{8}$$

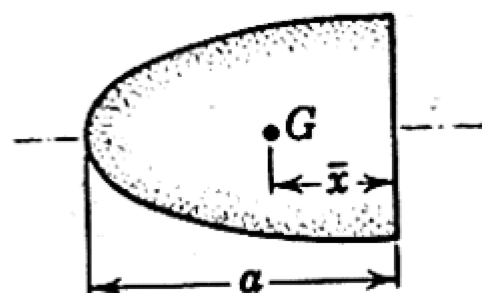


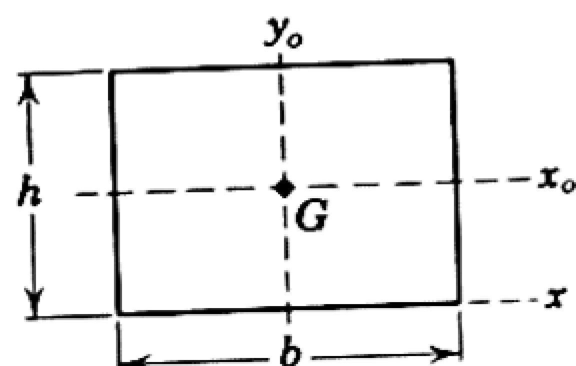
Table B5. Moments of Inertia of Areas

Rectangle

$$\bar{I}_x = \frac{bh^3}{12}$$

$$I_x = \frac{bh^3}{3}$$

$$J = \frac{bh}{12}(b^2 + h^2)$$

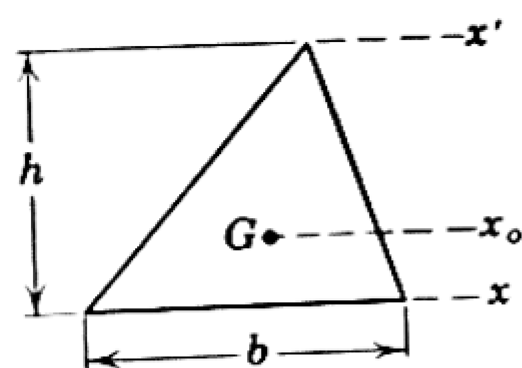


Triangle

$$\bar{I}_x = \frac{bh^3}{36}$$

$$I_x = \frac{bh^3}{12}$$

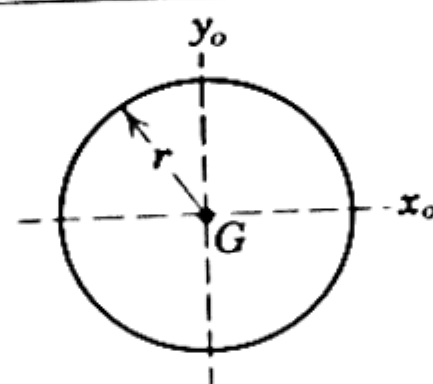
$$I_{x'} = \frac{bh^3}{4}$$



Circle

$$\bar{I}_x = \bar{I}_y = \frac{\pi r^4}{4}$$

$$J = \frac{\pi r^4}{2}$$



Ellipse

$$\bar{I}_x = \frac{\pi a b^3}{4}$$

$$\bar{I}_y = \frac{\pi a^3 b}{4}$$

$$J = \frac{\pi a b}{4}(a^2 + b^2)$$

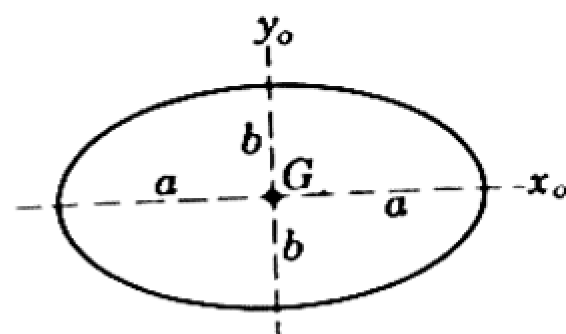


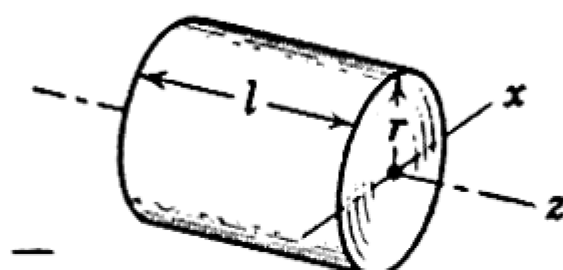
TABLE B6. MOMENTS OF INERTIA OF MASS

(m = mass of homogeneous solid shown)

Right Circular Cylinder

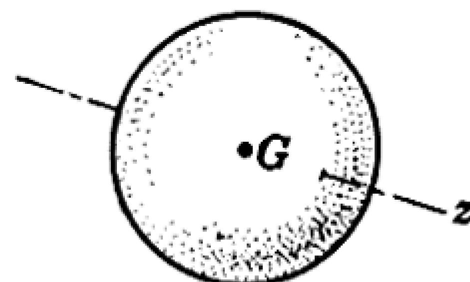
$I_z = \frac{1}{2}mr^2$

$I_x = \frac{1}{12}m(3r^2 + 4l^2)$



Sphere

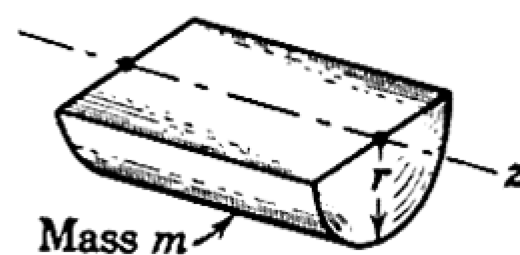
$I_z = \frac{2}{5}mr^2$



Semicylinder

$I_z = \frac{1}{2}(\frac{1}{2} \times 2mr^2)$

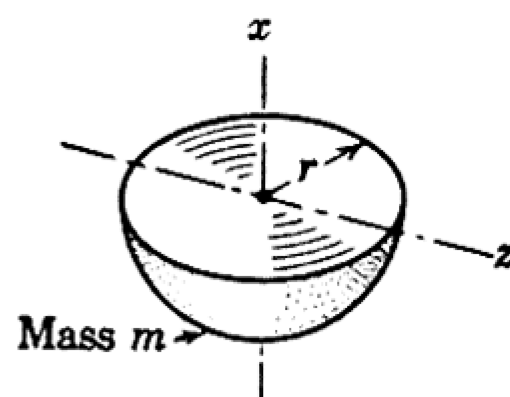
$= \frac{1}{2}mr^2$



Hemisphere

$I_x = I_z = \frac{1}{2}(\frac{2}{5} \times 2mr^2)$

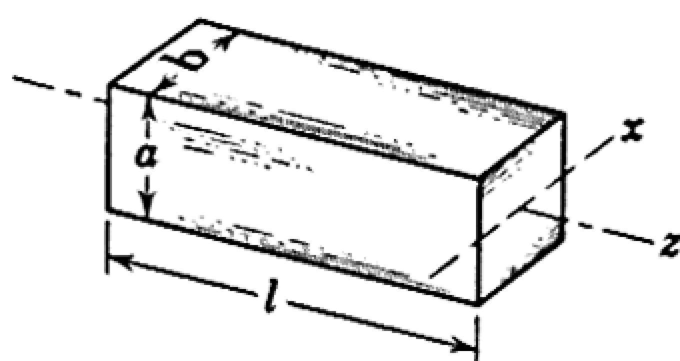
$= \frac{2}{5}mr^2$



Rectangular Parallelepiped

$I_z = \frac{1}{12}m(a^2 + b^2)$

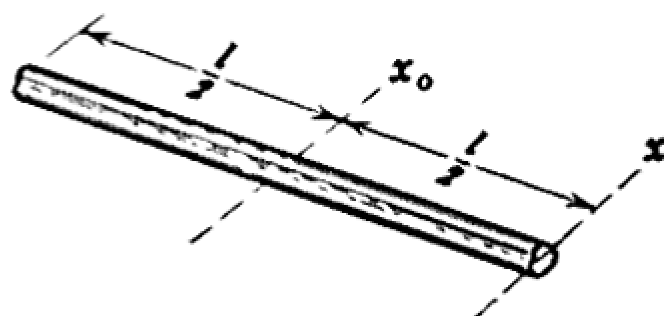
$I_x = \frac{1}{12}m(4l^2 + a^2)$



Uniform Slender Rod

$I_x = \frac{1}{3}ml^2$

$\bar{I}_x = \frac{1}{12}ml^2$



Right Circular Cone

$I_z = \frac{3}{10}mr^2$

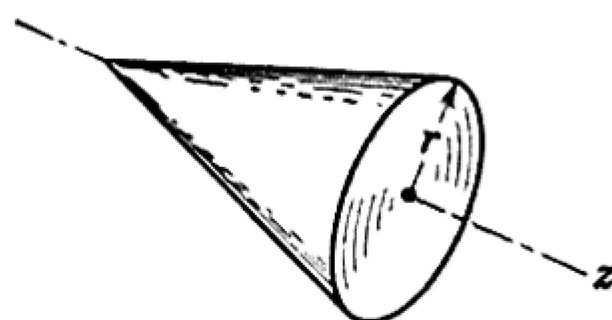
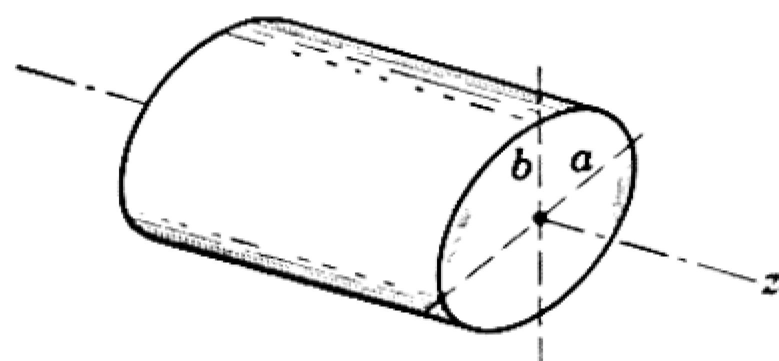


TABLE B6. MOMENTS OF INERTIA OF MASS—*Continued*

(m = mass of homogeneous solid shown)

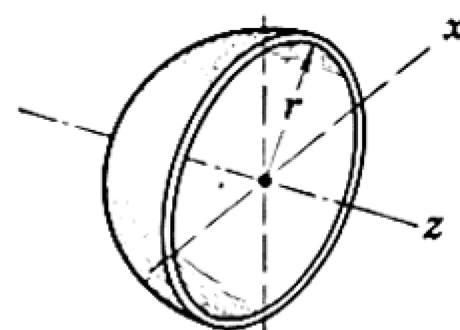
Elliptical Cylinder

$I_z = \frac{1}{4}m(a^2 + b^2)$



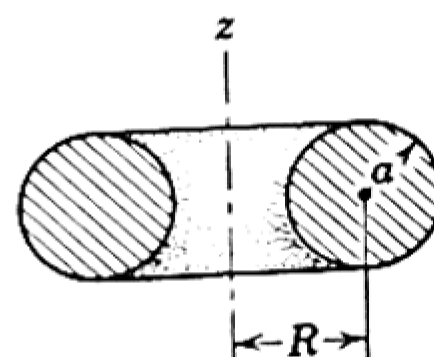
Hemispherical Shell

$I_x = I_z = \frac{2}{3}mr^2$



Torus (complete)

$I_z = m(R^2 + \frac{3}{4}a^2)$



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